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FLAT HOLOMORPHIC CONNECTIONS ON PRINCIPAL BUNDLES OVER A PROJECTIVE MANIFOLD

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Abstract. Let G be a connected complex linear algebraic group and $R_u(G)$ its unipotent radical. A principal G–bundle E_G over a projective manifold M will be called polystable if the associated principal $G/R_u(G)$–bundle is so. A G–bundle E_G over M is polystable with vanishing characteristic classes of degrees one and two if and only if E_G admits a flat holomorphic connection with the property that the image in $G/R_u(G)$ of the monodromy of the connection is contained in a maximal compact subgroup of $G/R_u(G)$.

1. Introduction

Let E be a holomorphic vector bundle over a compact connected Riemann surface M. A result due to Weil says that E admits a flat connection if and only if each direct summand of E is of degree zero [We], [At, p. 203, Theorem 10]. This criterion can be extended to holomorphic principal G–bundles over M, where G is a connected reductive linear algebraic group over $\mathbb{C}$ [AB]. The present work started by trying to find a criterion for the existence of a flat connection on principal bundles whose structure group is not reductive.

Let P be a connected complex linear algebraic group with $R_u(P)$ its unipotent radical. So then $L(P) := P/R_u(P)$ is the Levi group, which is reductive.

Given a holomorphic principal P–bundle E_P over a compact connected Riemann surface M, it is natural to conjecture that E_P admits a flat connection if and only if the corresponding principal $L(P)$–bundle $E_{L(P)}$, obtained by extending the structure group of E_P using the projection of P to $L(P)$, admits a flat connection. The simplest situation would be the following. Let F be a holomorphic subbundle of a holomorphic vector bundle E over M such that both F and E/F admit flat connections. Does this imply that E admits a flat connection that preserves the subbundle F? The answer is not known even in this special case.

On the other hand, there is a very rich theory of special connections on a fairly general class of bundles (both vector and principal).

In [NS] it was proved that a topologically trivial holomorphic vector bundle E over a compact connected Riemann surface admits a unitary flat connection if and only if E is polystable. In [Do1] it was proved by Donaldson that a vector bundle E over a smooth projective surface equipped with a Kähler metric admits a Hermitian–Yang–Mills connection if and only if E is polystable. In [Do2] this was

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extended to vector bundles over projective manifolds of arbitrary dimension. Uhlenbeck and Yau in [UY] proved a more general result; they proved the existence of a Hermitian–Yang–Mills connection for polystable vector bundles over an arbitrary compact Kähler manifold. In particular, a vector bundle E over a projective manifold, or more generally over a compact Kähler manifold, admits a unitary flat connection if and only if E is polystable and the first and second rational Chern classes of E vanish.

Let G be a connected reductive linear algebraic group over $\mathbb{C}$. The notion of polystability of a G–bundle was introduced in [Ra1]. In [RS] it was proved that a principal G–bundle E_G over a projective manifold M admits a Hermitian–Yang–Mills connection if and only if E_G is polystable. When M is a Riemann surface was proved in [Ra1].

So, in particular, a principal G–bundle E_G over a projective manifold admits a flat holomorphic connection with monodromy contained in a maximal compact subgroup of G if and only if E_G is polystable and all the (rational) characteristic classes of E_G of degree one and degree two vanish.

Here we consider an arbitrary linear algebraic group P over $\mathbb{C}$, not necessarily reductive. As before, $L(P) = P/R_u(P)$ is the maximal reductive quotient of P.

Let M be a connected projective manifold equipped with a fixed polarization. A principal P–bundle E_P over M will be called polystable if the associated principal $L(P)$–bundle $E_{L(P)} = (E_P \times L(P))/P$ over M is polystable. A Geometric Invariant Theoretic justification of this definition can be found in [Dr].

Let E_P be a polystable principal P–bundle over M with vanishing rational characteristic classes of both degrees one and two. We prove that E_P admits a flat holomorphic connection ∇ with the property that the image of the monodromy of ∇ in $L(P)$, by the natural projection of P to $L(P)$, is contained in a compact subgroup of $L(P)$ (Theorem 3.1).

The converse is also true. If E_P admits a flat holomorphic connection whose monodromy has the above property, then E_P is polystable and all the (rational) characteristic classes of E_P of positive degree vanish.

