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# ON IRREDUCIBLE REPRESENTATIONS OF $SO_{2n+1} \times SO_{2m}$

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ABSTRACT. In this paper, we study the restriction of irreducible representations of the group $SO_{2n+1} \times SO_{2m}$ to a spherical subgroup.

In a previous paper [G-P], we studied the question of the restriction of irreducible admissible representations $\pi$ of the quasi-split group $G = SO_N \times SO_{N-1}$ over a local field to the diagonally embedded subgroup $H = SO_{N-1}$. We conjectured the dimension of the space $\mathrm{Hom}_H(\pi, \mathbb{C})$ in terms of the signs of symplectic root numbers [G-P, Conjecture 10.7].

In this paper, we generalize the previous framework by considering irreducible admissible representations $\pi$ of the quasi-split group $G = SO_{2n+1} \times SO_{2m}$. We construct a spherical subgroup $H$ of $G$, as well as a character $\Theta\colon H \longrightarrow \mathbb{C}^*$, and present a conjecture for the dimension of the space $\mathrm{Hom}_H(\pi, \Theta)$. The subgroup $H$ is isomorphic to a semi-direct product $S \ltimes N$, where $S$ is the smaller orthogonal group in the pair and $N$ is a unipotent subgroup of the larger orthogonal group, and $\Theta$ is a generic, $S$-invariant, character of $N$. The dimension of $\mathrm{Hom}_H(\pi, \Theta)$ is again predicted by the signs of symplectic root numbers.

Turning the situation around, we obtain (via the Langlands parametrization) a representation-theoretic interpretation of the signs $\epsilon(M) = \pm 1$, for all symplectic representations $M$ of the Weil-Deligne group of a local field of the form $M = M_1 \otimes M_2$, where $M_1$ is symplectic and $M_2$ is orthogonal of even dimension.

Our conjecture was motivated by recent work of Ginzburg, Piatetski-Shapiro, and Rallis, and of Soudry. They have shown that $\dim \mathrm{Hom}_H(\pi, \Theta) \leq 1$ for all irreducible representations $\pi$ of $G$ in the non-Archimedean case, and have exhibited non-zero linear forms in this space for certain generic representations $\pi$. We wish to thank them for explaining their results to us, at a conference in April of 1992 at Ohio State. We also wish to thank Casselman and Wallach, who showed us how to properly formulate our conjectures in the Archimedean case, using the canonical smooth Fréchet globalization of $\pi$ with moderate growth, and Serre for his remarks on self-dual irreducible representations.

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TABLE OF CONTENTS



1. **Quadratic spaces.** In this section, we review some basic material on quadratic spaces over a commutative field $k$. We assume, for simplicity, that $\mathrm{char}(k) \neq 2$ throughout this paper.

Let $V$ be a finite dimensional vector space over $k$, and let $q\colon V \longrightarrow k$ be a quadratic form. Let $\langle v, w\rangle = q(v+w) - q(v) - q(w)$ be the symmetric bilinear form on $V$ associated to $q$. We say $q$ is *non-degenerate* if the pairing $\langle\ ,\ \rangle$ gives an isomorphism from $V$ to its dual space $V^\vee$; equivalently, the form $q$ is non-degenerate if the variety defined by the equation $q = 0$ is non-singular in $\mathbb{P}(V)$. A quadratic space $V$ is by definition a finite dimensional $k$-vector space together with a non-degenerate quadratic form.

The quadratic space $k$ of dimension 1 with form $q(x) = ax^2, a \neq 0$, will be denoted $\langle a\rangle$. If $V_1$ and $V_2$ are quadratic spaces over $k$, the orthogonal direct sum $V = V_1 \oplus V_2$ is the quadratic space with form $q(v_1 + v_2) = q_1(v_1) + q_2(v_2)$. We say two quadratic spaces $V$ and $V'$ are *isomorphic* if there is a $k$-linear isomorphism $T\colon V \longrightarrow V'$ which satisfies $q'(Tv) = q(v)$ for all $v \in V$. Then any quadratic space $V$ is isomorphic to an orthogonal direct sum of lines [S1; Chapter IV]

$$V \simeq \langle a_1\rangle \oplus \langle a_2\rangle \oplus \cdots \oplus \langle a_N\rangle \quad N = \dim V. \tag{1.1}$$

We will write

$$N = \dim V = \begin{cases} 2n \\ 2n+1 \end{cases} n \geq 0 \tag{1.2}$$

depending on the parity of $N$. Define the normalized determinant $d(V)$ of a quadratic space by the formula

$$d(V) \equiv (-1)^n \det(\langle e_i, e_j\rangle) \quad \text{in } k^*/k^{*2} \tag{1.3}$$

where $\{e_i\}$ is any basis for $V$. If $\{e_i\}$ is an orthogonal basis giving the isomorphism (1.1), we have $d(V) \equiv (-1)^n a_1 \cdots a_N \pmod{k^{*2}}$. We call $d(V)$ the *discriminant* of the space $V$.

An isotropic subspace $X \subset V$ is a subspace on which the form $q$ is identically zero. We have $\dim X \leq n$ for any isotropic subspace $X$; if $V$ contains an isotropic subspace $X$ with $\dim X = n$, we say the space $V$ is *split*. In this case, $V$ contains a pair of isotropic subspaces $(X, X')$ with $\dim X = \dim X' = n$, $X \cap X' = 0$, and $X + X'$ non-degenerate [S1; Chapter IV, 1.3]. More precisely, one can construct bases $\{e_i\}$ of $X$ and $\{e_i'\}$ of $X'$ such that $\langle e_i, e_j' \rangle = \delta_{ij}$.

If $\dim V$ is even and $V$ is split, then $V \simeq X + X'$. Hence $V$ is unique up to isomorphism and $d(V) \equiv 1$. If $\dim V$ is odd and $V$ is split, then $V \simeq (X + X') \oplus \langle a \rangle$. The isomorphism class of $V$ is determined by $d(V) \equiv a$.

If $\dim V = 2n$ and $V$ contains an isotropic subspace $X$ with $\dim X = n - 1$, we say that $V$ is *quasi-split*. Then $V \simeq (X + X') \oplus Y$ as above, with $\dim Y = 2$. The isomorphism class of $V$ is determined by the isomorphism class of $Y$, and $V$ is split if and only if $d(Y) \equiv d(V) \equiv 1$.

The possibilities for the 2-dimensional quadratic space $Y$ are as follows. Let $E = k[x]/\big(x^2 - d(Y)\big)$ be the quadratic $k$-algebra with discriminant $\equiv d(Y)$, let $\mathrm{N}\colon E \longrightarrow k$ be the norm, and let $c$ be a class in $k^*/\mathrm{N}E^*$. Then $cE = \{\alpha \in E, q(\alpha) = c \cdot \mathrm{N}(\alpha)\}$ is a quadratic space of dimension 2 and discriminant $\equiv d(Y)$, and $Y$ is isomorphic to precisely one of the spaces $cE$. Hence, a quasi-split even dimensional orthogonal space $V$ is determined up to isomorphism by $d(V)$ and a class $c$ in $k^*/\mathrm{N}E^*$, where $E = k[x]/\big(x^2 - d(V)\big)$.

2. **Orthogonal groups.** Let $V$ be a quadratic space over $k$. The orthogonal group $O(V)$ is the algebraic group of all linear transformations $T\colon V \longrightarrow V$ which satisfy $q(Tv) = q(v)$ for all $v \in V$. This forces $\det T = \pm 1$. The special orthogonal group $\mathrm{SO}(V)$ is the subgroup of those orthogonal transformations which satisfy $\det T = 1$. Then $\mathrm{SO}(V)$ is connected and reductive over $k$ [B1; §23.4].

The group $\mathrm{SO}(V)$ is split over $k$ if and only if the space $V$ is split. If $\dim V = 2n$, the group $\mathrm{SO}(V)$ is quasi-split over $k$ (*i.e.* contains a $k$-rational Borel subgroup) if and only if the space $V$ is quasi-split; in this case $\mathrm{SO}(V)$ is split by the extension $E = k\big(\sqrt{d(V)}\big)$.

Let $\bar{k}$ denote a separable closure of $k$, and put $\Gamma = \mathrm{Gal}(\bar{k}/k)$. If $G$ is an algebraic group over $k$, we denote the set $H^1\big(\Gamma, G(\bar{k})\big)$ in non-abelian Galois cohomology by $H^1(k, G)$ (*cf.* [S2; Chapter 1,§2]).

PROPOSITION 2.1. *The classes in $H^1\big(k, O(V)\big)$ correspond bijectively to the isomorphism classes of quadratic spaces $V'$ over $k$ with* $\dim V' = \dim V$.

*The classes in $H^1\big(k, \mathrm{SO}(V)\big)$ correspond bijectively to the isomorphism classes of quadratic spaces $V'$ over $k$ with* $\dim V' = \dim V$ *and* $d(V') \equiv d(V)$.

PROOF. The first statement is established in [S2; Chapter 3, Proposition 4]. When $\dim V \geq 1$, we have an exact sequence

$$1 \longrightarrow \mathrm{SO}(V) \longrightarrow O(V) \xrightarrow[\det]{} \langle \pm 1 \rangle \longrightarrow 1$$

of algebraic groups over $k$, which gives rise to an exact sequence of pointed sets:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{SO}(V)(k) & \longrightarrow & O(V)(k) & \xrightarrow[\det]{} & \langle \pm 1 \rangle & \longrightarrow & H^1\big(k, \mathrm{SO}(V)\big) \\
& & & & & & & & \downarrow \\
& & & & k^*/k^{*2} \simeq H^1(k, \pm 1) & & \xleftarrow[\beta]{} & & H^1\big(k, O(V)\big)
\end{array}$$

Since there are reflections in $O(V)(k)$, the determinant is surjective on $k$-valued points, and $H^1\big(k, \mathrm{SO}(V)\big)$ is the subset of classes $V'$ in $H^1\big(k, O(V)\big)$ which satisfy $\beta(V') \equiv 1 \pmod{k^{*2}}$. It is known, see [S3] or [Sp], that

$$\beta(V') \equiv d(V')/d(V). \tag{2.2}$$

This proves the second statement.

