ANNALES DE L'INSTITUT FOURIER

INDRANIL BISWAS

Determinant bundle over the universal moduli space of vector bundles over the Teichmüller space

Annales de l'institut Fourier, tome 47, nº 3 (1997), p. 885-914.

<http://www.numdam.org/item?id=AIF_1997__47_3_885_0>

Ann. Inst. Fourier, Grenoble
47, 3 (1997), 885–914

DETERMINANT BUNDLE OVER THE UNIVERSAL MODULI SPACE OF VECTOR BUNDLES OVER THE TEICHMÜLLER SPACE

by Indranil BISWAS

1. INTRODUCTION

Let X be a compact Riemann surface of genus $g \geqslant 2$, equipped with a compatible Riemannian metric. Fix a point $x_0 \in X$. Let $\mathcal{N}$ denote the moduli space of stable vector bundles on X of rank r and determinant $\mathcal{O}_X(d.x_0)$.

Let p be the projection of $X \times \mathcal{N}$ onto $\mathcal{N}$. On $X \times \mathcal{N}$ there is a natural universal adjoint bundle $\mathcal{E}$. The top exterior product of the first direct image

$$\Theta := \bigwedge^{\text{top}} (R^1 p_* \mathcal{E})$$

is an ample line bundle on $\mathcal{N}$. (The line bundle Θ is the anti-canonical bundle of $\mathcal{N}$.)

By a celebrated theorem of Quillen, there is a natural hermitian metric on Θ, such that the curvature of its hermitian connection is a multiple of a natural 2-form on $\mathcal{N}$ obtained after identifying $\mathcal{N}$ with a space of equivalence classes of irreducible $SU(r)$ representations of $\pi_1(X - x_0)$. This identification of $\mathcal{N}$ with a unitary representation space is provided by a well-known theorem of Narasimhan and Seshadri.

The primary aim here is to address the question of the dependence of the Quillen hermitian structure of the line bundle Θ over $\mathcal{N}$ on the

Key words: Moduli space – Determinant bundle – Quillen metric.
Math. classification: 32G13 – 14D20.

conformal structure of the Riemann surface X. Towards this goal, we set forth a systematic study of the universal moduli space and the Quillen determinant bundle over the Teichmüller space.

The space of all marked conformal structures of genus g is parameterized by a space what is known as the Teichmüller space; we shall denote this space by $\mathcal{T}$. Over $\mathcal{T}$ there is a natural universal family, say $\mathcal{C}$, of Riemann surfaces.

We construct a "universal moduli space", say $\mathcal{N}_{\mathcal{T}}$, over $\mathcal{T}$ such that for any $t \in \mathcal{T}$, the fiber $\mathcal{N}_t$ is the moduli space of stable bundles over the Riemann surface represented by the point t. Then we construct a "universal adjoint bundle", say $\mathcal{E}_{\mathcal{T}}$, over the fiber product $\mathcal{C} \times_{\mathcal{T}} \mathcal{N}_{\mathcal{T}}$.

Let p_2 denote the natural projection of $\mathcal{C} \times_{\mathcal{T}} \mathcal{N}_{\mathcal{T}}$ onto $\mathcal{N}_{\mathcal{T}}$. Generalizing the definition of Θ above, let us define

$$\Theta_{\mathcal{T}} := \bigwedge^{\mathrm{top}} (R^1 p_{2*} \mathcal{E}_{\mathcal{T}})$$

to be the line bundle on $\mathcal{N}_{\mathcal{T}}$.

By a construction of Bismut, Gillet and Soulé (which is a generalization of the construction of Quillen), we have a hermitian metric on the line bundle $\Theta_{\mathcal{T}}$.

Bismut, Gillet and Soulé in [BGS1] give a general formula for the curvature of the hermitian line bundle they construct (which is known as the *local Riemann-Roch formula*).

We show that in our particular situation, the curvature form of $\Theta_{\mathcal{T}}$ coincides with a natural $(1,1)$-form on $\mathcal{N}_{\mathcal{T}}$. In particular, we show that the curvature is positive semi-definite.

There is a natural action of the mapping class group, denoted by $\mathcal{MCG}^1_g$, on $\mathcal{T}$ such that the quotient is the moduli space of Riemann surfaces of genus g.

We show that the action of $\mathcal{MCG}^1_g$ lifts to all the objects over $\mathcal{T}$ that we construct. In particular, over the smooth locus of the moduli space of Riemann surfaces there is a "universal moduli space of vector bundles", and a determinant line bundle over this "universal moduli space". There is a natural hermitian connection on this line bundle whose curvature is computed Theorem 5.4; the curvature turns out to be semi-positive.

Acknowledgments. — The author is grateful to the referee for some useful remarks, and also for pointing out several references. The author is grateful to the Institut Fourier and the Académie des Sciences, Paris, for their hospitality and support during the writing of this paper.

2. PRELIMINARIES

In this section we shall recall some known facts about the Teichmüller spaces and the moduli spaces of vector bundles over Riemann surfaces.

2.1. Some facts about the Teichmüller space.

Let S be a compact connected oriented C^∞ surface of genus $g \geqslant 2$. Fix a point $s_0 \in S$.

The space, denoted by $\mathrm{Com}(S)$, of all complex structures on S compatible with the orientation of S, has a natural structure of an infinite dimensional complex Fréchet manifold. The group, $\mathrm{Diff}^+(S, s_0)$, of all orientation preserving diffeomorphisms of S fixing the point $s_0 \in S$, has a natural action on $\mathrm{Com}(S)$ given by the push-forward of a complex structure by a diffeomorphism. Consider the subgroup

$$\mathrm{Diff}_0^+(S, s_0) \subset \mathrm{Diff}^+(S, s_0)$$

consisting of all those diffeomorphisms of (S, s_0) which are homotopic to the identity map, with a homotopy preserving the base point. The *Teichmüller space* for (S, s_0),

$$\mathcal{T}_g^1 := \mathcal{T}_g^1(S, s_0)$$

is the quotient $\mathrm{Com}(S)/\mathrm{Diff}_0^+(S, s_0)$. Since the action of any $g \in \mathrm{Diff}^+(S, s_0)$ preserves the complex structure of $\mathrm{Com}(S)$, there is a natural induced complex structure on $\mathcal{T}_g^1$.

Consider triplets of the form (X, x_0, f), where X is a Riemann surface of genus g, $x_0 \in X$, and $f : X \longrightarrow S$ is a diffeomorphism such that $f(x_0) = s_0$. Identify (X, x_0, f) with (Y, y_0, g) if there is a bi-holomorphic map $h : X \longrightarrow Y$ such that $h(x_0) = y_0$, and the diffeomorphism $g \circ h \circ f^{-1} \in \mathrm{Diff}_0^+(S, s_0)$. The Teichmüller space $\mathcal{T}_g^1$ is the moduli space of equivalence classes of such triplets.

There is a *universal Riemann surface*

$$\pi : \mathcal{C}_g^1 \longrightarrow \mathcal{T}_g^1 \tag{2.1}$$

along with a holomorphic section $\zeta : \mathcal{T}_g^1 \longrightarrow \mathcal{C}_g^1$ such that $\pi \circ \zeta = id$. We shall briefly describe the construction of $\mathcal{C}_g^1$.

On the Cartesian product $S \times \mathrm{Com}(S)$ there is a tautological complex structure determined by the condition that for any $c \in \mathrm{Com}(S)$, the subset

$$S \times c \subset S \times \mathrm{Com}(S)$$

is a complex submanifold with the complex structure c; and for any $s \in S$, the subset $s \times \mathrm{Com}(S)$ is a complex submanifold. This implies that the projection

$$p_2 : S \times \mathrm{Com}(S) \longrightarrow \mathrm{Com}(S)$$

is a holomorphic map. The group $\mathrm{Diff}_0^+(S, s_0)$ acts on both the components S and $\mathrm{Com}(S)$. Consider the diagonal action of $\mathrm{Diff}_0^+(S, s_0)$ on $S \times \mathrm{Com}(S)$. This action preservers the complex structure of $S \times \mathrm{Com}(S)$. The universal Riemann surface $\mathcal{C}_g^1$ is the quotient space

$$(S \times \mathrm{Com}(S))/\mathrm{Diff}_0^+(S, s_0).$$

The projection p_2 induces the complex submersion π. Since the elements of $\mathrm{Diff}_0^+(S, s_0)$ preserve the point s_0, the section of the projection p_2 defined by s_0 descends to a section of the submersion π in (2.1). This is the section ζ mentioned earlier.

For any $t \in T$, after choosing a lift $\bar{t}$ of t in $\mathrm{Com}(S)$, the Riemann surface $\pi^{-1}(t)$ can be identified with S; this identification will be denoted by f_t. The point t represents the equivalence class of the triplet $(\pi^{-1}(t), \zeta(t), f_t)$.

Define $\mathcal{T}_g := \mathcal{T}(S)$ to be the *Teichmüller space for the surface S*. By definition, $\mathcal{T}_g = \mathrm{Com}(S)/\mathrm{Diff}_0^+(S)$, where $\mathrm{Diff}_0^+(S)$ is the group of all orientation preserving diffeomorphisms of S homotopic to the identity map. There is a natural holomorphic submersion of $\mathcal{T}_g^1$ onto $\mathcal{T}_g$ such that any fiber is bi-holomorphic to the open unit disc in $\mathbb{C}$.

For any $t := (X, f) \in \mathcal{T}_g$, the holomorphic tangent space $T_t\mathcal{T}_g$ is canonically identified with $H^1(X, T_X)$. Using Serre duality we have, the following identification of the cotangent space:

$$T_t^*\mathcal{T}_g = H^0(X, K_X^2)$$

where K_X denotes the holomorphic cotangent bundle of X. The Poincaré metric on X, denoted by g_X, is the unique metric with curvature -1. On

$H^0(X, K_X^2)$ the pairing

$$(\alpha\,,\,\beta) \longmapsto \int_X \alpha \otimes \bar{\beta} \otimes g_X^{-1} \tag{2.2}$$

defines a hermitian metric. (Note that $\alpha \otimes \bar{\beta} \otimes g_X^{-1}$ is a $(1,1)$-form on X.) The Riemannian metric on $\mathcal{T}_g$ thus obtained is a Kähler metric, and is called the *Weil-Petersson metric.* Let ω_{wp} denote the hermitian $(1,1)$-form on $\mathcal{T}_g$ for the Weil-Petersson metric.

2.2. Some facts about the moduli space of vector bundles.

Let X be a Riemann surface of genus $g \geqslant 2$. Fix a point $x_0 \in X$.

Let $M(r,d)$ denote the moduli space of isomorphism classes of stable bundles on X of rank r and degree d. We shall always assume that $-r < d \leqslant 0$. Note that after tensoring with a line bundle, the degree of a vector bundle can always be made to lie in this range.