Theorem 3.1 was proved in [Si] for the case where P is a parabolic subgroup of $\mathrm{GL}(n, \mathbb{C})$. What we do here is to derive Theorem 3.1 for general P using [Si]. Although the construction of the Hermitian–Yang–Mills connection done in [Do2], [UY] and [RS] do not need any assumption on the Chern classes, the construction of connection done in [Si] for parabolic subgroups of $\mathrm{GL}(n, \mathbb{C})$ crucially uses the assumption that the first and the second Chern classes of the vector bundle vanish.

The Hermitian–Yang–Mills connection on a polystable vector (or principal) bundle constructed in [Do2], [UY], [RS] is unique. A polystable vector (or principal) bundle admits exactly one such connection. Since we quotient P by its unipotent radical, it may happen that E_P has two distinct flat connections sharing the property that the image of the monodromy in $L(P)$ lies in a compact subgroup. In other words, the uniqueness statement fails in Theorem 3.1. See Remark 3.7 for the details.

Theorem 3.1 implies that the conjecture stated in the beginning is true if $E_{L(P)}$ is polystable. In other words, if E_P is a principal P–bundle over a compact connected Riemann surface such that the corresponding principal $L(P)$–bundle $E_{L(P)}$ is polystable and admits a flat holomorphic connection, then E_P itself admits a flat holomorphic connection.

As a corollary of Theorem 3.1, any principal P–bundle over a projective manifold M, where P is unipotent, admits a flat holomorphic connection (Corollary 3.8).

For P solvable, a principal P–bundle E_P over M admits a flat holomorphic connection if and only if for every character χ of P the associated line bundle $(E_P \times \mathbb{C})/P$ over M is of degree zero (Corollary 3.9).

2. Preliminaries

Let M be a connected smooth projective variety over $\mathbb{C}$. Fix an ample line bundle ζ over M. The degree of a coherent sheaf F on M is defined to be

$$\deg(F) := \int_M c_1(F) c_1(\zeta)^{d-1} \in \mathbb{Z},$$

where $d = \dim M$.

Let G be a connected linear algebraic group over $\mathbb{C}$. An algebraic *principal G–bundle* over M is a smooth complex variety E_G equipped with a right action

$$\psi : E_G \times G \longrightarrow E_G$$

of G, and a surjective affine submersion

$$\phi : E_G \longrightarrow M$$

such that

(1) $\phi \circ \psi$ coincides with $\phi \circ p_1$, where p_1 is the natural projection of $E_G \times G$ to E_G;
(2) the map to the fiber product over M

$$(\psi\,, p_1) : E_G \times G \longrightarrow E_G \times_M E_G$$

is an isomorphism.

Note that the first condition implies that ϕ is G–equivariant (with the trivial action on M), and the second condition implies that G acts freely transitively on the fibers of ϕ.

Let E_G be a principal G–bundle over M. All principal bundles, with an algebraic group as the structure group, considered here will be algebraic.

For a closed algebraic subgroup G' of G and a nonempty Zariski open subset $U \subseteq M$, a *reduction of structure group* over U of E_G to G' is defined by giving an algebraic section of E_G/G' over U. If σ is such a section and

$$q : E_G \longrightarrow E_G/G'$$

is the quotient map, then $q^{-1}(\text{image}(\sigma))$ is clearly a principal G'–bundle over U. Conversely, all G'–bundles $E_{G'} \subset E_G|_U$ are obtained this way.

Let the above considered group G be reductive. A *parabolic subgroup* of G is a Zariski closed algebraic proper subgroup $P \subset G$ such that the quotient G/P is complete. Let

$$L(P) := P/R_u(P)$$

be the *Levi factor* of P, where $R_u(P)$ is the unipotent radical. We recall that $R_u(P)$ is the largest connected normal unipotent subgroup of P [Hu, p. 125]. If we fix a maximal torus T of G contained in P, then the projection of P to $L(P)$ has a canonical splitting, that is, P is the semi–direct product of $L(P)$ and $R_u(P)$. Indeed, after fixing T the Levi factor, $L(P)$ is identified with the maximal reductive subgroup of P invariant under the adjoint action of T on P.

We will briefly recall the definition of (semi)stability of a principal G–bundle over M (see [Ra2], [RS] for the details).