We end with some remarks on conjugacy classes in the groups $O(V)(k)$ and $\mathrm{SO}(V)(k)$. Recall that $\bar{k}$ is a separable closure of $k$, and $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Let $\bar{V} = V \otimes \bar{k}$. Recall that an orthogonal transformation $T \in O(V)(k)$ is semi-simple if it is diagonalizable on $\bar{V}$.

PROPOSITION 2.3. *A semi-simple orthogonal transformation $T$ is conjugate to its inverse $T^{-1}$ in the group $O(V)(k)$.*

PROOF. This is clear when $\dim V \leq 1$. It is also true when $\dim V = 2$. Indeed, in this case the group $\mathrm{SO}(V)$ is a 1-dimensional torus. Any $T$ in $O(V)(k) - \mathrm{SO}(V)(k)$ satisfies $T^2 = 1$, and $TST^{-1} = S^{-1}$ for any $S \in \mathrm{SO}(V)(k)$.

If the result holds for the spaces $V_1$ and $V_2$, it also holds for transformations $T$ of the orthogonal direct sum $V = V_1 \oplus V_2$ which stabilize the summands. Hence it is true for transformations of $V$ which stabilize an orthogonal decomposition into lines and planes.

For general semi-simple orthogonal transformations $T$ of $V$, let $\bar{V}_\alpha$ be the subspace of $\bar{V}$ where $T$ acts by $\alpha \in \bar{k}^*$. Both $V_1$ and $V_{-1}$ are defined over $k$, and the restriction of $T$ to $V_{\pm 1}$ is equal to its inverse there. If $\alpha^2 \neq 1$, the subspace $\bar{V}_\alpha$ is isotropic. Indeed, if $v \in \bar{V}_\alpha$, then $q(v) = q(Tv) = q(\alpha v) = \alpha^2 q(v)$. Clearly, the sum $\bar{W}_\alpha = \bar{V}_\alpha + \bar{V}_{\alpha^{-1}}$ is non-degenerate. The subspace $W_\alpha$ can be defined over the subfield $K_\alpha$ of $\bar{k}$, fixed by the subgroup of $\Gamma$ fixing the unordered pair $\{\alpha, \alpha^{-1}\}$. $W_\alpha$ is the direct sum of $m_\alpha \geq 0$ 2-dimensional quadratic spaces over $K_\alpha$ which are stable under $T$, and the restriction of $T$ to $W_\alpha$ lies in $\mathrm{SO}(W_\alpha)(K_\alpha)$.

By the preliminary result, the restriction of $T$ to $W_\alpha$ is conjugate to $T^{-1}$ in $O(W_\alpha)(K_\alpha)$. More precisely, $T^{-1} = A_\alpha T A_\alpha^{-1}$ with $\det(A_\alpha) = (-1)^{m_\alpha}$. The orthogonal direct sum $W_{\Gamma\alpha}$ of the distinct $\Gamma$-conjugates $W_{\sigma\alpha}$ of $W_\alpha$ is defined over $k$, and the sum of the conjugates of $A_\alpha$ gives an element $A_{\Gamma\alpha}$ on $W_{\Gamma\alpha}$ with $\det(A_{\Gamma\alpha}) = (-1)^{\frac{1}{2}\dim W_{\Gamma\alpha}}$ and the conjugation formula $T^{-1} = A_{\Gamma\alpha} T A_{\Gamma\alpha}^{-1}$.

Since $V$ is the orthogonal direct sum of the spaces $V_{\pm 1}$ and the spaces $W_{\Gamma\alpha}$, this completes the proof.

COROLLARY 2.4. *Let $T$ be a semi-simple transformation in* $\mathrm{SO}(V)(k)$. *If* $\dim V \not\equiv 2 \pmod 4$, *then $T$ is conjugate to $T^{-1}$ in* $\mathrm{SO}(V)(k)$. *If* $\dim V \equiv 2 \pmod 4$, *then $T$ is conjugate to $ST^{-1}S^{-1}$ in* $\mathrm{SO}(V)(k)$, *where $S \in O(V)(k)$ has* $\det S = -1$.

PROOF. This follows from the proof of Proposition 2.3, using the formula $\det(A_{\Gamma\alpha}) = (-1)^{\frac{1}{2}\dim W_{\Gamma\alpha}}$, and summing over the spaces $W_{\Gamma\alpha}$.

NOTE 2.5. If $k$ is perfect, *every* element $T \in O(V)(k)$ is conjugate to its inverse. For unipotent elements, or equivalently, nilpotent elements in the Lie algebra, see [S-S; p. 259]. For another proof of this result, and a generalization, see [MVW] and [FZ].

3. **Relevant pairs.** Let $V$ be a quadratic space over $k$ and let $W$ be a quadratic space which embeds as a subspace of $V$. Fix an embedding and define

$$W^{\perp} = \{v \in V : \langle v, w\rangle = 0 \text{ for all } w \in W\}. \tag{3.1}$$

This is a quadratic space, whose isomorphism class depends only on the pair $(V, W)$ and not on the embedding chosen, by Witt's theorem [S1; Chapter IV, Theorem 3]. We have

$$V \simeq W \oplus W^{\perp}. \tag{3.2}$$

We say the pair $(V, W)$ is *relevant* if $W$ embeds in $V$ and $W^{\perp}$ is a split quadratic space of odd dimension over $k$. Assume this is the case. Since $\dim V - \dim W = \dim W^{\perp}$ is odd, exactly one of the spaces in a relevant pair is odd dimensional. We denote this odd dimensional space by $U_1$ and its discriminant by $d(U_1)$. We denote the even dimensional space in a relevant pair by $U_2$, and its discriminant by $d(U_2)$.

Now fix integers $2n+1 \geq 1$, $2m \geq 0$ and discriminants $d_1, d_2 \in k^*/k^{*2}$. We claim there is exactly one relevant pair of quasi-split spaces $(V, W)$ with

$$\begin{cases} \dim U_1 = 2n+1 & d(U_1) \equiv d_1 \\ \dim U_2 = 2m & d(U_2) \equiv d_2. \end{cases} \tag{3.3}$$

Indeed, consider the pair of quasi-split spaces:

$$\begin{cases} U_1 = (X_1 + X_1') \oplus \langle d_1\rangle \\ U_2 = (X_2 + X_2') \oplus d_1 E \end{cases} \tag{3.4}$$

where $X_1$ and $X_1'$ are dual isotropic subspace of dimension $n$, $X_2$ and $X_2'$ are dual isotropic subspaces of dimension $m-1$, and $E = k[x]/(x^2 - d_2)$.

PROPOSITION 3.5. *The spaces $U_1$ and $U_2$ defined in (3.4) give a relevant pair $(V, W)$ with invariants (3.3). They are the unique relevant pair of quasi-split spaces with these invariants.*

PROOF. The two dimensional space $d_1 E$ is isomorphic to the sum $\langle d_1\rangle \oplus \langle -d_1 d_2\rangle$, as it represents $d_1$ and has discriminant $\equiv d_2$.

If $n \leq m-1$, then $U_1 \oplus (X + X') \oplus \langle -d_1 d_2\rangle$ is isomorphic to $U_2$, with $X$ and $X'$ dual isotropic spaces of dimension $m-n-1$. Hence $W^{\perp} = (X+X') \oplus \langle -d_1 d_2\rangle$ is split. If $n \geq m$, then $U_2 \oplus (X + X') \oplus \langle d_1 d_2\rangle$ is isomorphic to $U_1$, with $X$ and $X'$ dual isotropic spaces of dimension $n-m$. This follows from the fact that $d_1 E \oplus \langle d_1 d_2\rangle = \langle d_1\rangle \oplus \langle -d_1 d_2\rangle \oplus \langle d_1 d_2\rangle$ is a split 3-dimensional space of discriminant $\equiv d_1$. Hence $W^{\perp} = (X + X') \oplus \langle d_1 d_2\rangle$ is split.

If $U_2' = (X_2 + X_2') \oplus cE$ is quasi-split with $(U_1, U_2')$ relevant, the above argument when combined with Witt cancellation, shows that $c \equiv d_1 \pmod{\mathrm{N}E^*}$. Hence the quasi-split relevant pair is unique.

COROLLARY 3.6. *If $(U_1, U_2)$ are the quasi-split relevant pair defined in (3.4), then the pairs $(U_1, 0)$ and $(U_2, \langle d_1\rangle)$ are both relevant.*

PROOF. This is clear.

4. **The spherical subgroup.** Let $(V, W)$ be a relevant pair of quadratic spaces over $k$, and let $G$ be the algebraic group $\mathrm{SO}(V) \times \mathrm{SO}(W)$. Our aim in this section is to define a connected algebraic subgroup $H$ of $G$, as well as a homomorphism of algebraic groups $\ell: H \longrightarrow \mathbb{G}_a$. We will show that the pair $(H, \ell)$ is well-defined up to conjugacy in $G$. When $G$ is quasi-split, $H$ is a spherical subgroup in the sense of Brion [Br].

Write $V = W \oplus W^\perp$, with $W^\perp$ split of dimension $2r + 1$. We may write $W^\perp = (X + X') \oplus \langle a \rangle$ with $X$ and $X'$ dual isotropic subspaces of dimension $r$. Let $P$ be the parabolic subgroup of $\mathrm{SO}(V)$ which fixes the isotropic subspace $X$, and let $M$ be the Levi subgroup of $P$ which fixes both $X$ and $X'$. Then $M$ acts on the quadratic space $Y = (X+X')$, which has codimension 1 in $W^\perp$.