Let $\mathcal{R}(r,d)$ denote the subspace of $\mathrm{Hom}^{\mathrm{ir}}(\pi_1(X - x_0), U(r))/U(r)$ consisting of all those representations whose holonomy along the oriented loop around x_0 is

$$\exp(-2\pi d\sqrt{-1}/r).I_{r\times r} \in \mathrm{center}(U(r)).$$

A fundamental theorem of Narasimhan and Seshadri, [NS], identifies $M(r,d)$ with $\mathcal{R}(r,d)$.

For $\alpha \in \mathcal{R}(r,d)$, let $\mathcal{U}_\alpha$ denote the local system on X given by the adjoint action on the Lie algebra $u(r)$. The tangent space of $T_\alpha\mathcal{R}(r,d)$ is $H^1(X, \mathcal{U}_\alpha)$. For $v, w \in T_\rho\mathcal{R}(r,d)$, the pairing

$$(v,w) \longmapsto \int_X \mathrm{trace}(v \cup w) \tag{2.3}$$

defines a symplectic form on $\mathcal{R}(r,d)$ [G], which is actually a Kähler form on $M(r,d)$. We shall denote this Kähler form by Ω. The form Ω was first constructed in [AB], and is a special case of a very general construction of Weil-Petersson form in ([ST], page 703, Theorem 2).

Choose and fix a metric on the Riemann surface X; for example, the Poincaré metric.

Fix a C^∞ hermitian vector bundle V of rank r and degree d on X. Let $\Omega^{p,q}(V)$ denote the space of all smooth (p,q)-forms on X with values in V. The group of all smooth automorphisms of V will be denoted by $\mathcal{G}$.

Let $\mathcal{A}$ denote the space of all holomorphic structures on V, which is an affine space for the vector space $\Omega^{0,1}(\mathrm{End}(V))$.

Consider the open subset $\mathcal{A}^s \subset \mathcal{A}$ consisting of all holomorphic structures such the corresponding holomorphic bundle is a stable bundle. The group $\mathcal{G}$ acts on $\mathcal{A}$, and it preserves the subset $\mathcal{A}^s$. The quotient $\mathcal{A}^s/\mathcal{G}$ is $M(r,d)$ [BR, Proposition 3.7].

Let p_i, $i = 1,2$, denote the projection of the Cartesian product $X \times \mathcal{A}$ to its i-th factor.

On $p_1^*(V)$ we have a holomorphic structure defined by the operator $\overline{\partial}_V$ which acts on any section s of $p_1^*(V)$ in the following way:

$$(\overline{\partial}_T s)(x, \overline{\partial}_V) := \overline{\partial}_V(s|_{\{X\times\overline{\partial}_V\}})(x) + \overline{\partial}_{\mathcal{A}}(s|_{\{x\times\mathcal{A}\}})(\overline{\partial}_V). \tag{2.4}$$

For any $\overline{\partial} \in \mathcal{A}$, let ∇ denote the corresponding holomorphic hermitian connection on V. The connection, ∇^T, on $p_1^*(V)$ defined by

$$(\nabla^T s)(x, \overline{\partial}) := \nabla(s|_{\{X\times\overline{\partial}\}})(x) + d(s|_{\{x\times\mathcal{A}\}})(\overline{\partial}) \tag{2.5}$$

is the holomorphic hermitian connection for the holomorphic structure $\overline{\partial}_T$ and the obvious pullback hermitian metric.

It is easy to check that the natural action of $\mathcal{G}$ on $p_1^*(V)$ preserves the holomorphic structure $\overline{\partial}_T$. Taking the quotient of $P(p_1^*(V))$ by $\mathcal{G}$ we get the universal projective bundle, denoted by P, on $X \times M(r,d)$.

Let $\mathcal{G}_{\mathcal{U}} \subset \mathcal{G}$ be the subgroup consisting of all unitary automorphisms. The action of $\mathcal{G}_{\mathcal{U}}$ on $p_1^*(V)$ preserves both the holomorphic structure $\overline{\partial}_T$ and the connection ∇^T.

A hermitian connection on V is called a *Yang-Mills connection* if the curvature is of the form $\lambda.\omega.Id_V$, where ω is the Kähler form on X, and λ is a constant on X.

Let $\mathcal{A}_H \subset \mathcal{A}$ be the set of all irreducible Yang-Mills connections on V. Using the action of any $\mathcal{G}_{\mathcal{U}}$ on the restriction of $p_1^*(V)$ to $X \times \mathcal{A}_H$, we get a reduction of the structure group of the universal projective bundle, P, to the projective unitary group $PU(r) \subset U(r)$. Moreover, since the action of $\mathcal{G}_{\mathcal{U}}$ preserve the operators $\overline{\partial}_T$ and ∇^T defined earlier, we have the following lemma:

LEMMA 2.6. — *The projective bundle P is equipped with a holomorphic connection, denoted by ∇, such that ∇ is the extension of a $PU(r)$ connection on the above reduction of structure group of P to $PU(r)$.*

Consider the following subvariety of $M(r,d)$

$$N(r,d) := \{E \in M(r,d) \mid \overset{r}{\wedge} E = \mathcal{O}_X(d.x_0)\}. \tag{2.7}$$

Let p'_2 denote the projection of $X \times N(r,d)$ onto $N(r,d)$. The sheaf $p'_{2*}\mathcal{E}^0$ on $N(r,d)$ is locally free of rank $(r^2-1)(g-1)$. Define

$$\Theta' := \overset{\text{top}}{\wedge} R^1 p'_{2*}\mathcal{E}^0. \tag{2.8}$$

(Note that the 0-th direct image $R^0 p'_{2*}\mathcal{E}^0$ is zero.) Let $i : N(r,d) \longrightarrow M(r,d)$ be the inclusion.

We shall denote the Quillen metric on Θ' by h_Q.

The Theorem 0.1 of [BGS1] gives the following:

Proposition 2.9. — *The curvature of the hermitian metric h_Q on Θ' is $4\pi r\sqrt{-1}.i^*\Omega$, where Ω is the form defined in (2.3).*

3. THE UNIVERSAL MODULI SPACE AND THE UNIVERSAL BUNDLE

In this section we shall carry out the constructions of Section 2.2 for the universal Riemann surface $\mathcal{C}_g^1$ defined in Section 2.1.

3.1. The universal moduli space.

The existence of the universal moduli over $\mathcal{T}_g^1$ follows from a very general construction carried out in [ST]. A simple construction that we give bellow will be used in later computations.

As the first step towards constructing the universal moduli space as a complex manifold, we shall construct the underlying topological manifold.

We continue with the notation of Section 2.1. Consider $\mathrm{Hom}^{\mathrm{ir}}(\pi_1(S-s_0), U(r))/U(r)$, the space of equivalence classes of irreducible representations of $\pi_1(S-s_0)$ in $U(r)$. This space does not depend upon the choice of the base point needed in order to define the fundamental group $\pi_1(S-s_0)$. Define $\mathcal{R}(r,d)$ to be the subspace of $\mathrm{Hom}^{\mathrm{ir}}(\pi_1(S-s_0), U(r))/U(r)$ consisting of all homomorphisms such that the holonomy along the oriented loop around s_0 is $\exp(-2\pi d\sqrt{-1}/r)I_{r\times r}$.

For any $\alpha := (X, x_0, f) \in \mathcal{T}_g^1$, let $\mathcal{R}(r,d)_\alpha$ denote the representation space (as above) for X. Using f, the space $\mathcal{R}(r,d)_\alpha$ gets identified with $\mathcal{R}(r,d)$. For (X, x_0, f') equivalent to α, since f is homotopic to f', the above identification of $\mathcal{R}(r,d)_\alpha$ with $\mathcal{R}(r,d)$ is independent of the choice of the diffeomorphism f in the equivalence class. Thus, for another $\beta := (Y, y_0, g) \in \mathcal{T}_g^1$, the spaces $\mathcal{R}(r,d)_\alpha$ and $\mathcal{R}(r,d)_\beta$ are identified, since both are individually identified with $\mathcal{R}(r,d)$.

In Section 2.2 we mentioned that $\mathcal{R}(r,d)$ is naturally diffeomorphic the moduli space of stable bundles. So from the above it follows that if the universal moduli space on $\mathcal{T}_g^1$ exists then it must be diffeomorphic to $\mathcal{T}_g^1 \times \mathcal{R}(r,d)$.

However, given any $\alpha \in \mathcal{T}_g^1$, there is some neighborhood $U \subset \mathcal{T}_g^1$, of α, such that the for the family of Riemann surfaces $\mathcal{C}_g^1|_U \longrightarrow U$, the corresponding family of moduli spaces $M_U(r,d) \longrightarrow U$ exists as a complex manifold. From the previous discussion we conclude that $M_U(r,d)$ is canonically diffeomorphic to $U \times \mathcal{R}(r,d)$. This identification equips the product $U \times \mathcal{R}(r,d)$ with a complex structure. For two such sets

$$U \times \mathcal{R}(r,d) \quad \text{and} \quad U' \times \mathcal{R}(r,d)$$

the two complex structures on the intersection $(U \cap U') \times \mathcal{R}(r,d)$ actually match. So we have a complex structure on $\mathcal{T}_g^1 \times \mathcal{R}(r,d)$; this complex manifold will be called *the universal moduli space*, and it will be denoted by $\mathcal{M}(r,d)$.

This universal moduli space has the following property which is obvious from its construction: Let U be a complex submanifold of $\mathcal{T}_g^1$ (of any possible dimension); and let E be a rank r holomorphic bundle on the restriction $\mathcal{C}_g^1|_U$ such that for any $u \in U$, the restriction, E_u, of E to $\mathcal{C}_g^1|_u$ is a stable bundle of rank r and degree d. For any $\alpha \in \mathcal{T}_g^1$, by using the natural identification of the moduli space $M_\alpha(r,d)$ with $\mathcal{R}(r,d)$, we get a C^∞ map which is called the *classifying map*

$$\Gamma_E \,:\, U \longrightarrow \mathcal{T}_g^1 \times \mathcal{R}(r,d)$$

for the family of bundles E; in other words, $\Gamma_E(u)$ is the point corresponding to the bundle E_u. The complex structure of $\mathcal{M}(r,d)$ has the property that this classifying map Γ_E is actually a holomorphic map.

Now from the definition of the complex structure on $\mathcal{M}(r,d)$ it is clear that the obvious projection

$$p_1 \,:\, \mathcal{M}(r,d) \longrightarrow \mathcal{T}_g^1$$

is a holomorphic submersion. The following theorem gives further properties of the complex structure on $\mathcal{M}(r,d)$.

THEOREM 3.1. — *For any $\nu \in \mathcal{R}(r,d)$, the subset $\mathcal{T}_g^1 \times \nu \subset \mathcal{M}(r,d)$ is a complex submanifold of $\mathcal{M}(r,d)$.*

Before we prove this theorem, let us show how this theorem gives an explicit description of the complex structure on $\mathcal{M}(r,d)$.