Let E_G be a principal G–bundle over M and let $\sigma : U \longrightarrow (E_G/P)|_U$ be a reduction of structure group of E_G to a parabolic subgroup $P \subset G$ over a nonempty Zariski open subset U of M. Note that if the complement $M \setminus U$ is of codimension at least two, then the direct image $\iota_* F$ is a coherent sheaf on M, where F is any coherent sheaf on U and ι is the inclusion map of U in M.

The principal G–bundle E_G is called *stable* (respectively, *semistable*) if for any reduction of structure group of E_G to any maximal parabolic subgroup P over a Zariski open subset U, with $\mathrm{codim}(M \setminus U) \geq 2$, the inequality

$$\deg(\iota_* \sigma^* T_{\mathrm{rel}}) > 0$$

(respectively, $\deg(\iota_* \sigma^* T_{\mathrm{rel}}) \geq 0$) is valid; here T_{rel} denotes the relative tangent bundle for the projection $E_G/P \longrightarrow M$.

Let Z_0 denote the connected component of the center of the reductive group G containing the identity element.

Let $E_P \subset E_G$ be a reduction of structure group of a principal G–bundle E_G to a parabolic subgroup P (not necessarily maximal) over a Zariski open subset $U \subset M$, with $\mathrm{codim}(M \setminus U) \geq 2$. Let χ be a nontrivial character of P trivial on Z_0 and dominant with respect to a Borel subgroup contained in P. So

$$E_P(\chi) := \frac{E_P \times \mathbb{C}}{P}$$

is a line bundle over U, where the quotient is for the diagonal action of P with P acting on $\mathbb{C}$ through χ. More precisely, that action of any $g \in P$ sends a point $(y\,,c) \in E_P \times \mathbb{C}$ to $(yg\,,\chi(g^{-1})c)$. The G–bundle E_G is stable (respectively, semistable) if and only if

$$\deg(\iota_* E_P(\chi)) < 0$$

(respectively, $\deg(\iota_* E_P(\chi)) \leq 0$) in every such situation for every parabolic subgroup P of G [Ra1], [RS, p. 22]. We recall that a character χ of P trivial on Z_0 is dominant if and only if the dual of the line bundle over G/P associated to χ admits nonzero global sections.

Let $E_P \subset E_G$ be a reduction of structure group of E_G over M to a parabolic subgroup $P \subset G$. This reduction is called *admissible* if for every character χ of P trivial on Z_0, the associated line bundle $E_P(\chi)$ is of degree zero [Ra2, p. 307, Definition 3.3].

We will explain the notion of admissibility by giving a couple of examples. Let V (respectively, W) be a holomorphic vector bundle over M of rank m (respectively, n) and positive (respectively, negative) degree. The vector bundle $V \oplus W$ defines a principal $\mathrm{GL}(m+n, \mathbb{C})$–bundle, and the direct sum decomposition gives a reduction of structure group to $\mathrm{GL}(m, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$. Note that $\mathrm{GL}(m, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ is a Levi subgroup of the parabolic subgroup of $\mathrm{GL}(m+n, \mathbb{C})$ defined by all $(m+n) \times (m+n)$ invertible matrices A satisfying the condition that $A_{ij} = 0$ whenever $i > m$ and $j \leq m$. Consider the character χ of $\mathrm{GL}(m, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ that sends any $(A_1\,, A_2)$ to $(\det A_1)^n/(\det A_2)^m$. (Note that the group of characters of a parabolic subgroup coincide with the group of characters of its Levi factor; this follows from the fact that a unipotent group does not have any nontrivial character.) Clearly χ vanishes on the center of $\mathrm{GL}(m+n, \mathbb{C})$. From the condition on the degrees of V and W it follows immediately that the line bundle over M associated to the $\mathrm{GL}(m,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$–bundle $V \oplus W$ for the character χ is of positive degree. In fact the degree of the line bundle is $n \cdot \deg(V) - m \cdot \deg(W)$. Therefore, this is not an admissible reduction. On the other hand, if $\mathrm{rank}(W)\deg(V) = \mathrm{rank}(V)\deg(W)$, then the decomposition $V \oplus W$ is an admissible reduction.