We have $P = M \ltimes N_P$, where $N_P$ is the unipotent radical of $P$. The group $M$ is isomorphic to $\mathrm{GL}(X) \times \mathrm{SO}(Y^\perp)$, and $N_P$ sits in an exact sequence of $M$-modules

$$0 \longrightarrow \overset{2}{\Lambda} X \longrightarrow N_P \longrightarrow X \otimes Y^\perp \longrightarrow 0.$$

The subspace $W$ has codimension 1 in $Y^\perp$.

If $r = 0$, we put $H = \mathrm{SO}(W)$ embedded diagonally in $G$ and $\ell = 0$. If $r \geq 1$, let $X_1 \subseteq X$ be a hyperplane and $\ell_1: X \longrightarrow \mathbb{G}_a$ a non-zero homomorphism which vanishes on $X_1$. Let $\ell_W: Y^\perp \longrightarrow \mathbb{G}_a$ be a non-zero homomorphism which is zero on the hyperplane $W$. Consider the composite map

$$m: N_P \longrightarrow N_P^{\mathrm{ab}} = X \otimes Y^\perp \underset{\ell_1 \otimes \ell_W}{\longrightarrow} \mathbb{G}_a.$$

The subgroup of $M$ which fixes the map $m$ is $\mathrm{GL}(X)_{\ell_1} \times \mathrm{SO}(W)$, where $\mathrm{GL}(X)_{\ell_1}$ is the subgroup of $\mathrm{GL}(X)$ which fixes the linear form $\ell_1$. This "mirabolic" subgroup of $\mathrm{GL}(X) \simeq \mathrm{GL}_r$ contains a maximal unipotent subgroup $N_r$: namely, let $N_r$ be the unipotent radical of the Borel subgroup $B_r$ which stabilizes complete flag in $X$ of the form $X \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_r = 0$. We define the subgroup $H$ of $P$ by

$$H = \big(N_r \times \mathrm{SO}(W)\big) \ltimes N_P. \tag{4.1}$$

Then $H$ embeds in $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$, using the obvious projection from $H$ onto the second factor $\mathrm{SO}(W)$.

Let $\ell_r: N_r \longrightarrow \mathbb{G}_a$ be a homomorphism which is non-trivial when restricted to each simple root space for $\mathrm{GL}_r$ in $N_r$. There is then a unique homomorphism

$$\ell: H \longrightarrow \mathbb{G}_a \tag{4.2}$$

which is equal to $\ell_r$ on the subgroup $N_r$, equal to zero on the subgroup $\mathrm{SO}(W)$, and equal to $m$ on the subgroup $N_P$.

This completes the description of the pair $(H, \ell)$. When $r = 0$, $H = \mathrm{SO}(W)$ is reductive. When $\dim W \leq 1$, $H$ is a maximal unipotent subgroup of $\mathrm{SO}(V)$ and $\ell: H \longrightarrow \mathbb{G}_a$ is non-trivial when restricted to each simple root space. In general, $H \simeq \mathrm{SO}(W) \ltimes N$ with $N$ unipotent in $\mathrm{SO}(V)$, and $\ell: H \longrightarrow \mathbb{G}_a$ is the extension of a generic functional on $N$ which is $\mathrm{SO}(W)$-invariant.

We give the structure of $H$ in one mixed case, when $W$ is split of dimension 2 and $\dim V = 2n+1 \geq 5$. Then $\mathrm{SO}(W) \simeq \mathbb{G}_m$ and $H \simeq \mathbb{G}_m \ltimes N$, with $N$ of codimension 1 in a maximal unipotent subgroup of the split group $\mathrm{SO}(V)$. If $V$ has basis $\langle v_1, \dots, v_n; v'_1, \dots, v'_n; v \rangle$ giving the decomposition $(X + X') \oplus \langle a \rangle$ and $W = \langle v_n, v'_n \rangle$, then $N$ contains all positive root spaces, except for the space with character $e_n$. An element $\lambda \in \mathbb{G}_m$ acts trivially on the root spaces of $N$ with character $e_i - e_j, e_i + e_j, e_i$ $1 \leq i < j < n$. It acts by $\lambda$ on the root spaces with character $e_i + e_n$, and by $\lambda^{-1}$ on the root spaces with character $e_i - e_n$. The map $\ell: H \longrightarrow \mathbb{G}_a$ is non-trivial on the simple root spaces $e_1 - e_2, e_2 - e_3, \dots, e_{n-2} - e_{n-1}$.

PROPOSITION 4.3. *The pair $(H, \ell)$ is uniquely determined up to conjugacy in the group* $\mathrm{SO}(V)$ *by the pair* $(V, W)$.

PROOF. Let $(\tilde{H}, \tilde{\ell})$ be another pair, constructed as above using the different choices $\tilde{X}, \tilde{Y} = \tilde{X} + \tilde{X}', \tilde{X} \supset \tilde{X}_1 \supset \cdots \supset \tilde{X}_r = 0$, $\tilde{\ell}_1$, $\tilde{\ell}_W$, and $\tilde{\ell}_r$.

We choose $g_1 \in \mathrm{SO}(V)$ so that $g_1 \tilde{X} = X$; this element exists by Witt's extension theorem and is uniquely determined up to left multiplication by $P$. Since $g_1 \tilde{X}'$ is then isotropic and dual to $X$, we may choose $g_2$ in $P$ so that $g_2 g_1 \tilde{X}' = X'$. The element $g_2 g_1$ is now unique up to left multiplication by $M = \mathrm{GL}(X) \times \mathrm{SO}(Y^{\perp})$.

Next choose $g_3$ in $M$ so that $g_3 g_2 g_1$ maps the flag $\tilde{X}_i$ of $\tilde{X}$ to the flag $\tilde{X}$ of $X$, hence maps the linear forms $\tilde{\ell}_1$ and $\tilde{\ell}_W$ to multiples of the linear forms $\ell_1$ and $\ell_W$. Then $\tilde{m}$ is mapped to a multiple of $m$, and $g_3 g_2 g_1$ is unique up to left multiplication by $B_r \times \mathrm{SO}(W)$.

Finally, choose $g_4$ in $B_r$ so that $g_4 g_3 g_2 g_1$ maps $\tilde{\ell}_r$ to a multiple of the form $\ell_r$, and choose $g_5$ in the center $\mathbb{G}_m$ of $B_r$ so that $g_5 g_4 g_3 g_2 g_1 = g$ maps $\tilde{\ell}$ to $\ell$. Then $g$ conjugates $(\tilde{H}, \tilde{\ell})$ to $(H, \ell)$, and is well-determined up to left multiplication by an element in the subgroup $N_r \times \mathrm{SO}(W)$ of $H$ which fixes the form $\ell$.

PROPOSITION 4.4. *Assume that $G$ is quasi-split. Then $H$ has an open orbit on the flag variety $F = G/B$ of Borel subgroups of $G$, with trivial stability subgroup.*

PROOF. If $r = 0$, so $H = \mathrm{SO}(W) = \mathrm{SO}_{N-1}$ and $G = \mathrm{SO}_{N-1} \times \mathrm{SO}_N$, this follows from [Br; Theorem, p. 190]. When $r \geq 1$, one can use [Br; Proposition I.1, p. 191] to reduce to the reductive case. In the notation of [Br], the parabolic subgroup $Q$ of $G$ whose unipotent radical contains $N = N_r \ltimes N_P$ and whose Levi subgroup $L$ contains $\mathrm{SO}(W)$ is the stabilizer of a complete flag of the form $X \supset X_1 \supset X_2 \supset \cdots \supset 0$ in $V$.

Since $\dim H = \dim(G/B)$, the stability subgroup is finite. One can show it is trivial by explicitly giving the flags in the open orbit.

5. **Admissible representations.** We henceforth assume that $k$ is a local field, so $k^+ = \mathbb{G}_a(k)$ is a locally compact, non-discrete, topological group. Let $\underline{G}$ be a connected, reductive group over $k$. The group $G = \underline{G}(k)$ of $k$-rational points is then locally compact. In this section, we describe the (conjectural) Langlands parameters of irreducible, admissible complex representations of $G$.

We begin by specifying what we mean by an admissible representation. When $k$ is non-Archimedean, an admissible representation is a homomorphism $\pi: G \longrightarrow \mathrm{GL}(E)$, where

$E$ is a complex vector space. We insist that the map $G \times E \longrightarrow E$ is continuous, when $E$ is given the discrete topology, so every vector in $E$ has an open stabilizer in $G$. Finally, if $K$ is any compact open subgroup of $G$, we insist that the vector space $\mathrm{Hom}_K(E_i, E)$ is finite-dimensional, for every irreducible representation $E_i$ of $K$. The obvious map of $K$-modules

$$\bigoplus_i \mathrm{Hom}_K(E_i, E) \otimes E_i \longrightarrow E \tag{5.1}$$

is then an isomorphism.

The admissible representation $\pi$ is irreducible if there are no proper subspaces of $E$ stable under $G$, and two representations $E$ and $F$ are isomorphic if there is a linear isomorphism $E \longrightarrow F$ which intertwines the action of $G$.

When $k = \mathbb{R}$ and $k = \mathbb{C}$, the definition of an admissible representation of $G$ is more involved. We will work in the category of smooth, Fréchet representations of moderate growth (*cf.* [C], [W, Chapter 11]). An admissible representation is a homomorphism $\pi: G \longrightarrow \mathrm{GL}(E)$, where $E$ is a complex Fréchet space. We insist that the associated map $G \times E \longrightarrow E$ is continuous. Moreover, for all vectors $v$ in $E$ and continuous linear forms $v'$ in $E'$, we insist that the matrix coefficient

$$F = F_{v,v'}(g) = \langle gv, v' \rangle : G \longrightarrow \mathbb{C}$$

is a $C^\infty$-function of moderate growth on $G$. The latter condition means that

$$|F(g)| \leq C \cdot \|g\|^N \quad g \in G$$

where $\| \quad \|: G \longrightarrow \mathbb{R}_+$ is a norm on $G$, and $C = C_{v,v'}$, $N = N_{v,v'}$ are constants (which also depend on the choice of $\| \quad \|$) [W, Chapter 11]. We remark that once the condition of moderate growth is satisfied for all matrix coefficients, then it is also satisfied for all the derivatives of matrix coefficients. Finally, if $K$ is a maximal compact subgroup, we insist that the vector space $\mathrm{Hom}_K(E_i, E)$ is finite dimensional, for every irreducible representation $E_i$ of $K$.