For any $\mu := (\alpha, \rho) \in \mathcal{T}_g^1 \times \mathcal{R}(r,d)$, the (real) tangent space has the decomposition

$$T_\mu(\mathcal{T}_g^1 \times \mathcal{R}(r,d)) = T_\alpha \mathcal{T}_g^1 \oplus T_\rho \mathcal{R}(r,d).$$

The complex manifold structure on $\mathcal{T}_g^1$ gives an automorphism (the almost complex structure)

$$A \in \mathrm{Aut}(T_\alpha \mathcal{T}_g^1)$$

such that $A^2 = -Id$. Take any $\alpha := (X, x_0, f) \in \mathcal{T}_g^1$. The space $\mathcal{R}(r,d)_\alpha$ is identified with the moduli of rank r and degree d stable bundles on X. This induces a complex structure on $\mathcal{R}(r,d)_\alpha$. Now, the identification of $\mathcal{R}(r,d)_\alpha$ with $\mathcal{R}(r,d)$ (using f) gives a complex structure on $\mathcal{R}(r,d)$. Using this complex structure, we have

$$B_\alpha \in \mathrm{Aut}(T_\rho \mathcal{R}(r,d))$$

such that $B_\alpha^2 = -Id$. Using the decomposition $T_\mu(\mathcal{T}_g^1 \times \mathcal{R}(r,d)) = T_\alpha \mathcal{T}_g^1 \oplus T_\rho \mathcal{R}(r,d)$, define

$$J(\alpha) := A \oplus B_\alpha \tag{3.2}$$

to be the automorphism of $T_\mu(\mathcal{T}_g^1 \times \mathcal{R}(r,d))$. Clearly, $J(\alpha)^2 = -Id$. In other words, we have constructed an almost complex structure on $\mathcal{T}_g^1 \times \mathcal{R}(r,d)$, which we shall denote by J.

The submanifold $\alpha \times \mathcal{R}(r,d) \subset \mathcal{M}(r,d)$ is obviously a complex submanifold, and from Theorem 3.1, $\mathcal{T}_g^1 \times \rho$ is a complex submanifold of $\mathcal{M}(r,d)$. Hence we have the following corollary of Theorem 3.1.

COROLLARY 3.3. — *The almost complex structure for the complex manifold $\mathcal{M}(r,d)$ is J (defined in (3.2)). In particular, the almost complex structure J is integrable.*

So, in view of Corollary 3.3, Theorem 3.1 can be taken as a construction of the complex structure of the universal moduli space.

Proof of Theorem 3.1. — Before actually proving the theorem, here is the strategy of the proof. Given a point $\alpha \in \mathcal{T}_g^1$ we shall show that there is an open set $U \subset \mathcal{T}_g^1$ containing α such that over the family of Riemann surfaces

$$\mathcal{C}_g^1|_U \longrightarrow U$$

there is a holomorphic bundle $E \longrightarrow \mathcal{C}_g^1|_U$ such that for any $\beta := (Y, y_0, g) \in U$, the restriction $E|_\beta \longrightarrow Y$ is the stable bundle corresponding to the representation ν (in the statement of 3.1). (As noted earlier, using g the representation ν gives a natural element of $\mathrm{Hom}^{\mathrm{ir}}(\pi_1(Y - y_0), U(r))/U(r)$ with holonomy $2\pi d/r.I_{r\times r}$ around y_0.) We earlier observed that the classifying map is holomorphic. In particular, the classifying map

$$f_E : U \longrightarrow \mathcal{M}(r, d)$$

for the family E is holomorphic. But the image of f_E, from the property of E, is the submanifold $U \times \nu$ of $\mathcal{M}(r, d)$. Thus, $U \times \nu$ is a holomorphic submanifold of $\mathcal{M}(r, d)$. This would complete the proof of Theorem 3.1.

So the point is to construct the holomorphic bundle E.

First we shall consider the case where the degree $d = 0$.

Take any β and ν as above. The local system on $Y - y_0$ given by ν naturally extends across y_0, since the holonomy around y_0 is trivial. Let V_β denote the rank r local system on Y obtained this way. The stable bundle on Y corresponding to ν is the holomorphic bundle given by V_β.

For the projection π defined in (2.1) — since the fibers are connected and $\mathcal{T}_g^1$ is contractible — the long homotopy exact sequence implies that the inclusion of a fiber in $\mathcal{C}_g^1$ induces an isomorphism of the fundamental groups. Using this isomorphism, ν gives an element

$$\bar{\nu} \in \mathrm{Hom}^{\mathrm{ir}}(\pi_1(\mathcal{C}_g^1), U(r))/U(r).$$

(Since the holonomy of ν is trivial around s_0, we may consider it as an element of $\mathrm{Hom}^{\mathrm{ir}}(\pi_1(S), U(r))/U(r)$.) Let $\bar{V}_{\bar{\nu}}$ denote the local system on $\mathcal{C}_g^1$ given by $\bar{\nu}$. Clearly the restriction of $\bar{V}_{\bar{\nu}}$ to any Riemann surface $\pi^{-1}(\beta)$ is the local system V_β.

Let E be the holomorphic bundle on $\mathcal{C}_g^1$ corresponding to the local system $\bar{V}_{\bar{\nu}}$.

For any $\beta := (Y, y_0, g)$, we noted that the restriction of $\bar{V}_{\bar{\nu}}$ to Y is the local system V_β defined earlier. This implies that the restriction of E

to Y is the stable bundle on Y corresponding to ν. So E has the required property which implies that the classifying map $\mathcal{T}_g^1 \longrightarrow \mathcal{M}(r,d)$, given by $\beta \longmapsto (\beta, \nu)$ is holomorphic. This completes the proof for the case $d = 0$.

Now assume that $d \neq 0$. We shall complete proof of the theorem by reducing this case to the earlier case of $d = 0$. The following proposition will be used for that purpose.

PROPOSITION 3.4. — *For any integer $r \geqslant 1$ there is a Galois cover $\lambda : \tilde{S} \longrightarrow S$ with Galois group $\mathbb{Z}/r$ which is totally ramified over s_0.*

Proof of Proposition 3.4. — Let $D^2 := \{z \in \mathbb{C} \mid |z|^2 \leqslant 1\}$ denote the closed unit disc in $\mathbb{C}$ and let $D_0^2 := \{z \in \mathbb{C} \mid |z|^2 < 1\}$ be its interior. Take a disc in S

$$f : D^2 \longrightarrow S$$

with $f(1) = s_0$. Consider the manifold with boundary $S' := S - f(D_0^2)$. Take the disjoint union of r copies of S':

$$S^r := \bigcup_{j=1}^{r} S'_j$$

where each S'_j is a copy of S'. For any integer j with $1 \leqslant j \leqslant r-1$, and any $t = \exp(2\pi\sqrt{-1}\theta) \in \partial D^2$ with $0 \leqslant \theta \leqslant 1/2$, identify the point $f(t)$ in the component S'_j with the point $f(t^{-1})$ in the component S'_{j+1}; also, identify the point $f(t)$ (t as above) in S'_r with the point $f(t^{-1})$ in S'_1. Let $\tilde{S}$ denote the quotient space (of S^r) obtained using the above identifications. Let S_0 denote the quotient of $S - f(D_0^2)$ obtained by identifying $f(t) \in f(\partial D^2)$ with $f(t^{-1})$. It is easy to see that there is a natural projection of $\tilde{S}$ onto S_0. This projection is a Galois covering with Galois group $\mathbb{Z}/r$, and it is totally ramified over (the images of) s_0 and $f(-1)$. But S_0 is a compact oriented two manifold of genus g. Hence S_0 is diffeomorphic to S. So, composing the above projection with a diffeomorphism from S_0 to S which takes the image of s_0 (in S_0) to s_0, we get the required covering λ in the statement of the proposition. □

Continuing with the proof of Theorem 3.1, for β and ν as earlier, recall the construction of the corresponding stable bundle. The representation ν gives a rank r local system on $Y - y_0$. The local system gives a holomorphic vector bundle on $Y - y_0$. Now, using the local system this vector bundle is extended to Y. See Section 1 of [MS] for the details of this extension. Of course, such extensions were carried out earlier in a much more general situation by Deligne [K].

Fix once and for all a cover $\lambda : \tilde{S} \longrightarrow S$ of degree r given by Proposition 3.4. Denote $\lambda^{-1}(s_0)$ by $\tilde{s}$.

Take any $\beta := (Y, y_0, g) \in \mathcal{T}_g^1$. Using the diffeomorphism g we may pullback the covering λ to a covering

$$g^*\lambda : \tilde{Y} \longrightarrow Y. \tag{3.5}$$

Let $\bar{g} : \tilde{Y} \longrightarrow \tilde{S}$ be the diffeomorphism induced by g. It is easy to see that the pointed Riemann surface $(\tilde{Y}, \bar{g}^{-1}(\tilde{s}))$ does not depend upon the choice of g in the equivalence class. We shall denote $\bar{g}^{-1}(\tilde{s})$ by $\tilde{y}$.

For $\nu \in \mathcal{R}(r,d)$ let V_ν be the stable bundle on Y corresponding to ν. Let $\lambda^*(\nu) \in \mathrm{Hom}^{\mathrm{ir}}(\pi_1(\tilde{Y} - \tilde{y}), U(r))/U(r)$ denote the pullback using the homomorphism

$$(g^*\lambda)_* : \pi_1(\tilde{Y} - \tilde{y}) \longrightarrow \pi_1(Y - y_0)$$

note that this corresponds to pullback of the local system. From the condition on ramification of λ, the holonomy around the point $\tilde{y}$ of the local system on $\tilde{Y} - \tilde{y}$ given by $\lambda^*(\nu)$ is identity. So the local system extends to $\tilde{Y}$. Let W_ν be the holomorphic vector bundle on $\tilde{Y}$ for this local system. The Galois group $\mathbb{Z}/r$ acts as automorphisms of on $\tilde{Y}$, and, since W_ν corresponds to a local pullback local system, the action of $\mathbb{Z}/r$ on $\tilde{Y}$ lifts to automorphisms of W_ν. (That ν is only a local system on the complement $Y - y_0$ and not on the whole of Y is reflected in the fact that the isotropy, $\mathbb{Z}/r$, of $\tilde{y}$ acts nontrivially on the fiber $W_\nu|_{\tilde{y}}$.) So $\mathbb{Z}/r$ acts on the direct image $(g^*\lambda)_*W_\nu$ (defined in (3.5)). It can be checked that V_ν is the invariant subsheaf of $(g^*\lambda)_*W_\nu$.

We shall do the above constructions for a family of Riemann surfaces.

For $U \subset \mathcal{T}_g^1$ let $\mathcal{C}|_U$ denote the restriction of the family of Riemann surfaces $\mathcal{C}_g^1$ to U.

Let U be an contractible open set containing $\alpha := (X, x_0, f)$ such that we have a Galois cover

$$\Gamma : \tilde{\mathcal{C}} \longrightarrow \mathcal{C}|_U \tag{3.6}$$

with Galois group $\mathbb{Z}/r$ such that it is totally ramified over the divisor in $\mathcal{C}|_U$ given by the image of ζ (in (2.1)). So $\tilde{\mathcal{C}}$ is a family of Riemann surfaces parameterized by U; for $\beta \in U$, the Riemann surface over β, for this family, will be denoted by $\tilde{\mathcal{C}}|_\beta$.