Since the action of $G$ on $E$ is smooth, we may differentiate to get an action of the complex Lie algebra $\mathfrak{g} = \mathrm{Lie}(G) \otimes_k \mathbb{C}$. The space of $K$-finite vectors

$$E_K = \bigoplus_i \mathrm{Hom}_K(E_i, E) \otimes E_i \tag{5.2}$$

embeds as a dense subspace of $E$, which is stable under $\mathfrak{g}$. This is the infinitesimal representation associated to $E$, which is an admissible $(\mathfrak{g}, K)$-module. The admissible representation $\pi$ is irreducible if there are no proper, closed subspaces of $E$ stable under $G$, and two representations $E$ and $F$ are isomorphic if there is a continuous linear isomorphism $E \longrightarrow F$ which intertwines the action of $G$. One can also state these conditions algebraically in terms of the $(\mathfrak{g}, K)$-modules $E_K$ and $F_K$.

Recall that if $\pi: G \longrightarrow \mathrm{GL}(E)$ is an irreducible admissible representation, one can construct another irreducible, admissible representation $\pi^\vee: G \longrightarrow \mathrm{GL}(E^\vee)$ called the *contragredient* of $\pi$. In the non-Archimedean case, $E^\vee$ is the subspace of vectors in the algebraic dual $E^* = \mathrm{Hom}(E, \mathbb{C})$ which have an open stabilizer in $G$. We have an isomorphism of $K$-modules $E^\vee \simeq \bigoplus \mathrm{Hom}_K(E_i, E) \otimes E_i^*$. In the Archimedean case, $E^\vee$ is the canonical Fréchet globalization [C], which is smooth of moderate growth, of the dual $(\mathfrak{g}, K)$-module $E_K^\vee = \bigoplus \mathrm{Hom}_K(E_i, E) \otimes E_i^*$. In all cases, $(E^\vee)^\vee$ is canonically isomorphic to $E$, and the natural $G$-invariant pairing $E \otimes E^\vee \longrightarrow \mathbb{C}$ is non-degenerate and unique up to scaling.

Now assume that $G = \mathrm{SO}(V)(k)$. Let $E$ be an irreducible, admissible representation of $G$, let $E^\vee$ be its contragredient, and let $E'$ be the conjugate of $E$ by the outer action of reflections in $O(V)(k)$ on $G$.

PROPOSITION 5.3. *If* $\dim V \not\equiv 2 \pmod 4$, *then* $E^\vee \simeq E$. *If* $\dim V \equiv 2 \pmod 4$, *then* $E^\vee \simeq E'$.

PROOF. This follows from 2.4–2.5 as in [MVW, theorem on p. 91] using results of Bernstein and Zelevinsky [BZ, Theorems 6.3 and 6.5]. If the characteristic of $k$ is zero, in which case the character of an irreducible admissible representation is known to be represented by a locally integrable function by the work of Harish-Chandra, it follows directly from Corollary 2.4.

QUESTION. When $\dim V \not\equiv 2 \pmod 4$, we obtain a $G = \mathrm{SO}(V)(k)$ invariant pairing $\langle\, ,\, \rangle : E \otimes E \longrightarrow \mathbb{C}$, which is unique up to scalars. Hence $\langle v, w\rangle = \pm\langle w, v\rangle$ for all $v, w \in E$. Is this pairing always symmetric? We can prove this in the Archimedean case, using Vogan's theory of minimal $K$-types, and in the non-Archimedean case for unramified representations $E$. More generally, let $G$ be a semi-simple group over $k$. There is an element $h$ in the centre of the $G$, of order 1 or 2, with the property that a finite dimensional irreducible algebraic representation $F$ of $G$ which is isomorphic to its dual is orthogonal if $h = 1$ on $F$, and symplectic if $h = -1$ on $F$ (*cf.* [St, Lemma 79]). Does this criterion extend to self-dual irreducible admissible representations $E$? We note that $h = 1$ in $\mathrm{SO}(V)$.

We now turn to a discussion of Langlands parameters for irreducible representations of a general reductive group $G$. Let $\bar{k}$ be a separable closure of $k$ and $\Gamma = \mathrm{Gal}(\bar{k}/k)$. The $L$-group ${}^LG$ of $G$ is a semi-direct product ${}^LG = {}^\vee G \rtimes \Gamma$, where ${}^\vee G$ is a connected, reductive group over $\mathbb{C}$ with the dual root datum to $G$ over $\bar{k}$. Let $W(k)'$ be the Weil-Deligne group of $k$ (*cf.* [T, §4]). A Langlands parameter $\varphi$ for $G$ is a homomorphism

$$\varphi : W(k)' \longrightarrow {}^LG \tag{5.4}$$

satisfying certain additional conditions [B2, 8.2]. Two parameters are considered equivalent if they are conjugate under ${}^\vee G$. Associated to $\varphi$ is the finite component group:

$$A_\varphi = \pi_0(C_\varphi), \tag{5.5}$$

where $C_\varphi$ is the centralizer in ${}^\vee G$ of the image of $\varphi$.

Langlands (*cf.* [B2, Chapter III]) has conjecturally associated to each equivalence class of parameters $\varphi$ a finite set $\Pi_\varphi(G)$ of irreducible, admissible representations of $G$

up to isomorphism, called the $L$-packet of $\varphi$. The $L$-packets should be disjoint, and should exhaust the irreducible, admissible representations of $G$. This correspondence has been established when $k = \mathbb{R}$ or $\mathbb{C}$, and for some fairly simple groups $G$ over non-Archimedian fields $k$, like tori and $\mathrm{GL}(n)$ (for some $n$).

Vogan has refined the Langlands parametrization as follows. Assume that $G$ is quasi-split over $k$, and fix a generic character $\Theta_0$ of the unipotent radical of a Borel subgroup. We say a group $G'$ over $k$ is a *pure inner form* of $G$ if it is an inner form and the associated cohomology class in $H^1(k, G/Z)$ lifts to $H^1(k, G)$. To specify a pure inner form $G'$, one must specify the lifted class in $H^1(k, G)$. The parameter $\varphi$ for $G$ may also be a parameter for $G'$. If so, we let $\Pi_\varphi(G')$ be the Langlands $L$-packet; if not, we let $\Pi_\varphi(G')$ be the empty set.

The Vogan $L$-packet $\Pi_\varphi$ of $\varphi$ is the disjoint union of representations of distinct groups:

$$\Pi_\varphi = \bigcup_{H^1(k,G)} \Pi_\varphi(G') \tag{5.6}$$

Vogan conjectures [V] that there is a bijection between the elements of $\Pi_\varphi$ and the irreducible characters $\chi$ of the component group $A_\varphi$. If $\varphi$ is a generic parameter (*cf.* [G-P; §2]), the unique $\Theta_0$-generic representation in $\Pi_\varphi(G)$ should correspond to the trivial character $\chi_0$ of $A_\varphi$.

For more information on Vogan $L$-packets, and a recipe for the group $G'$ which acts on the representation $\pi(\varphi, \chi)$ in $\Pi_\varphi$, see [G-P; §3–4]. We will assume the conjectures of Langlands and Vogan for the group $G = \mathrm{SO}(W) \times \mathrm{SO}(V)$ in all that follows. For more discussion of parameters in this case, see [G-P; §6].

6. **The conjecture.** In this section, $k$ is a local field, with $\mathrm{char}(k) \neq 2$. Let $(V, W)$ be a relevant pair of orthogonal spaces over $k$. Since we will consider all pure inner forms of the group $\mathrm{SO}(V) \times \mathrm{SO}(W)$ simultaneously, there is no loss of generality, by Proposition 3.5, in assuming that the spaces $V$ and $W$ are both quasi-split and given by (3.4). We henceforth do so.

We change notation slightly from §4, and let $G$ denote the $k$-rational points of the group $\mathrm{SO}(V) \times \mathrm{SO}(W)$, $H$ denote the $k$-rational points of the spherical subgroup constructed in (4.1), and $\ell: H \longrightarrow k$ the homomorphism of (4.2) on $k$-rational points. Then $G$ is locally compact, $H$ is a closed subgroup, and $\ell$ is a continuous homomorphism. The pair $(H, \ell)$ is well-defined up to $G$-conjugacy; in particular, $(H, \alpha \cdot \ell)$ is conjugate to $(H, \ell)$ for any $\alpha \in k^*$.

Let $\psi: k \longrightarrow S^1$ be a non-trivial additive character, and let

$$\Theta = \psi \circ \ell: H \longrightarrow S^1 \tag{6.1}$$

be the corresponding homomorphism of $H$. Since any other choice of $\psi$ has the form $\psi'(x) = \psi(\alpha x)$ for $\alpha \in k^*$, the pair $(H, \Theta)$ is well-defined up to $G$-conjugacy.

Let $\mathbb{C}(\Theta)$ denote the 1-dimensional representation of $H$ given by (6.1), and let $\pi: G \longrightarrow \mathrm{GL}(E)$ be an admissible representation of $G$. We define

$$\mathrm{Hom}_H(\pi, \Theta) = \mathrm{Hom}_H\big(E, \mathbb{C}(\Theta)\big) \tag{6.2}$$

as the complex vector space of all $H$-invariant continuous linear maps from $E$ to $\mathbb{C}(\Theta)$. Equivalently, $\mathrm{Hom}_H(\pi, \Theta)$ is the subspace of all continuous linear forms on $E$ on which $H$ acts by the character $\Theta^{-1}$.

By the previous remarks, the isomorphism class of the complex vector space $\mathrm{Hom}_H(\pi, \Theta)$ depends only on the representation $\pi$ up to isomorphism, not on the specific realization $E$ or on the choices made in defining $H$ and $\Theta$. In particular, the integer $\dim \mathrm{Hom}_H(\pi, \Theta)$ is an invariant of the isomorphism class of $\pi$. It *does*, however, *depend* on the particular quasi-split relevant pair $(V, W)$ used to define $H$, although this is not apparent in our notation. Indeed, the isomorphism class of $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$ determines the invariants $2n + 1$, $2m$, and $d_2$ in (3.3), but the choice of $d_1$ is arbitrary. If $(V', W')$ is a quasi-split relevant pair with $G' \simeq G$ but $d_1' \not\equiv d_1$, the subgroup $H'$ of $G'$ will usually *not* be conjugate to $H$ in $G$.

We take this dependence into account by *also* using the relevant pair $(V, W)$ to define a generic character

$$\Theta_0 \colon U \longrightarrow S^1 \tag{6.3}$$

of a maximal unipotent subgroup of $G$ (= the unipotent radical of a Borel subgroup). Again the pair $(U, \Theta_0)$ is well-defined up to $G$-conjugacy, so the complex vector space $\mathrm{Hom}_U(\pi, \Theta_0)$ is well-defined up to isomorphism. A distinct quasi-split relevant pair $(V', W')$ with $G'$ isomorphic to $G$ will usually give a distinct generic character $\Theta_0'$.

The definition of $\Theta_0$ is as follows. By Corollary 3.6, the pairs $(U_1, 0)$ and $(U_2, \langle d_1 \rangle)$ are both relevant. In these pairs, $\dim W \leq 1$. Hence the first gives, as spherical subgroups, $(H_1, \ell_1)$ with $H_1$ a maximal unipotent subgroup of $\mathrm{SO}(U_1)$ and $\ell_1$ a generic linear functional of $H_1$. Likewise, the second gives, as spherical subgroup, $(H_2, \ell_2)$ with $H_2$ a maximal unipotent subgroup of $\mathrm{SO}(U_2)$ and $\ell_2$ a generic linear functional of $H_2$. We take $U = H_1 \times H_2$ in $G$, and $\Theta_0 = \psi \circ (\ell_1 \times \ell_2)$.

We now turn to a discussion of Langlands parameters $\varphi$ and Vogan $L$-packets $\Pi_\varphi$ for the group $G$. Let $M_1$ be a complex symplectic space of dimension $2n = \dim(U_1) - 1$, and let $M_2$ be a complex orthogonal space of dimension $2m = \dim(U_2)$. A Langlands parameter for $G$ may be viewed as a homomorphism [G-P; 6.1–6.2, 7.4–7.5]

$$\varphi \colon W(k)' \longrightarrow \mathrm{Sp}(M_1) \times O(M_2) \tag{6.4}$$

which satisfies the condition

$$\mathrm{kernel}(\det \varphi | M_2) = \mathrm{N}E^* \subset k^* = W(k)^{\mathrm{ab}} \tag{6.5}$$

with $E = k[x]/(x^2 - d_2)$. Two parameters are equivalent if and only if they are conjugate under the subgroup ${}^\vee G = \mathrm{Sp}(M_1) \times \mathrm{SO}(M_2)$.

The group $A_\varphi$ is an elementary abelian 2-group, whose rank depends on the number of distinct irreducible symplectic summands of $M_1$ and the number of distinct irreducible orthogonal summands of $M_2$ [G-P; Corollary 6.6, Corollary 6.7]. Hence $\hat{A}_\varphi = \mathrm{Hom}(A_\varphi, \langle \pm 1 \rangle)$.

In [G-P; §10] we constructed a character

$$\chi : A_\varphi \longrightarrow \langle \pm 1 \rangle \tag{6.6}$$

using the theory of symplectic root numbers. Specifically if $a$ is an involution in $\mathrm{Sp}(M_1) \times \mathrm{SO}(M_2) = {}^\vee G$ which centralizes the image of $\varphi$, we defined [G-P; 10.2]:

$$\chi(a) = \epsilon(M^{a_1 \otimes a_2 = -1}) \det(M_2)(-1)^{\frac{1}{2}\dim M_1^{a_1=-1}} \det(M_2^{a_2=-1})(-1)^{\frac{1}{2}\dim M_1}. \tag{6.7}$$

Here $M = M_1 \otimes M_2$ is the tensor product representation of $W(k)'$, and $\epsilon(M) = \pm 1$ is the symplectic root number defined, as $\epsilon(M \otimes \| \quad \|^{1/2})$, in [G; §3]. Then $\chi(a)$ depends only on the class of $a$ in $\pi_0(C_\varphi) = A_\varphi$, and $\chi(ab) = \chi(a)\chi(b)$ is a homomorphism.

We will use the generic character $\Theta_0$ of $G$ defined following (6.3) to normalize the Vogan correspondence $\Pi_\varphi \leftrightarrow \hat{A}_\varphi$. Specifically, if the parameter $\varphi$ is generic, the representation $\pi(\varphi, \chi_0)$ corresponding to the trivial character $\chi_0$ of $A_\varphi$ is the $\Theta_0$-generic representation in $\Pi_\varphi(G)$. Then the character $\chi$ defined in (6.7) determines a unique representation $\pi(\varphi, \chi)$ in $\Pi_\varphi$, by the Vogan correspondence.

If $G_\alpha = \mathrm{SO}(V_\alpha) \times \mathrm{SO}(W_\alpha)$ is a pure inner form of $G$, then by Proposition 2.1 we have:

$$\begin{cases} \dim V_\alpha = \dim V & d(V_\alpha) \equiv d(V) \\ \dim W_\alpha = \dim W & d(W_\alpha) \equiv d(W). \end{cases} \tag{6.8}$$

If the pair $(V_\alpha, W_\alpha)$ is relevant, we may define the spherical subgroup $H_\alpha$ of $G_\alpha$, and a character $\Theta_\alpha = \psi \circ \ell_\alpha : H_\alpha \longrightarrow S^1$ using the results of §4. Hence we have a well-defined complex vector space $\mathrm{Hom}_{H_\alpha}(\pi_\alpha, \Theta_\alpha)$ for any irreducible representation $\pi_\alpha$ of $G_\alpha$ which occurs in $\Pi_\varphi$. If $G_\alpha$ does *not* come from a relevant pair, we adopt the convention that $\mathrm{Hom}_{H_\alpha}(\pi_\alpha, \Theta_\alpha) = 0$ for all representations $\pi_\alpha$ of $G_\alpha$ in $\Pi_\varphi$ (as $H_\alpha$ and $\Theta_\alpha$ are not defined).

CONJECTURE 6.9. *Let $\varphi$ be a Langlands parameter for $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$ and let $\Pi_\varphi$ be the Vogan L-packet of $\varphi$.*

*(1) The complex vector space $L_\varphi = \bigoplus_{\pi_\alpha \in \Pi_\varphi} \mathrm{Hom}_{H_\alpha}(\pi_\alpha, \Theta_\alpha)$ has dimension $\leq 1$.*

*(2) If $\varphi$ is generic, then $\dim(L_\varphi) = 1$. The representation $\pi_\alpha = \pi(\varphi, \chi)$ determined by the symplectic root number character $\chi$ of (6.7) satisfies $\dim \mathrm{Hom}_{H_\alpha}(\pi_\alpha, \Theta_\alpha) = 1$.*

In particular, this conjecture implies that whenever

$$\mathrm{Hom}_U\big(\pi(\varphi, \chi_0), \Theta_0\big) \neq 0 \tag{6.10}$$

then we should have:

$$\mathrm{Hom}_{H_\alpha}\big(\pi(\varphi, \chi), \Theta_\alpha\big) \neq 0. \tag{6.11}$$

By [G-P, Conjecture 2.6], we expect (6.10) to hold when the adjoint $L$-function of $\varphi$ is regular at the point $s = 1$.

NOTE 6.12. We have been assuming that char$(k) \neq 2$ for simplicity throughout, but with more care one can also formulate Conjecture 6.9 for local fields of characteristic $= 2$.

NOTE 6.13. We expect that the formalism of Vogan $L$-packets makes sense also for $O(V)$ when the dimension of $V$ is even, say $2n$. As in the case of SO$(V)$, $O(2n, \mathbf{C})$ could be taken as the $L$-group of $O(V)$ but this time the equivalence of the parameters is up to conjugation by $O(2n, \mathbf{C})$, and the component group $A_\varphi$ of a parameter $\varphi$ is taken to be the group of connected components of the centraliser of $\varphi$ in $O(2n, \mathbf{C})$.

7. **Evidence.** We now present some preliminary evidence for Conjecture 6.9, retaining the notation of the previous section.

PROPOSITION 7.1 [G-PS-R]. *Assume that $k$ is non-Archimedean. If $\pi$ is an irreducible, admissible representation of $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$, then* $\dim \mathrm{Hom}_H(\pi, \Theta) \leq 1$.

This result is proved using ideas of J. Bernstein, who independently treated the case when $\dim W^\perp = 1$. What is initially proved is that:

$$\dim \mathrm{Hom}_H(\pi, \Theta) \cdot \dim \mathrm{Hom}_H(\pi^\vee, \Theta^{-1}) \leq 1 \tag{7.2}$$

where $\pi^\vee$ is the contragradient of $\pi$. Using Proposition 5.3, one can show that the two vector spaces appearing in (7.2) have the same dimension (*cf.* [Ro1]).