Since fibers of $\pi \circ \Gamma$ are connected and U is contractible, we have that $\pi_1(\tilde{\mathcal{C}})$ is canonically isomorphic to the fundamental group of a fiber of $\pi \circ \Gamma$ where the isomorphism is given by the inclusion of the fiber. So there is an one-to-one correspondence between local systems on $\tilde{\mathcal{C}}$ and local systems on a fiber of $\pi \circ \Gamma$. We saw that the local system $\lambda^*(\nu)$ extends across the puncture; and hence, using the above correspondence, we have a local system $\bar{V}_{\bar{\nu}}$ on $\tilde{\mathcal{C}}$. Let V denote the holomorphic bundle on $\tilde{\mathcal{C}}$ given by $\bar{V}_{\bar{\nu}}$. It is easy to see that the holomorphic bundle on $\mathcal{C}|_U$ given by the invariant subsheaf of the direct image sheaf, namely

$$E := (\Gamma_* V)^{\mathbb{Z}/r}$$

has the required property. This completes the proof of Theorem 3.1. □

In (2.3) we defined the symplectic form, Ω, on $\mathcal{R}(r,d)$ constructed in [G]. Let $p_2^*\Omega$ denote the pullback of Ω to $\mathcal{M}(r,d)$, using the projection to the second factor.

LEMMA 3.7. — *The 2-form $p_2^*\Omega$ on $\mathcal{M}(r,d)$ is of type $(1,1)$, and also it is a closed positive semi-definite form.*

We note that this form $p_2^*\Omega$ is a special case of a very general Weil-Petersson form constructed in [ST].

Proof of Lemma 3.7. — Since Ω is a closed form, the pullback form, $p_2^*\Omega$, is also closed.

Since $p_2^*\Omega$ is a real form, in order to prove that it is $(1,1)$ type, it is enough to show that the $(2,0)$ part of $p_2^*\Omega$, denoted by $(p_2^*\Omega)^{2,0}$, vanishes.

Take any $\bar{m} := (\alpha, \nu) \in \mathcal{M}(r,d)$. From Corollary 3.3, the holomorphic tangent vector space

$$T^{1,0}_{\bar{m}}\mathcal{M}(r,d) = T^{1,0}_{\alpha}\mathcal{T}^1_g \oplus T^{1,0}_{\nu}M_{\alpha}(r,d) \tag{3.8}$$

where $M_{\alpha}(r,d)$ is the moduli space of stable vector bundles over the pointed Riemann surface $\mathcal{C}^1_g|_{\alpha}$. For any $v \in T^{1,0}_{\alpha}\mathcal{T}^1_g$, we have $d(p_2)(v) = 0$ (as an element of $T_{\nu}\mathcal{R}(r,d) \otimes \mathbb{C}$). So, in order to prove that $(p_2^*\Omega)^{2,0}(\bar{m}) = 0$, it is enough to show that $p_2^*\Omega(u,v) = 0$ for $v, w \in T^{1,0}_{\nu}M_{\alpha}(r,d)$. But Ω is a $(1,1)$-form on $M_{\alpha}(r,d)$. So we have $p_2^*\Omega(u,v) = 0$.

To complete the proof we have show that $p_2^*\Omega(w,\bar{w}) \geqslant 0$ for any $w \in T^{1,0}_{\bar{m}}\mathcal{M}(r,d)$.

In view of the earlier remark that $d(p_2)(T^{1,0}_{\alpha}\mathcal{T}^1_g) = 0$, it is enough to show the above inequality for any $w \in T^{1,0}_{\nu}M_{\alpha}(r,d)$. But Ω is a Kähler form on $M_{\alpha}(r,d)$, and hence the proof of the lemma is completed. □

We note that from the proof of Lemma 2.6 it follows that for any Kähler form Ω' on $\mathcal{T}_g^1$, the form $p_1^*\Omega' + p_2^*\Omega$ is a Kähler form on $\mathcal{M}(r,d)$.

3.2. The universal projective bundle over $\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)$.

We shall construct a universal projective bundle over the fiber product $\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)$. As the first step we shall construct the underlying topological projective bundle of the universal projective bundle.

DEFINITION 3.9. — *Let $\pi : P \longrightarrow M$ be a principal G bundle on a manifold M equipped with a foliation $F \subset TM$ (i.e., F is an integral subbundle of the tangent bundle). A partial connection on P along F is a C^∞ lift*

$$\tilde{\pi} : \pi^* F \longrightarrow TP$$

of the differential $d\pi$ (i.e., $d\pi \circ \tilde{\pi}$ is identity on F) which is equivariant for the action of G on P [KT]. A flat partial connection is a partial connection such that the lift $\tilde{\pi}$ preserves the Lie bracket. This is equivalent to the condition that the image of $\tilde{\pi}$ is an integrable subbundle of TP. Note that in the special case where $F = TM$, a (flat) partial connection is a (flat) connection in the usual sense.

For any $\rho \in \mathcal{R}(r,d)$ we have a flat principal $U(r)$ bundle, U_ρ, on $S - s_0$. Consider the extension of structure group of U_ρ to $PU(r)$ given by the projection $U(r) \longrightarrow PU(r)$. The holonomy around s_0 of the connection on U_ρ is in the center of $U(r)$. Hence the principal $PU(r)$ bundle extends to S; let $P(\rho)$ denote the projective bundle on S thus obtained. The flat connection U_ρ induces a flat $PU(r)$ connection on $P(\rho)$, which we shall denote by ∇^ρ.

Let $\mathcal{F}$ denote the foliation on $S \times \mathcal{R}(r,d)$ along the S direction; in other words, the leaves of $\mathcal{F}$ are the fibers of the natural projection of $S \times \mathcal{M}(r,d)$ onto $\mathcal{M}(r,d)$.

On $S \times \mathcal{R}(r,d)$ there is a C^∞ projective bundle, $P(S)$, equipped with a flat partial connection, $\nabla(\mathcal{F})$, along $\mathcal{F}$ such that for any $\rho \in \mathcal{R}(r,d)$, the restriction of the pair $(P(S), \nabla(\mathcal{F}))$ to $S \times \rho$ is isomorphic to the pair $(P(\rho), \nabla^\rho)$ defined above. Before constructing the bundle $P(S)$, we first note that any two projective bundles on $S \times \mathcal{R}(r,d)$ with this property are canonically isomorphic. Indeed, any automorphism of $P(\rho)$ preserving the connection ∇^ρ must be the trivial automorphism. Hence for another projective bundle $(P(S)', \nabla(\mathcal{F})')$, there is a unique isomorphism between the restrictions to $S \times \rho$ of $P(S)'$ and $P(S)$ respectively, which takes the connection $\nabla(\mathcal{F})$ to $\nabla(\mathcal{F})'$. Hence $P(S)$ and $P(S)'$ are canonically isomorphic. Moreover, the isomorphism between them takes the partial connection $\nabla(\mathcal{F})$ to the partial connection $\nabla(\mathcal{F})'$.

After putting a complex structure on the surface S we may invoke Lemma 2.6 which says that for a Riemann surface X there is a universal projective bundle on $X \times M(r,d)$ with a $PU(r)$ connection.

Let $(P(S), \nabla)$ be the projective bundle with $PU(r)$ connection on $S \times \mathcal{M}(r,d)$ for the chosen complex structure on S. The partial connection on $P(S)$ along $\mathcal{F}$ induced by ∇ is denoted by $\nabla(\mathcal{F})$. Clearly the pair $(P(S), \nabla(\mathcal{F}))$ satisfy the above conditions. The point is: though the connection ∇ depends upon the conformal structure on S, the partial connection does not depend upon the conformal structure.

The group $\mathrm{Diff}_0^+(S, s_0)$ (defined in Section 2.1) acts on $S \times \mathcal{R}(r,d)$ by the combination of the tautological action on S and the trivial action on $\mathcal{R}(r,d)$. We shall describe an action of $\mathrm{Diff}_0^+(S, s_0)$ on the bundle $P(S)$ which is lift of the above action on $S \times \mathcal{R}(r,d)$. Take any $f \in \mathrm{Diff}_0^+(S, s_0)$; for $\rho \in \mathcal{R}(r,d)$ consider the pullback bundle $f^*P(\rho)$ equipped with the pullback connection $f^*\nabla^\rho$. Since $f \in \mathrm{Diff}_0^+(S, s_0)$, the two bundles with flat connections, namely $(P(\rho), \nabla^\rho)$ and $(f^*P(\rho), f^*\nabla^\rho)$, are isomorphic. Indeed, as f is homotopic to the identity map, their holonomies are conjugate to each other. However, the pair $(P(\rho), \nabla^\rho)$ do not admit any nontrivial automorphism, since the connection is irreducible. So there is a unique isomorphism between $(P(\rho), \nabla^\rho)$ and $(f^*P(\rho), f^*\nabla^\rho)$. Consider the diffeomorphism $f \times Id$ of $S \times \mathcal{R}(r,d)$. It is easy to check that the isomorphism

$$I(f) \,:\, (f \times Id)^* P(\rho) \longrightarrow P(\rho)$$

obtained above has the following property: for $g \in \mathrm{Diff}_0^+(S, s_0)$, the equality

$$I(f) \circ I(g) \;=\; I(f \circ g)$$

holds. In other words, we have lift of the action of $\mathrm{Diff}_0^+(S, s_0)$ on $S \times \mathcal{R}(r,d)$ to the pair $(P(S), \nabla(\mathcal{F}))$, i.e., a lift to an action on $P(S)$ which preserves the partial connection $\nabla(\mathcal{F})$.

Recall the construction of the universal Riemann surface $\mathcal{C}_g^1$ in Section 2.1. Let p_{12} denote the projection of $S \times \mathcal{R}(r,d) \times \mathrm{Com}(S)$ onto $S \times \mathcal{R}(r,d)$. Consider the projective bundle with flat partial flat $PU(r)$ connection

$$(p_{12}^* P(S)\,,\, p_{12}^* \nabla(\mathcal{F})) \longrightarrow S \times \mathcal{R}(r,d) \times \mathrm{Com}(S). \tag{3.10}$$

The group $\mathrm{Diff}_0^+(S, s_0)$ acts on $S \times \mathcal{R}(r,d) \times \mathrm{Com}(S)$ by the combination of the previous action on $S \times \mathcal{R}(r,d)$ and the push-forward of complex structure on S (defined in Section 2.1) by a diffeomorphism. Note that the action of $\mathrm{Diff}_0^+(S, s_0)$ on $(P(S), \nabla(\mathcal{F}))$ induces action of $\mathrm{Diff}_0^+(S, s_0)$ on $(p_{12}^* P(S), p_{12}^* \nabla(\mathcal{F}))$.

Consider the projection
(3.11)
$$\big(p_{12}^* P(S)\,,\, p_{12}^* \nabla(\mathcal{F})\big)/\mathrm{Diff}_0^+(S, s_0) \longrightarrow \big(S \times \mathcal{R}(r,d) \times \mathrm{Com}(S)\big)/\mathrm{Diff}_0^+(S, s_0)$$
$$= \mathcal{C}_g^1 \times \mathcal{R}(r,d) = \mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d).$$

(In section 3.1 we saw that $\mathcal{M}(r,d) = \mathcal{T}_g^1 \times \mathcal{R}(r,d)$.) It is easy to that the map in (3.11) gives a projective bundle on $\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)$ equipped with a flat partial $PU(r)$ connection along the fibers of the projection of $\mathcal{C}_g^1$ onto $\mathcal{T}_g^1$; we shall denote this projective bundle with partial connection by $(\mathcal{P}(r,d), \nabla(\mathrm{par}))$.

Let $q_2 : \mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d) \longrightarrow \mathcal{M}(r,d)$ be the projection onto the second factor. For any element
$$(t, \rho) \in \mathcal{T}_g^1 \times \mathcal{R}(r,d) = \mathcal{M}(r,d)$$
the restriction of $\mathcal{P}(r,d)$ to $q_2^{-1}(t,\rho)$ is the projective bundle $P(\rho)$ defined earlier, and the restriction of $\nabla(\mathrm{par})$ is the partial connection ∇^ρ.

Using an earlier argument, namely any automorphism of $P(\rho)$ preserving the connection ∇^ρ must be the trivial automorphism (which was used in the proof of uniqueness of $(P(S), \nabla)$), any two projective bundles on $\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)$ with partial connections satisfying the above condition must be canonically isomorphic. Thus we have established the uniqueness of the projective bundle $\mathcal{P}(r,d)$.

Remark 3.12. — We shall give another construction of the pair $(\mathcal{P}(r,d), \nabla(\mathrm{par}))$. The projection $\gamma : \mathrm{Com}(S) \longrightarrow \mathcal{T}_g^1$ admits local sections. Take a covering $\{U_i\}$ of $\mathcal{T}_g^1$ by open sets such that over each U_i there is a smooth section s_i of γ. Consider the disjoint union
$$\bigcup_i (P(S) \times U_i).$$
On the intersection $U_i \cap U_j$ the difference of the two sections s_i and s_j is given by a map
$$g_{ij} : U_i \cap U_j \longrightarrow \mathrm{Diff}_0^+(S, s_0).$$

Recall the action of $\text{Diff}_0^+(S, s_0)$ on $P(S)$. Using the map g_{ij} we may glue $P(S) \times U_i$ and $P(S) \times U_j$ along $P(S) \times (U_i \cap U_j)$. The resulting space is $\mathcal{P}(r,d)$. The action of $\text{Diff}_0^+(S, s_0)$ on $P(S)$ preserves the partial connection $\nabla(\mathcal{F})$. This implies that the partial connection on the projective bundle $P(S) \times U_i$ over $S \times \mathcal{R}(r,d) \times U_i$ given by the pullback of $\nabla(\mathcal{F})$ is preserved by the above gluing. Hence $\mathcal{P}(r,d)$ gets an partial connection, which is $\nabla(\text{par})$.

We shall now put a $PU(r)$ connection and a holomorphic structure on the projective bundle $\mathcal{P}(r,d)$ over $\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)$ constructed above.

Let

$$q : \mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d) \longrightarrow \mathcal{T}_g^1 \tag{3.13}$$

denote the obvious projection. For any $t \in \mathcal{T}_g^1$, let $(X, x_0) = (\pi^{-1}(t), \zeta(t))$ be the pointed Riemann surface over t for the family of pointed Riemann surfaces $(\mathcal{C}_g^1, \zeta)$ (defined in Section 2.1). We may restrict the pair $(\mathcal{P}(r,d), \nabla(\text{par}))$ to the complex submanifold $q^{-1}(t)$ of $\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)$; let $(\mathcal{P}(r,d)_t, \nabla(\text{par})_t)$ denote this restriction.

Recall Lemma 2.6 – the pair $(\mathcal{P}(r,d)_t, \nabla(\text{par})_t)$ is the universal projective bundle P in Lemma 2.6 equipped with the partial connection induced by the connection on P. Indeed, as we have seen earlier, this property of the partial connection $\nabla(\mathcal{F})$ uniquely fixes the projective bundle $P(S)$. Since both $\nabla(\text{par})_t$ (from the construction, $\nabla(\text{par})_t$ is same as $\nabla(\mathcal{F})$) and the partial connection on P, induced by the connection in Lemma 2.6, have the property of $\nabla(\mathcal{F})$, we get that the projective bundles $\mathcal{P}(r,d)_t$ and P are canonically isomorphic. Moreover, this isomorphism takes the partial connection $\nabla(\text{par})_t$ to the partial connection induced by the connection on P given by Lemma 2.6.

Using the above identification with P, we conclude that the bundle $\mathcal{P}(r,d)_t$ is equipped with $PU(r)$ connection, which we shall denote by ∇_t, and a holomorphic structure compatible with ∇_t.

There is a natural isomorphism between $\pi_1(S)$ and $\pi_1(\mathcal{C}_g^1)$. Indeed, $\pi_1(S)$ has an isomorphism with the fundamental group of any Riemann surface in the family $\mathcal{C}_g^1$, which in turn is isomorphic to $\pi_1(\mathcal{C}_g^1)$, with an isomorphism given by the long homotopy exact sequence for the fibration π.

Take a $\rho \in \mathcal{R}(r,d)$; using the above isomorphism, ρ gives

$$\bar{\rho} \in \text{Hom}^{\text{ir}}(\pi_1(\mathcal{C}_g^1), PU(r))/PU(r).$$

(Note that ρ gives an element of $\mathrm{Hom}^{\mathrm{ir}}(\pi_1(S), PU(r))/PU(r)$.) Let

$$\big(P(\mathcal{T}_g^1, \rho)\,,\ \nabla(\mathcal{T}_g^1, \rho)\big) \tag{3.14}$$

be the projective bundle on $\mathcal{C}_g^1$ equipped with flat $PU(r)$ connection given by $\bar{\rho}$. The flat connection $\nabla(\mathcal{T}_g^1, \rho)$ induces a holomorphic structure on $P(\mathcal{T}_g^1, \rho)$.

Let

$$\tau\,:\,\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d) \longrightarrow \mathcal{R}(r,d) \tag{3.15}$$

be the composition of the projection to $\mathcal{M}(r,d)$ with the projection of $\mathcal{M}(r,d) = \mathcal{T}_g^1 \times \mathcal{R}(r,d)$ to the second factor $\mathcal{R}(r,d)$.

Theorem 3.1 says that the fibers of the projection $\mathcal{M}(r,d) \longrightarrow \mathcal{R}(r,d)$ are complex submanifolds of $\mathcal{M}(r,d)$. So $\tau^{-1}(\rho)$ is a complex submanifold for any $\rho \in \mathcal{R}(r,d)$. Corollary 3.3 implies that the map

$$\mathcal{C}_g^1 \longrightarrow \tau^{-1}(\rho)$$

defined by $c \longmapsto (c, \rho)$ is a bi-holomorphism.

Let $(\mathcal{P}(r,d)^\rho, \nabla(\mathrm{par})^\rho)$ denote the restriction of $(\mathcal{P}(r,d), \nabla(\mathrm{par}))$ to $\tau^{-1}(\rho)$. Note that the partial connection $\nabla(\mathrm{par})^\rho$ is an actual connection on $\mathcal{P}(r,d)^\rho$. Clearly, the two projective bundles with flat connections on $\tau^{-1}(\rho)$, namely

$$\big(P(\mathcal{T}_g^1, \rho)\,,\ \nabla(\mathcal{T}_g^1, \rho)\big) \quad \text{and} \quad \big(\mathcal{P}(r,d)^\rho\,,\ \nabla(\mathrm{par})^\rho\big)$$

are canonically identified. (As before, the isomorphism is determined by the condition that the connection $\nabla(\mathcal{T}_g^1, \rho)$ is taken to $\nabla(\mathrm{par})^\rho$.) Using this identification, the bundle $\mathcal{P}(r,d)^\rho$ gets a holomorphic structure.

Take any $c \in \mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)$. Denote $q(c) \in \mathcal{T}_g^1$ and $\tau(c) \in \mathcal{R}(r,d)$ by t and ρ respectively, where q and τ are defined in (3.13) and (3.15) respectively. The intersection $q^{-1}(t) \cap \tau^{-1}(\rho)$ is a copy of the Riemann surface $\pi^{-1}(t)$. The restrictions of the two projective bundles with connections, $(\mathcal{P}(r,d)_t, \nabla_t)$ and $(\mathcal{P}(r,d)^\rho, \nabla(\mathrm{par})^\rho)$, to the Riemann surface $\pi^{-1}(t)$ are identified with the flat projective bundle on $\pi^{-1}(t)$ given by the representation $\rho \in \mathcal{R}(r,d)$. In particular, the above two restrictions coincide on $q^{-1}(t) \cap \tau^{-1}(\rho)$.

Clearly,

$$T_c(\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)) \,=\, T_t(q^{-1}(t)) \,+\, T_\rho(\tau^{-1}(\rho)).$$

(Note that this is not a direct sum.) We saw that the two projective bundles with $PU(r)$ connections, namely

$$(\mathcal{P}(r,d)_t, \nabla_t) \quad \text{and} \quad (\mathcal{P}(r,d)^\rho, \nabla(\mathrm{par})^\rho)$$

coincide on $q^{-1}(t) \cap \tau^{-1}(\rho)$. From this we conclude that there is a unique $PU(r)$ connection, $\nabla(P)$, on the projective bundle $\mathcal{P}(r,d)$ over $\mathcal{P}(r,d)$ such that the restriction of $\nabla(P)$ to any $q^{-1}(t)$ is ∇_t, and the restriction of $\nabla(P)$ to any $\tau^{-1}(\rho)$ is $\nabla(\mathrm{par})^\rho$.

LEMMA 3.16. — *The connection $\nabla(P)$ obtained above induces a holomorphic structure on the projective bundle $\mathcal{P}(r,d)$.*

Remark 3.17. — If the connection $\nabla(P)$ induces a holomorphic structure then the holomorphic structure is uniquely determined by the following property: for any $t \in \mathcal{T}_g^1$ and $\rho \in \mathcal{R}(r,d)$ the holomorphic structures on the restrictions, namely $\mathcal{P}(r,d)_t$ and $\mathcal{P}(r,d)^\rho$ respectively, are precisely the holomorphic structures obtained above.

Proof of Lemma 3.16. — Let

$$K(\nabla(P)) \in \Omega^2\big(\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)\,,\, Ad(\mathcal{P}(r,d))\big)$$

be the curvature of the connection $\nabla(P)$. Since a C^∞ connection on a complex manifold whose curvature is of type $(1,1)$ gives a holomorphic structure on the bundle [Ko, Ch. I, Proposition 3.7], to prove the lemma we have to show that $K(\nabla(P))$ is of the type $(1,1)$.

Let $\mathcal{C}_D \longrightarrow D$ be a holomorphic family of Riemann surfaces of genus g parameterized by the open disc D. Let s be a section for this family. Let E be a holomorphic vector bundle of rank r on $\mathcal{C}_D$ such that for any $t \in D$, the restriction E_t to the Riemann surface $\mathcal{C}_t$ is a stable bundle of degree d.