PROPOSITION 7.3. *Assume that* $\dim W \leq 1$. *Then $H$ is the unipotent radical $U$ of a Borel subgroup of $G = \mathrm{SO}(V) \times \mathrm{SO}(W) \simeq \mathrm{SO}(V), \Theta = \Theta_0$, and $\chi = \chi_0$. Hence the conjectured implication (6.10) $\Rightarrow$ (6.11) is true.*

PROOF. This is clear, as $\dim W \leq 1$ implies that $M = M_1 \otimes M_2 = 0$.

PROPOSITION 7.4. *If* $\dim V \leq 3$, *then Conjecture 6.9 is true. If* $\dim V = 4$ *and* $\dim W = 1$, *Conjecture 6.9 is true. If* $\dim V = 4$ *and* $\dim W = 3$, *then* $\dim(L_\varphi) \leq 1$ *for all parameters $\varphi$, with equality holding when $\varphi$ is generic.*

PROOF. When $\dim V \leq 4$, the theory of $L$-packets is known by reduction to the group $\mathrm{GL}_2$ (*cf.* [G-P; §15]), and there is a unique $\Theta_0$-generic element in each generic packet for $G$. Hence, by Proposition 7.3, the only difficult cases are when $\dim V = 3, \dim W = 2$ and when $\dim V = 4, \dim W = 3$. In the first case, the conjecture is true by work of Tunnell [Tu] and H. Saito [Sa]. In the second, the results on $\dim(L_\varphi)$ are due to Prasad [P1, P2].

PROPOSITION 7.5 [So1, So2]. *Assume that $\varphi_2 = \sigma \oplus \sigma^\vee$, where $\sigma\colon W(k)' \longrightarrow \mathrm{GL}(P) = \mathrm{GL}_n(\mathbf{C})$ has no orthogonal direct summands. Then $\chi = \chi_0$, and if $\varphi$ is generic,* $\mathrm{Hom}_H(\pi_0, \Theta)$ *is 1-dimensional.*

PROOF. Write $M_2 = P \oplus P^\vee$. Since $P$ has no orthogonal direct summands, $A_{\varphi_2} = 1$ [G-P; Corollary 7.7]. Hence

$$\chi(a) = \chi(a_1, 1) = \epsilon(M_1^{a_1=-1} \otimes M_2)$$

as $\det M_2 = 1$.

But by hypothesis

$$M_1^{a_1=-1} \otimes M_2 = M_1^{a_1=-1} \otimes (P \oplus P^\vee)$$
$$= (M_1^{a_1=-1} \otimes P) \oplus (M_1^{a_1=-1} \otimes P)^\vee.$$

Hence by [G; Proposition 3.15] we have

$$\chi(a) = \det(M_1^{a_1=-1} \otimes P)(-1)$$
$$= \det M_1^{a_1=-1}(-1)^{\dim P} \det P(-1)^{\dim M_1^{a_1=-1}}.$$

Since $M_1^{a_1=-1}$ is symplectic, it has trivial determinant and even dimension. Hence $\chi(a) = 1$, so $\chi = \chi_0$ is the trivial character.

Now assume $\varphi$ is generic, and let $\pi_1$ be the unique generic representation of $SO(V_1)$ in $\Pi_{\varphi_1}$. The unique representation $\pi_2$ in $\Pi_{\varphi_2}$ may be constructed as follows. Let $\pi(\sigma)$ be the irreducible representation of $GL_n(k)$ corresponding to the Langlands parameter $\sigma$, and extend $\pi(\sigma)$ to a representation of a parabolic subgroup $P$ in $SO(U_2)$ which stabilizes an isotropic $n$-plane. Then $\pi_2 = \mathrm{Ind}_P^{SO(U_2)} \pi(\sigma)$, which is irreducible given our hypotheses on $\sigma$.

But Soudry shows [So1, So2] that $\mathrm{Hom}_H(\pi_1 \otimes \pi_2, \Theta)$ has dimension 1, by an explicit integral representation of the linear form. Since $\pi_0 = \pi_1 \otimes \pi_2$ is the unique generic representation in $\Pi_\varphi$, we are done.

When $\dim W = 2$ and $d(W) \equiv 1$, *every* parameter $\varphi_2$ has the form $\sigma \oplus \sigma^\vee$, where $\sigma$ is a character of $W(k)^{\mathrm{ab}} = k^*$. The argument in Proposition 7.5 works even in the case when $\sigma^2 = 1$, as $A_{\varphi_2} = 1$ in all cases. Hence we obtain the following.

COROLLARY 7.6. *Assume that* $\dim W = 2$ *and that* $d(W) \equiv 1$. *Then* $\chi = \chi_0$, *and when* $\varphi_1$ *is generic,* $\mathrm{Hom}_H(\pi_0, \Theta)$ *is* 1*-dimensional.*

A description of the subgroup $H$ and the character $\Theta$ when $\dim W = 2, d(W) \equiv 1$ was given preceding Proposition 4.3.

8. **The pure inner form $G_\alpha$.** Let $(V, W)$ be a quasi-split relevant pair over the local field $k$. If $\varphi$ is a generic parameter for $G = SO(V) \times SO(W)$, then Conjecture 6.9 describes a distinguished representation $\pi_\alpha = \pi(\varphi, \chi)$ in the Vogan $L$-packet $\Pi_\varphi$. This is a representation of some pure inner form $G_\alpha = SO(V_\alpha) \times SO(W_\alpha)$ of $G$, and the conjecture predicts that the pair $(V_\alpha, W_\alpha)$ is *relevant*. We verify this prediction below.

To describe the set $H^1(k, G)$ of pure inner forms of $G$, we recall the spinor covering of $SO(V)$:

$$1 \longrightarrow \langle \pm 1 \rangle \longrightarrow \mathrm{Spin}(V) \longrightarrow SO(V) \longrightarrow 1. \tag{8.1}$$

Taking the coboundary map in non-abelian Galois cohomology gives a map of pointed sets

$$\delta : H^1\big(k, SO(V)\big) \longrightarrow H^2(k, \langle \pm 1 \rangle) = \mathrm{Br}_2(k) \tag{8.2}$$

where $\mathrm{Br}(k) = H^2(k, \mathbb{G}_m)$ is the Brauer group of $k$. There is a similar map on the set $H^1\big(k, \mathrm{SO}(W)\big)$ of pure inner forms of $\mathrm{SO}(W)$, so we obtain a map of pointed sets:

$$\begin{aligned} \Delta\colon H^1(k, G) &\longrightarrow \mathrm{Br}_2(k) \times \mathrm{Br}_2(k) \\ (V_\alpha, W_\alpha) &\longmapsto \big(\delta(V_\alpha), \delta(W_\alpha)\big). \end{aligned} \tag{8.3}$$

We exploit this map, and the fact that $k$ is a local field, to prove the following.

PROPOSITION 8.4. *If $k = \mathbb{C}$, then $\mathrm{Br}_2(k) = 1$. In all other cases $\mathrm{Br}_2(k) \simeq \langle \pm 1 \rangle$.*

*If $k \neq \mathbb{R}$, the map $\Delta$ of (8.3) is an injection, and the pair $(V_\alpha, W_\alpha)$ is relevant if and only if $\delta(V_\alpha) = \delta(W_\alpha)$ in $\mathrm{Br}_2(k)$.*

PROOF. The calculation of $\mathrm{Br}(k)$ is one of the central results in local classfield theory (*cf.* [T; §1]).

To establish the properties of $\Delta$, we recall the Hasse-Witt invariant $e(V) \in \mathrm{Br}_2(k)$ of a quadratic space over a field $k$, with $\mathrm{char}(k) \neq 2$. Choose an orthogonal basis for $V$, so $V \simeq \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_N \rangle$ as in (1.1) and define

$$e(V) = \prod_{i<j} (a_i, a_j) \quad \text{in } \mathrm{Br}_2(k) \tag{8.5}$$

where $(\,,\,)\colon k^*/k^{*2} \times k^*/k^{*2} \longrightarrow \mathrm{Br}_2(k)$ is the Hilbert symbol (= cup-product on $H^1(k, \langle \pm 1 \rangle)$). This invariant is independent of the choice of orthogonal basis used in its definition. If $k$ is a non-Archimedean local field with $\mathrm{char}(k) \neq 2$, a quadratic space $V$ over $k$ is determined up to isomorphism by the three invariants $\dim(V) \geq 0, d(V) \in k^*/k^{*2}$ and $e(V) \in \mathrm{Br}_2(k)$. For proofs of these assertions, see [S; Chapter IV, §2].

Now assume that $V_\alpha$ is a pure inner form of $V$, so $\dim(V_\alpha) = \dim(V)$ and $d(V_\alpha) \equiv d(V)$. It is known, see [Sp] or [S3], that

$$\delta(V_\alpha) = e(V_\alpha)/e(V) \quad \text{in } \mathrm{Br}_2(k). \tag{8.6}$$

Hence the map $\delta$ of (8.2), and the map $\Delta$ of (8.3), are both injections when $k$ is non-Archimedean. They are clearly injections when $k = \mathbb{C}$.

We now investigate when the pair $(W_\alpha, V_\alpha)$ is relevant, *i.e.*, when we have an embedding $W_\alpha \hookrightarrow V_\alpha$ with $W_\alpha^\perp$ split. Assume $k$ is non-Archimedean and that an embedding exists. Then $W_\alpha^\perp$ is determined by the invariants $\dim(W_\alpha^\perp) = \dim(W^\perp), d(W_\alpha^\perp)$, and $e(W_\alpha^\perp)$. But $d(W_\alpha^\perp) \equiv \pm d(V_\alpha)/d(W_\alpha)$ where the sign depends on the dimensions of $(V_\alpha, W_\alpha)$ (mod 4). Since $d(V_\alpha)/d(W_\alpha) \equiv d(V)/d(W) \equiv \pm d(W^\perp)$ with the same sign, we have $d(W_\alpha^\perp) \equiv d(W^\perp)$. Hence, a necessary condition for $(W_\alpha, V_\alpha)$ to be relevant is that $W_\alpha \hookrightarrow V_\alpha$ and $e(W_\alpha^\perp) = e(W^\perp)$.