Using a theorem of Narasimhan and Seshadri, [NS], there is a reduction of the structure group of the projective bundle $P(E)$ to $PU(r)$. There is a unique connection on $Ad(E)$ (vector bundle of trace zero endomorphisms) compatible with both the holomorphic and the hermitian structures of $Ad(E)$; let $\nabla(D)$ denote this connection.

Fix a diffeomorphism, f, such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{C}_D & \xrightarrow{f} & S \times D \\ & \searrow \quad \swarrow & \\ & D & \end{array}$$

and $f^{-1}(s_0 \times D)$ is a complex submanifold of $\mathcal{C}_D$; for $t \in D$, let f_t denote the restriction of f to $\mathcal{C}_t$. Using f we have a holomorphic map

$$\bar{f} : D \longrightarrow \mathcal{T}_g^1$$

such that for any $t \in D$, the pair $(\mathcal{C}_t, s(t), f_t)$ is represented by $\bar{f}(t)$. Let f' denote the natural lift of $\bar{f}$ to a holomorphic map from $\mathcal{C}_D$ to $\mathcal{C}_g^1$. Let

$$\sigma_f : \mathcal{C}_D \longrightarrow \mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)$$

be the holomorphic map given by the family of bundles E.

It is easy to see that the pullback $\sigma_f^*(\mathcal{P}(r,d))$ is naturally isomorphic to $P(E)$, and the pullback connection $\sigma_f^*(\nabla(P))$ is $\nabla(D)$. But the curvature form of the connection $\nabla(D)$ is of the type $(1,1)$. Since σ_f is holomorphic, $\sigma_f^*(K(\nabla(P)))$ is of the type $(1,1)$ on $\mathcal{C}_D$.

Let $W \subset T_\alpha^{1,0}(\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d))$, where $\alpha := (t,m) \in \mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)$, be a subspace of rank two. Then either W can be realized as a subspace of $d(\sigma_f)(a)$, with $a \in \mathcal{C}_D$, where

$$d(\sigma_f) : T_a(\mathcal{C}_D) \otimes \mathbb{C} \longrightarrow T_\alpha(\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)) \otimes \mathbb{C}$$

is the differential, or W is a subspace of $T_m^{1,0}(\mathcal{M}(r,d))$. In the first case the contraction of $K(\nabla(P))(\alpha)$ with any element of $\bigwedge^2 W$ vanishes, since $\sigma_f^*(K(\nabla(P)))$ is of the type $(1,1)$. The curvature of the connection ∇_t on the bundle $\mathcal{P}(r,d)_t$ over $q^{-1}(t)$ (defined in (3.13)) is of type $(1,1)$. So, in the second case, also, the contraction of $K(\nabla(P))(\alpha)$ with any element of $\bigwedge^2 W$ vanishes. So the $(2,0)$-part of $K(\nabla(P))(\alpha)$ must be zero. Similarly it can shown that $(0,2)$-part of $K(\nabla(P))(\alpha)$ vanishes. This completes the proof of the lemma. □

We shall come back to $K(\nabla(P))$ in Section 5.

4. MAPPING CLASS GROUP ACTION

In this section we shall give actions of the mapping class group on $\mathcal{M}(r,d)$ and $\mathcal{P}(r,d)$.

4.1. Action on $\mathcal{M}(r,d)$.

We shall use the notation of Section 2.1.

The group $\mathrm{Diff}_0^+(S,s_0)$ is a normal subgroup of $\mathrm{Diff}^+(S,s_0)$. The quotient group

$$\mathcal{MCG}_g^1 := \mathrm{Diff}^+(S,s_0)/\mathrm{Diff}_0^+(S,s_0)$$

is known as the *mapping class group.*

The natural action of $\mathrm{Diff}^+(S,s_0)$ on $\mathrm{Com}(S)$ induces an action of $\mathcal{MCG}_g^1$ on $\mathcal{T}_g^1 := \mathrm{Com}(S)/\mathrm{Diff}_0^+(S,s_0)$. Let ρ_1 denote this action.

Any element $g \in \mathrm{Diff}^+(S,s_0)$ gives an automorphism of $\pi_1(S - s_0)$. So the diffeomorphism g gives an diffeomorphism of $\mathrm{Hom}^{\mathrm{ir}}(\pi_1(S-s_0),U(r))/U(r)$, which in turn induces a diffeomorphism of $\mathcal{R}(r,d)$. (This corresponds to taking the pullback of a local system using the diffeomorphism g^{-1}.) Clearly, for any $g \in \mathrm{Diff}_0^+(S,s_0)$, the induced diffeomorphism of $\mathcal{R}(r,d)$ is trivial. So we have an action of $\mathcal{MCG}_g^1$ on $\mathcal{R}(r,d)$, which we shall denote by ρ_2. Recall that as a real manifold, $\mathcal{M}(r,d) = \mathcal{T}_g^1 \times \mathcal{R}(r,d)$.

LEMMA 4.1. — *The action, $\rho_1 \times \rho_2$, of $\mathcal{MCG}_g^1$ on $\mathcal{M}(r,d)$ preserves both the complex structure and the $(1,1)$-form $p_2^*\Omega$.*

Proof. — Clearly the expression (2.3) is invariant under the action of any element of $\mathrm{Diff}^+(S,s_0)$. In other words, the form Ω on $\mathcal{R}(r,d)$ is invariant under the action ρ_2 of $\mathcal{MCG}_g^1$. This implies that the form $p_2^*\Omega$ is also invariant under the action $\rho_1 \times \rho_2$.

Now from the description of the complex structure on $\mathcal{M}(r,d)$ given in Theorem 3.1 and Corollary 3.3, we conclude that in order to prove that the action of $\mathcal{MCG}_g^1$ on $\mathcal{M}(r,d)$ preserves the complex structure of $\mathcal{M}(r,d)$, we must prove the following two statements:

(1) For any $g \in \mathcal{MCG}_g^1$ and $\nu \in \mathcal{R}(r,d)$, the translation $(\rho_1 \times \rho_2)(g)(\mathcal{T}_g^1,\nu)$ of the complex submanifold $(\mathcal{T}_g^1,\nu) \subset \mathcal{M}(r,d)$ is also a complex submanifold, with the translation map being holomorphic (on $(\mathcal{T}_g^1,\nu)$).

(2) For any $g \in \mathcal{MCG}_g^1$ and $t \in \mathcal{T}_g^1$, the translation $(\rho_1 \times \rho_2)(g)(t,\mathcal{R}(r,d))$ is a complex submanifold, with the translation map being holomorphic (on $(t,\mathcal{R}(r,d))$).

For a proof of the first statement note that $(\rho_1 \times \rho_2)(g)(\mathcal{T}_g^1,\nu) = (\mathcal{T}_g^1, \rho_2(g)\nu)$. Hence the statement (1) follows from Theorem 3.1 and the fact that the diffeomorphism of $\mathcal{T}_g^1$ given by the action of $g \in \mathcal{MCG}_g^1$ is a holomorphic automorphism.

For the second statement, from Corollary 3.3 we conclude that

$$(\rho_1 \times \rho_2)(g)(t, \mathcal{R}(r,d)) = (\rho_1(g)(t), \mathcal{R}(r,d))$$

is a complex submanifold of $\mathcal{M}(r,d)$. The Riemann surfaces given by $t \in \mathcal{T}_g^1$ and $\rho_1(g)(t)$ are isomorphic. The translation map on $(t, \mathcal{R}(r,d))$ is induced by a holomorphic automorphism (possibly identity map) of the Riemann surface given by t. Hence the translation map is holomorphic. This completes the proof of the lemma. □

The quotient $\mathcal{M}_g^1 := \mathcal{T}_g^1/\mathcal{MCG}_g^1$ is the moduli of Riemann surfaces of genus g with one marked point. Let

$$\xi : \mathcal{T}_g^1 \longrightarrow \mathcal{M}_g^1$$

denote the quotient map.

The obvious actions of the group of $\mathrm{Diff}^+(S, s_0)$ on S and $\mathrm{Com}(S)$ respectively, combine together to induce an action of $\mathcal{MCG}_g^1$ on the universal Riemann surface $\mathcal{C}_g^1$. It is easy to see that for any $g \in \mathcal{MCG}_g^1$, the diffeomorphism of $\mathcal{C}_g^1$, given by the action of g, is actually a holomorphic automorphism. Clearly the projection, π, of $\mathcal{C}_g^1$ to $\mathcal{T}_g^1$, is equivariant for the actions of $\mathcal{MCG}_g^1$.

For $g \geqslant 3$, the generic Riemann surface of genus g does not admit any nontrivial automorphism. For $g = 2$, the only nontrivial automorphism of the generic Riemann surface is the hyperelliptic involution.

Let $\mathcal{M}^0 \subset \mathcal{M}_g^1$ be the Zariski open set consisting of pairs (X, x_0) such that X does not have any nontrivial automorphism fixing x_0. From the above remark it follows that the set $\mathcal{M}^0$ is non-empty. The subset $\xi^{-1}(\mathcal{M}^0) \subset \mathcal{T}_g^1$ is obviously invariant under the action of $\mathcal{MCG}_g^1$. In fact, the action of $\mathcal{MCG}_g^1$ on $\xi^{-1}(\mathcal{M}^0)$ is free. The quotient

$$\mathcal{C}^0 := (\xi \circ \pi)^{-1}(\mathcal{M}^0)/\mathcal{MCG}_g^1$$

is the universal curve over $\mathcal{M}^0$.

The diagonal action of $\mathcal{MCG}_g^1$ on $\xi^{-1}(\mathcal{M}^0) \times \mathcal{R}(r,d)$ is free, since it is free on $\xi^{-1}(\mathcal{M}^0)$. The projection from the quotient manifold, namely

$$\Lambda : \mathcal{M}^0(r,d) := \big(\xi^{-1}(\mathcal{M}^0) \times \mathcal{R}(r,d)\big)/\mathcal{MCG}_g^1 \longrightarrow \mathcal{M}^0 \tag{4.2}$$

gives the universal moduli space for the universal curve $\mathcal{C}^0$ over $\mathcal{M}^0$.

It will be interesting to give an algebraic structure on the complex manifold $\mathcal{M}^0(r,d)$ such that the map Λ becomes an algebraic morphism.

4.2. Action on $\mathcal{P}(r,d)$.

The actions of $\mathcal{MCG}_g^1$ on $\mathcal{C}_g^1$ and $\mathcal{M}(r,d)$ together induce an action on the fiber product $\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)$; let ρ_3 denote this action.

Recall the construction of $\mathcal{P}(r,d)$ in (3.11). The group $\mathcal{MCG}_g^1$ acts naturally on all the three factors in (3.10), namely S, $\mathcal{R}(r,d)$ and $\mathrm{Com}(S)$. Using the uniqueness of $(P(S), \nabla(\mathcal{F}))$, the projective bundle equipped with connection, the action of $\mathcal{MCG}_g^1$ on $S \times \mathcal{R}(r,d) \times \mathrm{Com}(S)$ induces an action of $\mathcal{MCG}_g^1$ on the total space of $\mathcal{P}(r,d)$. This action on $\mathcal{P}(r,d)$ has the property that the projection of $\mathcal{P}(r,d)$ to $\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)$ is equivariant for the actions of $\mathcal{MCG}_g^1$.

Let ρ_4 denote the action of $\mathcal{MCG}_g^1$ on $\mathcal{P}(r,d)$ obtained this way.

LEMMA 4.3. — *The action, ρ_4, of $\mathcal{MCG}_g^1$ on $\mathcal{P}(r,d)$ preserves the $PU(r)$ connection $\nabla(P)$.*

Proof. — We need to establish the following two statements to prove the lemma:

(1) For any $t \in \mathcal{T}_g^1$, the identification of $\mathcal{P}(r,d)_t$ with $\mathcal{P}(r,d)_{\rho_1(t)}$, given by the action of $g \in \mathcal{MCG}_g^1$ on $\mathcal{P}(r,d)$, takes the connection ∇_t on $\mathcal{P}(r,d)_t$ to the connection $\nabla_{\rho_1(t)}$ on $\mathcal{P}(r,d)_{\rho_1(t)}$.

(2) For any $\nu \in \mathcal{R}(r,d)$, the identification between $\mathcal{P}(r,d)^{\nu}$ and $\mathcal{P}(r,d)^{\rho_2(\nu)}$ given by the action of $g \in \mathcal{MCG}_g^1$, takes the connection $\nabla(\mathrm{par})^{\nu}$ to the connection $\nabla(\mathrm{par})^{\rho_2(\nu)}$.

The Riemann surfaces corresponding to $t \in \mathcal{T}_g^1$ and $\rho_1(t)$ are isomorphic. The connection ∇_t is the connection obtained in Lemma 2.6. The uniqueness of this connection implies that — since the two Riemann surfaces are isomorphic — the connection ∇_t must be identified with $\nabla_{\rho_1(t)}$.

The connection $\nabla(\mathrm{par})^{\nu}$ on $\mathcal{P}(r,d)^{\nu}$ is given by the connection $\nabla(\mathcal{T}_g^1, \nu)$ (defined in (3.14)) using the identification of $\mathcal{P}(r,d)^{\nu}$ with $P(\mathcal{T}_g^1, \nu)$. It is easy to see that $\nabla(\mathcal{T}_g^1, \nu)$ is identified with $\nabla(\mathcal{T}_g^1, \rho_2(\nu))$ by the action of $g \in \mathcal{MCG}_g^1$. □

Taking the quotient by $\mathcal{MCG}_g^1$ of the restriction of $\mathcal{P}(r,d)$ to the open set

$$(\xi \circ q)^{-1}(\mathcal{M}^0) \subset \mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{M}(r,d)$$

(q defined in (3.13)) we get the universal projective bundle, $\mathcal{P}^0(r,d)$, over the fiber product $\mathcal{C}^0 \times_{\mathcal{M}^0} \mathcal{M}^0(r,d)$. Note that $\mathcal{P}^0(r,d)$ has a hermitian connection induced by $\nabla(P)$ using Lemma 4.3. The Lemma 3.16 implies that this connection induces a holomorphic structure on $\mathcal{P}^0(r,d)$.

5. THE DETERMINANT LINE BUNDLE

Let $\mathcal{N}(r,d) \subset \mathcal{M}(r,d)$ be the submanifold consisting of triplets of the form (X, x_0, E) such that

$$\overset{r}{\wedge} E = \mathcal{O}_X(d.x_0). \tag{5.1}$$

From the description of the complex structure of $\mathcal{M}(r,d)$ in Corollary 3.3 it is obvious that $\mathcal{N}(r,d)$ is a complex submanifold of $\mathcal{M}(r,d)$.

Let $\mathcal{R}_{SU} \subset \mathcal{R}(r,d)$ be the space of all $SU(r)$ representations of $\pi_1(S - s_0)$. Note that the holonomy around s_0 for a connection in $\mathcal{R}(r,d)$, namely $2\pi d/r.I_{r\times r}$, is in $SU(r)$.

In the above notation,

$$\mathcal{N}(r,d) = \mathcal{T}_g^1 \times \mathcal{R}_{SU}$$

in the identification $\mathcal{M}(r,d) = \mathcal{T}_g^1 \times \mathcal{R}(r,d)$.

We may restrict the $PGL(r)$ bundle with connection $(\mathcal{P}(r,d), \nabla(P))$ to $\mathcal{N}(r,d)$; this restriction is also denoted by $(\mathcal{P}(r,d), \nabla(P))$.

Let $Ad(\mathcal{P}(r,d))$ denote the adjoint bundle of $\mathcal{P}(r,d)$, i.e., the vector bundle associated to the adjoint action of $PGL(r)$ on its Lie algebra, namely $sl(r,\mathbb{C})$. Note that for any $t \in \mathcal{T}_g^1$, the restriction of $Ad(\mathcal{P}(r,d))$ to $q^{-1}(t)$ is the bundle $\mathcal{E}^0$ defined in Section 2.2.

Lemma 3.16 implies that the connection $\nabla(P)$ induces a holomorphic structure on $Ad(\mathcal{P}(r,d))$.

Following (2.8), define the holomorphic line bundle on $\mathcal{N}(r,d)$ given by the top exterior product of the first direct image

$$\Theta_{\mathcal{T}} := \overset{\text{top}}{\wedge} R^1 q_{2*} Ad(\mathcal{P}(r,d)) \tag{5.2}$$

where q_2 is the projection of $\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{N}(r,d)$ onto $\mathcal{N}(r,d)$. Note that $\Theta_{\mathcal{T}}$ is the relative anti-canonical bundle on $\mathcal{N}(r,d)$.

Since any endomorphism of a stable bundle is a multiplication by some scalar, the 0-th direct image $R^0 q_{2*} Ad(\mathcal{P}(r,d))$ is zero. So the line bundle $\Theta_{\mathcal{T}}$ is the dual of the determinant bundle for $Ad(\mathcal{P}(r,d))$ in the sense of [BGS1].

On a Riemann surface of genus $g \geqslant 2$ there is a unique Kähler metric of constant curvature (-1), which is known as the *Poincaré metric*. For a holomorphic family of Riemann surfaces, the Poincaré metric on each fiber patches up smoothly to give a smooth hermitian metric on the relative tangent bundle. So, in particular, we have a hermitian metric on the relative tangent bundle of the family of Riemann surfaces given by the map q_2.

The family given by q_2 is locally Kähler in the sense of [BGS1] (defined in page 50 there). Indeed, this family is the pullback of the family given by π (defined in (2.1)) using map projection q (defined in (3.13)). Since pullback of a locally Kähler family by a holomorphic map is clearly locally Kähler, we need to show that the family given by π is locally Kähler. However, the universal cover of $\mathcal{C}_g^1$ is $\mathcal{T}_g^2$; and on $\mathcal{T}_g^2$ there is a natural Weil-Petersson form which is invariant under the deck transformations. So $\mathcal{C}_g^1$ is a Kähler manifold. This implies that the family given by π is locally Kähler.

We note that from the construction in Section 1(d) of [BGS3], there is a hermitian metric on $\Theta_{\mathcal{T}}$; we shall denote this hermitian metric by H_Q.

We want to calculate the curvature of the hermitian connection for the hermitian metric H_Q.

In (2.2) we defined the Weil-Petersson form ω_{wp} (which is a Kähler form) on the Teichmüller space $\mathcal{T}$; and prior to that, we noted that there is a natural projection of $\mathcal{T}_g^1$ to $\mathcal{T}$. Let γ denote this projection. Let σ denote the obvious projection of $\mathcal{N}(r,d)$ onto $\mathcal{T}_g^1$. So

$$\bar{\omega}_{wp} := (\gamma \circ \sigma)^* \omega_{wp} \tag{5.3}$$

is a positive semi-definite closed $(1,1)$-form on $\mathcal{T}_g^1$.

Let $K(H_Q)$ denote the curvature of the hermitian connection on $\Theta_{\mathcal{T}}$ for the hermitian metric H_Q.

THEOREM 5.4. — *The curvature $(1,1)$-form $K(H_Q)$ on $\mathcal{N}(r,d)$ coincides with the following form*

$$\Phi := 4\pi r\sqrt{-1}.p_2^*\Omega + \frac{(r^2-1)\sqrt{-1}}{6\pi}.\bar{\omega}_{wp}$$

where $p_2^*\Omega$ and $\bar{\omega}_{wp}$ are as in Lemma 3.7 and (5.3) respectively. (We denote the restriction of $p_2^*\Omega$ on $\mathcal{M}(r,d)$ to the submanifold $\mathcal{N}(r,d)$, also, by $p_2^*\Omega$.)

Proof. — Recall that $\mathcal{N}(r,d) = \mathcal{T}_g^1 \times \mathcal{R}_{SU}$. For $t := (X, x_0, f) \in \mathcal{T}_g^1$, the inverse image $\sigma^{-1}(t)$ is the moduli space $N(r,d)$ defined in (2.7).

Clearly the restriction of the form $\bar{\omega}_{wp}$ to $\sigma^{-1}(t)$ is zero. The construction of the determinant bundle with hermitian structure in [BGS3] is compatible with respect to base change. So from Proposition 2.9 we obtain that the restrictions of the two $(1,1)$-forms, namely $K(H_Q)$ and Φ, to $\sigma^{-1}(t)$ coincide.

Take any $\rho \in \mathcal{R}_{SU}$. Recall the construction of the connection $\nabla(P)$ on $\mathcal{P}(r,d)$. Over the complex submanifold

$$\mathcal{C}_g^1 \times \rho \subset \mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{N}(r,d)$$

the connection $\nabla(P)$ is the flat unitary connection induced by ρ.

Let $Ad(\mathcal{P}(r,d))_\rho$ denote the restriction of $\mathcal{P}(r,d)$ to $\mathcal{C}_g^1 \times \rho$. Let $\nabla(P)_\rho$ denote the restriction of $\nabla(P)$ to $Ad(\mathcal{P}(r,d))_\rho$.

Recall the Chern-Weil construction of Chern forms from a connection. Since $\nabla(P)_\rho$ is flat, all Chern forms of degree more than one vanish identically. So the Chern character form of $\nabla(P)_\rho$ is simply the rank of $Ad(\mathcal{P}(r,d))$, which is (r^2-1).

We noted earlier that family of Riemann surfaces given by π is locally Kähler. So we can apply Theorem 0.1 of [BGS1] (page 51) to the family of Riemann surfaces $\mathcal{C}_g^1$ over $\mathcal{T}_g^1$, equipped with the Poincaré metric on the relative tangent bundle, and the hermitian bundle with connection $(\nabla(P)_\rho, Ad(\mathcal{P}(r,d))_\rho)$ over $\mathcal{C}_g^1$.