But by the definition (8.5), we have

$$e(V) = e(W) \cdot e(W^\perp) \cdot \big(\pm d(W), \pm d(W^\perp)\big)$$

where the signs again depend on the dimensions of $W$ and $W^\perp$ (mod 4). Since this identity also holds for the pair $W_\alpha \hookrightarrow V_\alpha$, with the same signs, we find that

$$\begin{aligned} e(W^\perp)/e(W_\alpha^\perp) &= e(V)e(W_\alpha)/e(W)e(V_\alpha) \\ &= \delta(W_\alpha)/\delta(V_\alpha) \quad \text{in } \mathrm{Br}_2(k). \end{aligned}$$

Hence a necessary condition for $(V_\alpha, W_\alpha)$ to be relevant is that $\delta(V_\alpha) = \delta(W_\alpha)$ in $\mathrm{Br}_2(k)$.

One checks that when $\delta(V_\alpha) = \delta(W_\alpha)$, an embedding $W_\alpha \hookrightarrow V_\alpha$ exists. Then (8.7) shows that $e(W_\alpha^\perp) = e(W^\perp)$, so $W_\alpha^\perp$ is split. Hence the condition is also sufficient for $(V_\alpha, W_\alpha)$ to be relevant.

REMARK 8.8. Assume $k = \mathbb{R}$, that $V$ has signature $(p, q)$, and that $W$ has signature $(r, s)$. If $(V_\alpha, W_\alpha)$ is a pure inner form with signatures $(p_\alpha, q_\alpha)$ and $(r_\alpha, s_\alpha)$ we have

$$\begin{cases} p_\alpha + q_\alpha = p + q \\ r_\alpha + s_\alpha = r + s \\ p_\alpha \equiv p \pmod 2 \\ r_\alpha \equiv r \pmod 2. \end{cases}$$

The pair $(V_\alpha, W_\alpha)$ is relevant if

$$\begin{cases} r_\alpha \leq p_\alpha \\ s_\alpha \leq q_\alpha \\ r - r_\alpha = p - p_\alpha \end{cases}$$

and we have

$$\Delta(V_\alpha, W_\alpha) = \left((-1)^{\frac{p-p_\alpha}{2}}, (-1)^{\frac{r-r_\alpha}{2}}\right).$$

Hence the condition $\delta(V_\alpha) = \delta(W_\alpha)$ is still necessary for the pair to be relevant, and is sufficient when $\dim V \leq 3$.

We now consider the pure inner form $G_\alpha = \mathrm{SO}(V_\alpha) \times \mathrm{SO}(W_\alpha)$ which acts on the distinguished representation $\pi_\alpha = \pi(\varphi, \chi)$ in the Vogan $L$-packet. By [G-P; §6], the odd dimensional space in this pair has $\delta(U_{1,\alpha}) = \chi(-1_{M_1}, 1_{M_2})$, and the even dimensional space in this pair has $\delta(U_{2,\alpha}) = \chi(1_{M_1}, -1_{M_2})$. But

$$\chi(-1, 1) = \chi(1, -1) = \epsilon(M) \cdot \det M_2(-1)^{\frac{1}{2}\dim M_1}$$

by definition (6.7). Hence

$$\delta(V_\alpha) = \delta(W_\alpha) = \epsilon(M) \cdot \det M_2(-1)^{\frac{1}{2}\dim M_1} \tag{8.9}$$

and the necessary condition for relevancy is met. From Proposition 8.4 we obtain the following:

COROLLARY 8.10. *Assume that k is non-Archimedean. Then the pair $(V_\alpha, W_\alpha)$ determined by the distinguished representation $\pi(\varphi, \chi)$ is relevant. The group $G_\alpha = \mathrm{SO}(V_\alpha) \times \mathrm{SO}(W_\alpha)$ acting on $\pi_\alpha = \pi(\varphi, \chi)$ is isomorphic to G if and only if $\epsilon(M) = \det M_2(-1)^{\frac{1}{2}\dim M_1}$.*

It still remains to prove the relevancy of the pair $(V_\alpha, W_\alpha)$ when $k = \mathbb{R}$, where the recipe for the group $G_\alpha$ acting on $\pi(\varphi, \chi)$ is more complicated. We have not checked this in all cases, but have verified it is true for discrete series parameters. For example, assume that $G = \mathrm{SO}(n+1, n) \times \mathrm{SO}(2, 0)$ and $\varphi$ is a discrete series parameter, given by the numerical invariants $\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$ $\alpha_i \in \frac{1}{2}\mathbf{Z}$ and $\beta \in \mathbf{Z}$ [G-P; 12.12]. The only relevant (non-trivial) pure inner form of $G$ is the group $G' = \mathrm{SO}(n-1, n+2) \times \mathrm{SO}(0, 2)$. If $0 \leq k \leq n$ is the unique integer such that $\alpha_k > |\beta| > \alpha_{k+1}$, then one finds that $\pi(\varphi, \chi)$ is a representation of $G$ when $k$ is even, and a representation of $G'$ when $k$ is odd.

9. **The principal series.** In this section, we consider Conjecture 6.9 in more detail for generic parameters $\varphi$ in the principal series for $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$. We show that $\chi = \chi_0$ (the trivial character of $A_\varphi$) in all cases, so the distinguished representation $\pi(\varphi, \chi)$ is predicted to be the unique $\Theta_0$-generic representation $\pi_0$ in the Langlands $L$-packet $\Pi_\varphi(G)$. In other words, whenever the complex vector space $\mathrm{Hom}_U(\pi, \Theta_0)$ is non-trivial, for $\pi \in \Pi_\varphi(G)$, the complex vector space $\mathrm{Hom}_H(\pi, \Theta)$ should *also* be non-trivial. This suggests that the $U$- and $H$-equivariant linear functionals on $\pi$ may be related by an averaging process.

We begin with a discussion of the principal series for general reductive groups. Let $\underline{G}$ be a quasi-split, connected reductive group over $k$. Let $\underline{B}$ be a Borel subgroup of $\underline{G}$, rational over $k$, and $\underline{T}$ a maximal torus (= Levi subgroup) of $\underline{B}$. Let $T \subset B \subset G$ be the associated groups of $k$-rational points. We say a Langlands parameter $\varphi$ for $G$ is in the *principal series* if the image of $\varphi$ lies in the subgroup ${}^LT = {}^\vee T \rtimes \Gamma$ of ${}^LG = {}^\vee G \rtimes \Gamma$. Since ${}^\vee T$ consists only of semi-simple elements, such a parameter determines a continuous homomorphism of the Weil group

$$\varphi\colon W(k) \longrightarrow {}^LT \tag{9.1}$$

up to conjugation by the Weyl group of ${}^\vee T$ in ${}^\vee G$.

A parameter $\varphi$ as in (9.1) corresponds [B2; §9] to a continuous homomorphism:

$$\rho = \rho(\varphi)\colon T \longrightarrow \mathbb{C}^* \tag{9.2}$$

up to conjugation by the normalizer of $T$ in $G$; the elements in the Langlands $L$-packet $\Pi_\varphi(G)$ are certain Jordan-Holder constituents of the normalized induced representation $\mathrm{Ind}_B^G\, \rho(\varphi)$. When $\varphi$ is generic, $\pi_\varphi(G)$ consists of *all* the distinct Jordan-Holder constituents of this induced representation.

NOTE 9.3. If $k$ is non-Archimedean, we expect that for general $\varphi\colon W(k) \longrightarrow {}^LT$, any Jordan-Holder constituent of the normalized induced representation $\mathrm{Ind}_B^G\big(\rho(\varphi)\big)$ belongs to an $L$-packet determined by $(\varphi, N)$ where $N$ is a nilpotent element in the Lie algebra of ${}^\vee G$ such that $\mathrm{Ad}\big(\varphi(w)\big)N = \|w\| N$ for all $w$ in $W(k)$. Moreover, at least one representation in the $L$-packet determined by $(\varphi, N)$ for any choice of $N$ as above, is a Jordan-Holder constituent of $\mathrm{Ind}_B^G\, \rho(\varphi)$. If $\rho(\varphi)$ is a regular character of $T$, this follows from the work of Rodier, *cf.* [Ro2], and for $G = \mathrm{GL}(n)$, this follows from the work of Zelevinsky, *cf.* [Ze].