In this situation, the curvature of the determinant bundle of $Ad\,(\mathcal{P}(r,d))_\rho$, equipped with the connection $\nabla(P)_\rho$, coincides with (r^2-1)-times the curvature of the determinant bundle for the trivial bundle over $\mathcal{C}_g^1$ equipped with the trivial metric. Indeed, the Chern character form of the trivial bundle with the trivial metric is 1. We noted that the Chern character form of $\nabla(P)_\rho$ is (r^2-1). So for these two different situations, namely $\nabla(P)_\rho$ and the trivial connection, the expression (0.3) in Theorem 0.1 of [BGS1] differ by the (multiplicative) factor (r^2-1).

Obviously the restriction of the differential form $p_2^*\Omega$ (as in (5.4)) to $\mathcal{T}_g^1 \times \rho \subset \mathcal{T}_g^1 \times \mathcal{R}_{SU}$ vanishes. So, in order to prove that the restrictions of the two $(1,1)$-forms, namely $K(H_Q)$ and Φ, coincide on $\mathcal{T}_g^1 \times \rho$, we must show that the curvature of the determinant line bundle on $\mathcal{T}_g^1$ for the trivial bundle on $\mathcal{C}_g^1$, equipped with the trivial metric, is

$$\frac{\sqrt{-1}}{6\pi}.\gamma^*(\omega_{wp}) \tag{5.5}$$

where γ as in (5.3).

However, this follows from Theorem 2 of [ZT] and the compatibility of the hermitian structure of the determinant bundle with respect to taking pullbacks. In fact, the earlier statement on the determinant of the trivial line bundle is proved in § 4.2 (page 184) of [ZT]. Note that there is a difference in sign between [ZT] and (5.5) above. This is because the determinant bundle that we defined is dual of the determinant bundle in [ZT] or [BGS1]. (The curvature form of the induced connection on the dual line bundle is (-1)-times the curvature form of the original connection. We choose the dual, since Θ' in (2.8) is ample.)

We note that the above mentioned result of [ZT] can also be found in [FS] where a very general theorem has been proved which is valid for the moduli space of any dimensional non-uniruled Kähler manifolds.

Take any $\alpha := (t,\rho) \in \mathcal{T}_g^1 \times \mathcal{R}_{SU} = \mathcal{N}(r,d)$. Let $v \in T_t^{\mathbb{C}}\mathcal{T}_g^1$ (resp. $w \in T_\rho^{\mathbb{C}}\mathcal{R}_{SU}$) be a (complex) tangent vector at t (resp. ρ). Let

$$\tilde{v} \in T_\alpha^{\mathbb{C}}\mathcal{N}(r,d) \qquad (\text{resp.}\, \tilde{w} \in T_\alpha^{\mathbb{C}}\mathcal{N}(r,d))$$

be the tangent vector at $\alpha \in \mathcal{N}(r,d)$ given by v (resp. w).

In order to complete the proof of the Theorem 5.4 we must show that

$$K(H_Q)(\alpha)(\tilde{v},\tilde{w}) = 0. \tag{5.6}$$

We apply Theorem 0.1 of [BGS1] to the family of Riemann surfaces

$$q_2 : \mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{N}(r,d) \longrightarrow \mathcal{N}(r,d)$$

equipped with Poincaré metric and the hermitian bundle $Ad(\mathcal{P}(r,d))$.

Let Td^j be the component of $Td(-R^Z/2\pi\sqrt{-1})$ (in the expression 0.3 of Theorem 0.1 of [BGS1]) of degree $2j$ (i.e., the component form of type (j,j)). Similarly, let Ch^j denote the component of $\exp(-L^\xi/2\pi\sqrt{-1})$ (in the expression 0.3 of Theorem 0.1 of [BGS1]) of type (j,j).

We shall denote the integration along fiber (the Gysin map) by q_{2*}.

So, Theorem 0.1 of [BGS1] applied to our situation says that

$$K(H_Q) = -2\pi\sqrt{-1}.q_{2*}\big(Td^2Ch^0 + Td^1Ch^1 + Td^0Ch^2\big). \tag{5.7}$$

Now, $Ad(\mathcal{P}(r,d))$ being an adjoint bundle, $Ch^1 = 0$; also $Td^0 = 1$ and $Ch^0 = (r^2 - 1)$.

The family of Riemann surface given by q_2 along with the Poincaré metric on the relative tangent bundle is the pullback of the family given by π (defined in (2.1)). Hence, the form Td^2 on $\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{N}(r,d)$ is the pullback of a form on $\mathcal{C}_g^1$ using the obvious projection. So the contraction of $q_{2*}Td^2$ with $\widetilde{\omega}$ vanishes.

Now from the definition of the connection $\nabla(P)$ we have the following: the restriction of the connection $\nabla(P)$ on $\mathcal{C}_g^1 \times_{\mathcal{T}_g^1} \mathcal{N}(r,d)$ to the submanifold $\mathcal{C}_g^1 \times \rho'$, where $\rho' \in \mathcal{R}_{\mathcal{SU}}$, is actually a flat connection. This implies that at each point of $\mathcal{N}(r,d)$ the form $q_{2*}Ch^2$ is a pullback of a form on $\mathcal{R}_{\mathcal{SU}}$. Thus the contraction of $q_{2*}Ch^2$ with $\widetilde{v}$ vanishes.

These observations, along with the expression (5.7) together imply the equality (5.6), and the proof of Theorem 5.4 is completed. □

In Lemma 3.7 we proved that the form $p_2^*\Omega$ is positive semi-definite. The form $\bar{\omega}_{wp}$, being a pullback of a Kähler form by a holomorphic map, is clearly positive semi-definite. Hence the first Chern form of the bundle $Ad(\mathcal{P}(r,d))$ for the curvature $K(H_Q)$ is a positive semi-definite form.

It is easy to see that the annihilator of the form $K(H_Q)$ is precisely the (complex) rank one subbundle of the tangent bundle of $\mathcal{N}(r,d)$ given by the relative tangent bundle of the projection

$$\gamma : \mathcal{T}_g^1 \longrightarrow \mathcal{T}.$$

Indeed, ω_{wp} being a Kähler form, the relative tangent bundle of γ is clearly the annihilator of the form $\gamma^*\omega_{wp}$. Now the nondegeneracy of Ω on $\mathcal{R}_{\mathcal{SU}}$ implies that the annihilator of $K(H_Q)$ is precisely the subbundle of $T^{\mathbb{C}}\mathcal{T}_g^1$ given by the relative tangent bundle for γ. (See the remark following the proof of Lemma 3.7.)

In Lemma 4.3 we saw that the action of the mapping class group, $\mathcal{MCG}_g^1$, on $\mathcal{P}(r,d)$ preserves the connection $\nabla(P)$. The action of $\mathcal{MCG}_g^1$ on $\mathcal{P}(r,d)$ induces an action of $\mathcal{MCG}_g^1$ on the bundle $\Theta_{\mathcal{T}}$, which preserves the holomorphic hermitian structure of $\Theta_{\mathcal{T}}$.

Recall $\mathcal{M}^0(r,d)$ defined in (4.2). Let $\mathcal{N}^0(r,d) \subset \mathcal{M}^0(r,d)$ be the complex submanifold given by the image (under the quotient map) of

$\mathcal{N}(r,d)$ defined in (5.1). Taking the quotient for the action of $\mathcal{MCG}_g^1$ on $\Theta_{\mathcal{T}}$, we get a holomorphic hermitian line bundle over $\mathcal{N}^0(r,d)$. Since $K(H_Q)$ is positive semi-definite, the Chern form of this determinant bundle over the universal moduli space $\mathcal{N}^0(r,d)$ for the universal curve $\mathcal{C}^0$, is positive semi-definite.

BIBLIOGRAPHY

[AB] M.F. ATIYAH, R. BOTT, The Yang-Mills equations over Riemann surfaces, Phil. Tran. Roy. Soc. Lond., A308 (1982), 523-615.

[BGS1] J.M. BISMUT, H. GILLET, C. Soulé, Analytic torsion and holomorphic determinant bundles I. Bott-Chern forms and analytic torsion, Commun. Math. Phy., 115 (1988), 49-78.

[BGS2] J.M. BISMUT, H. GILLET, C. SOULÉ, Analytic torsion and holomorphic determinant bundles II. Direct images and Bott-Chern forms, Commun. Math. Phy., 115 (1988), 79-126.

[BGS3] J.M. BISMUT, H. GILLET, C. SOULÉ, Analytic torsion and holomorphic determinant bundles III. Quillen metrices on holomorphic determinants, Commun. Math. Phy., 115 (1988) 301-351.

[BR] I. BISWAS, N. RAGHAVENDRA, Determinants of parabolic bundles on Riemann surface, Proc. Indian Acad. Sci., 103 (1993), 41-71.

[FS] A. FUJIKI, G. SCHUMACHER, The moduli space of extremal compact Kähler manifolds and generalized Weil-Petersson metric, Publ. R.I.M.S. Kyoto Univ., 26 (1990), 101-183.

[G] W. GOLDMAN, The symplectic nature of fundamental groups of surfaces, Adv. Math., 54 (1984), 200-225.

[K] N.M. KATZ, An overview of Deligne's work on Hilbert's twenty-first problem, Proc. Symp. Pure Math., 28 (1976), 537-557.

[KM] F. KNUDSEN, D. MUMFORD, The projectivity of the moduli space of stable curves I. Preliminaries on "det" and "div", Math. Scand., 39 (1976), 19-55.

[Ko] S. KOBAYASHI, Differential geometry of complex vector bundles, Publications of the Math. Soc. of Japan, Iwanami Schoten Pub. and Princeton Univ. Press (1987).

[KT] F. KAMBER, P. TONDEUR, Foliated bundles and characteristic classes, Lec. Notes in Math. 493, Springer-Verlag, Berlin-Heidelberg-New York, (1975).

[MS] V. MEHTA, C. SESHADRI, Moduli of vector bundles on curves with parabolic structures, Math. Ann., 248 (1980), 205-239.

[NS] M.S. NARASIMHAN, C. SESHADRI, Stable and unitary vector bundles on a compact Riemann surface, Ann. Math., 82 (1965), 540-567.

[Q] D. QUILLEN, Determinants of Cauchy-Riemann operators over a Riemann surface, Funct. An. Appl., 19 (1985), 31-34.

[S1] G. SCHUMACHER, Moduli of polarized Kähler manifolds, Math. Ann., 269 (1984), 137-144.

[ST] G. Schumacher, M. Toma, Moduli of Kähler manifolds equipped with Hermite-Einstein vector bundles, Rev. Roumaine Math. Pures Appl., 38 (1993), 703-719.

[ZT] P. Zograf, L. Takhtadzhyan, A local index theorem for families of $\overline{\partial}$-operators on Riemann surfaces, Russian Math. Surveys, 42–6 (1987), 169-190.

Manuscrit reçu le 24 avril 1996,
révisé le 2 septembre 1996,
accepté le 21 février 1997.

Indranil BISWAS,
Tata Institute of Fundamental Research
School of Mathematics
Homi Bhabha Road
Bombay 400005 (Inde).
indranil@@math.tifr.res.in