In the special case, where $G = \mathrm{SO}(W) \times \mathrm{SO}(V)$, the principal series parameters can be given quite explicitly, and Conjecture 6.9 takes a relatively simple form. A parameter $\varphi_1$ for the split odd orthogonal group $\mathrm{SO}(V_1) = \mathrm{SO}_{2n+1}$ is given by a collection of $n$ quasi-characters $(\chi_1, \ldots, \chi_n)$ of $k^*$, up to the action of the Weyl group of all permutations and inversions. The homomorphism $\varphi_1$ factors through $W(k)^{ab} = k^*$, and we have

$$(9.4) \qquad \varphi_1(\alpha) = \begin{pmatrix} \chi_1(\alpha) & & & & & \\ & \ddots & & & & 0 \\ & & \chi_n(\alpha) & & & \\ & & & \chi_n(\alpha)^{-1} & & \\ & 0 & & & \ddots & \\ & & & & & \chi_1(\alpha)^{-1} \end{pmatrix}$$

in $Sp(M_1) = Sp_{2n}(\mathbf{C})$, for $\alpha \in k^*$. The character $\rho$ is given on $T \subset SO_{2n+1}(k)$ by

$$(9.5) \qquad \rho \begin{pmatrix} \alpha_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & \alpha_n & & & & \\ & & & 1 & & & \\ & & & & \alpha_n^{-1} & & \\ & & 0 & & & \ddots & \\ & & & & & & \alpha_1^{-1} \end{pmatrix} = \chi_1(\alpha_1)\chi_2(\alpha_2)\cdots\chi_n(\alpha_n).$$

When $E = k$, so the even orthogonal group $SO(U_2) = SO_{2m}$ is split, a parameter $\varphi_2$ is given by a collection of $m$ quasi-character $(\eta_1, \dots, \eta_m)$ of $k^*$, up to the action of the Weyl group of all permutations, and all even inversions. Again $\varphi_2$ factors through $W(k)^{ab} = k^*$ and

$$(9.6) \qquad \varphi_2(\alpha) = \begin{pmatrix} \eta_1(\alpha) & & & & & \\ & \ddots & & & & 0 \\ & & \eta_m(\alpha) & & & \\ & & & \eta_m(\alpha)^{-1} & & \\ & 0 & & & \ddots & \\ & & & & & \eta_1(\alpha)^{-1} \end{pmatrix}$$

in $SO(M_2) = SO_{2m}(\mathbf{C})$. The character $\rho$ is given on $T \subset SO_{2m}(k)$ by

$$(9.7) \qquad \rho \begin{pmatrix} \alpha_1 & & & & \\ & \ddots & & & \\ & & \alpha_m & & \\ & & & \alpha_m^{-1} & \\ & & & & \ddots & \\ & & & & & \alpha_1^{-1} \end{pmatrix} = \eta_1(\alpha_1)\eta_2(\alpha_2)\cdots\eta_m(\alpha_m).$$

When $E \neq k$, so $d(U_2) \not\equiv 1$ and $SO(U_2)$ is quasi-split but not split, the group ${}^LT$ in (8.1) may be replaced by ${}^\vee T \rtimes \mathrm{Gal}(E/k) \simeq (\mathbf{C}^*)^{m-1} \times O_2(\mathbf{C})$. A parameter $\varphi_2$ is given by a collection of $(m-1)$ quasi-characters $(\eta_1, \dots, \eta_{m-1})$ of $k^*$, up to permutations and even inversions, and a quasi-character $\eta$ of $E^*/k^*$. These quasi-characters determine quasi-characters $\eta_i$ of $W(k)$, via the isomorphism $W(k)^{ab} = k^*$, and a quasi-character $\eta$

of $W(E)$ with trivial transfer to $W(k)$. Hence $\mathrm{Ind}_{W(E)}^{W(k)}\,\eta$ is a 2-dimensional representation of $W(k)$ with determinant equal to the quadratic character $\epsilon_{E/k}$ of $k^*/\mathrm{N}E^*$. We have

$$(9.8)\qquad \varphi_2 = \begin{pmatrix} \eta_1 & & & & & & \\ & \ddots & & & & & \\ & & \eta_{m-1} & & & & \\ & & & \mathrm{Ind}\,\eta & & & \\ & & & & \eta_{m-1}^{-1} & & \\ & & & & & \ddots & \\ & & & & & & \eta_1^{-1} \end{pmatrix}$$

in $O(M_2)$. The orthogonal representation $\mathrm{Ind}\,\eta$ is reducible if and only if $\eta^2 = 1$. In this case, $\eta = \eta_0 \circ \mathrm{N}_{E/k}$ with $\eta_0$ a quasi-character of $k^*/k^{*2}$ and $\mathrm{Ind}\,\eta \simeq \eta_0 \oplus \eta_0 \cdot \epsilon_{E/k}$.

PROPOSITION 9.9. *Let $\varphi = \varphi_1 \times \varphi_2$ be a principal series parameter for $G = \mathrm{SO}(U_1) \times \mathrm{SO}(U_2)$. Then $A_\varphi = A_{\varphi_1} \times A_{\varphi_2}$, where $A_{\varphi_1} = 1$, and $A_{\varphi_2}$ is given as follows:*

*a) If $E = k$, let $t$ be the number of distinct quasi-characters $\eta_i$ in $\varphi_2$ which satisfy $\eta_i^2 = 1$. Then $A_{\varphi_2} = 1$ if $t = 0$, and $A_{\varphi_2} \simeq (\mathbf{Z}/2)^{t-1}$ if $t \geq 1$.*

*b) If $E \neq k$ and $\eta^2 \neq 1$, let $t$ be the number of distinct quasi-characters $\eta_i$ in $\varphi_2$ which satisfy $\eta_i^2 = 1$. Then $A_{\varphi_2} \simeq \mathbf{Z}/2$ if $t = 0$, and $A_{\varphi_2} \simeq (\mathbf{Z}/2)^t$ if $t \geq 1$.*

*c) If $E \neq k$ and $\eta^2 = 1$, let $t$ be the number of distinct quasi-characters $\eta_i$ in $\varphi_2$ which satisfy $\eta_2^2 = 1$ and $\eta_i \neq \eta_0, \eta_0\epsilon_{E/k}$. Then*

$$A_{\varphi_2} \simeq (\mathbf{Z}/2)^{t+1}.$$

*The character $\chi$ of $A_\varphi$ defined using local root numbers in (6.7) is always equal to the trivial character $\chi_0$.*

PROOF. The determination of $A_{\varphi_1}$ follows from [G-P; Corollary 6.6] and the determination of $A_{\varphi_2}$ follows from [G-P; Corollary 7.7]. We now turn to the computation of $\chi$. First assume $E = k$, and for each distinct quasi-character $\eta_i$ with $\eta_i^2 = 1$, let $\delta_i$ be a simple reflection in the orthogonal group of the corresponding centralizer in $O(M_2)$ [G-P; Proposition 7.6]. The classes $a = \sum_{i=1}^t a_i\delta_i$ with $a_i$ in $\mathbf{Z}/2$ and $\sum_{i=1}^t a_i \equiv 0 \pmod 2$ represent the elements of the group $A_{\varphi_2} = A_\varphi$.

By definition (6.7), we have

$$\chi(\delta_i) = \epsilon(M_1 \otimes \eta_i)\eta_i(-1)^{\frac{1}{2}\dim M_1}.$$

But by (9.3), $M_1 \simeq P \oplus P^\vee$, so by [G; Proposition 3.15]

$$\begin{aligned}\epsilon(M_1 \otimes \eta_i) &= \det(P \otimes \eta_i)(-1)\\ &= \det P(-1) \cdot \eta_i(-1)^{\frac{1}{2}\dim M_1}\end{aligned}$$

Hence $\chi(\delta_i) = \det P(-1)$ is independent of $i$, so $\chi(a) = +1$ for all $a = \Sigma a_i\delta_i$ in $A_\varphi$.

When $E \neq k$ and $\eta^2 = 1$, the proof is exactly the same, with additional classes $\delta_0$ and $\delta_0'$ introduced for the quadratic characters $\eta_0$ and $\eta_0\epsilon_{E/k}$. When $\eta^2 \neq 1$, there is an additional $\delta$ in $A_{\varphi_2}$ corresponding to the element $-1 \in O_2(\operatorname{Ind}\eta)$ [G-P; Proposition 7.6]. We find that

$$\chi(\delta) = \epsilon(M_1 \otimes \operatorname{Ind}\eta) \cdot \epsilon_{E/k}(-1)^{\frac{1}{2}\dim M_1},$$

as $\epsilon_{E/k} = \det(\operatorname{Ind}\eta)$. Again $M_1 = P \oplus P^\vee$ by (9.3), so by [G; Proposition 3.15]

$$\begin{aligned}\epsilon(M_1 \otimes \operatorname{Ind}\eta) &= \det(P \otimes \operatorname{Ind}\eta)(-1)\\ &= \det(P)^2(-1) \cdot \det(\operatorname{Ind}\eta)^{\frac{1}{2}\dim M_1}(-1)\\ &= \epsilon_{E/k}(-1)^{\frac{1}{2}\dim M_1}.\end{aligned}$$

Hence $\chi = \chi_0$ in all cases.

We end by remarking that there is a simple criterion (*cf.* [G-P; Conjecture 2.6]) which should imply that $\varphi$ is generic, in the split case ($E = k$). Namely, for all roots $\alpha^\vee$ of $T^\vee$, one has the quasi-character $\varphi_{\alpha^\vee} = \alpha^\vee \circ \varphi\colon W(k)^{\mathrm{ab}} = k^* \longrightarrow \mathbf{C}^*$. The criterion is that $\varphi_{\alpha^\vee}$ is *not* equal to any of the following quasi-characters $\chi$ of $k^*$.

$$\text{(9.10)} \qquad \begin{cases} k = \mathbf{C} & \chi(z) = z^{-a}\bar{z}^{-b}\ a, b > 0 \\ k = \mathbf{R} & \chi(x) = x^{-a}\operatorname{sign}(x)\ a > 0 \\ k \text{ non-Archimedean} & \chi(x) = |x|^{-1} \end{cases}$$

Using (9.3) and the root system of type $C_n$, and (9.5) and the root system of type $D_m$, we find that $\varphi_{\alpha^\vee}$ runs through the quasi-characters

$$\text{(9.11)} \qquad \varphi_{\alpha^\vee} = \begin{cases} \chi_i^\pm \chi_j^\pm & 1 \le i \le j \le n \\ \eta_i^\pm \eta_j^\pm & 1 \le i < j \le m. \end{cases}$$

NOTE 9.12. When the character $\rho\colon T \longrightarrow \mathbf{C}^*$ associated to $\varphi$ is unitary, one has a great deal of information on the irreducible representations in the Langlands $L$-packet $\Pi_\varphi(G)$, via an analysis of natural intertwining operators of the unitary representation $\operatorname{Ind}_B^G \rho$ (*cf.* [Sh], [K], [K-Sh]). In particular, one can show that $\varphi$ is generic, as predicted by (9.9)–(9.10), and that $\Pi_\varphi(G)$ has the cardinality predicted by Proposition 9.8.

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