For any holomorphic vector bundle V over M of rank n, the vector bundle $V \oplus V^*$ has an $\mathrm{Sp}(2n,\mathbb{C})$ structure as well as an $\mathrm{SO}(2n,\mathbb{C})$ structure. The symplectic structure on $V \oplus V^*$ is defined by

$$(a_1\,, a_2) \times (a'_1\,, a'_2) \longmapsto a'_2(a_1) - a'_1(a_2)\,,$$

and the orthogonal structure is defined by

$$(a_1\,, a_2) \times (a'_1\,, a'_2) \longmapsto a'_2(a_1) + a'_1(a_2)\,.$$

The decomposition $V \oplus V^*$ defines a reduction of structure group of the $\mathrm{Sp}(2n,\mathbb{C})$ and the $\mathrm{SO}(2n,\mathbb{C})$–bundle to $\mathrm{GL}(n,\mathbb{C})$. Note that $\mathrm{GL}(n,\mathbb{C})$ sits inside $\mathrm{Sp}(2n,\mathbb{C})$ or $\mathrm{SO}(2n,\mathbb{C})$ as

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}.$$

Also, this homomorphism makes $\mathrm{GL}(n,\mathbb{C})$ a Levi subgroup of $\mathrm{Sp}(2n,\mathbb{C})$ as well as of $\mathrm{SO}(2n,\mathbb{C})$. Consider the character χ of $\mathrm{GL}(n,\mathbb{C})$ defined by $A \longmapsto \det A$. Note that the center of $\mathrm{Sp}(2n,\mathbb{C})$ or $\mathrm{SO}(2n,\mathbb{C})$ is a discrete group. The line bundle associated to $V \oplus V^*$ for this character is isomorphic to $\bigwedge^n V$. Therefore, the reduction of the $\mathrm{Sp}(2n,\mathbb{C})$ or $\mathrm{SO}(2n,\mathbb{C})$–bundle defined $V \oplus V^*$ is admissible if and only if $\deg(V) = 0$.

A principal G–bundle E_G over M is called *polystable* if either E_G is stable or there is a parabolic subgroup P of G and a reduction

$$E_{L(P)} \subset E_G$$

over M of structure group of E_G to the Levi factor $L(P)$ (recall that $L(P)$ can be realized as a subgroup of P) such that

(1) the principal $L(P)$–bundle $E_{L(P)}$ is stable;
(2) the extension of structure group of $E_{L(P)}$ to P, constructed using the inclusion of $L(P)$ in P, is an admissible reduction of E_G to P.

(See the following comment.)

For a principal H–bundle E_H and a homomorphism f of H to H', consider the action of H on $E_H \times H'$ defined by the condition that the action of any $g \in H$ sends a point $(y\,, h') \in E_H \times H'$ to $(yg\,, f(g^{-1})h')$. We recall that the quotient $(E_H \times H')/H$ for this action is a principal H'–bundle, which is known as the one obtained by *extending* the structure group of E_H using f. Now note that since $E_{L(P)}$ is a reduction of structure group of E_G to the subgroup $L(P)$, the principal P–bundle obtained by extending the structure group of $E_{L(P)}$, using the inclusion of $L(P)$ in P, is identified with a reduction of structure group of E_G to P.

It is easy to see that any polystable G–bundle is semistable.

Let E_P be a principal P–bundle over M, where P is a connected complex linear algebraic group. As before, set $L(P) := P/R_u(P)$, where $R_u(P)$ is the unipotent radical of P (see [Bo, p. 157, §11.21], [Hu, §19.5] for its definition and properties). So, $L(P)$ is a connected complex reductive algebraic group. Let $E_{L(P)}$ denote the principal $L(P)$–bundle obtained by extending the structure group of E_P using the projection of P to $L(P)$.

Definition 2.1. The P–bundle E_P will be called *stable* (respectively, *semistable*) if the $L(P)$–bundle $E_{L(P)}$ is stable (respectively, semistable). Similarly, E_P will be called *polystable* if $E_{L(P)}$ is polystable.

We note that for Geometric Invariant Theoretic quotients of polarized varieties using nonreductive group actions, the semistability of a point is defined in terms of the Levi group. See [Dr, p. 769, Définition 1] for the definition of (semi)stability for the action of a group which is not necessarily reductive.

From the above definition it follows that if P is a solvable group, then E_P is stable. Indeed, in that case $L(P)$ is isomorphic to a product of copies of $\mathbb{C}^*$; any principal bundle with a product of copies of $\mathbb{C}^*$ as structure group is stable (as $\mathbb{C}^*$ does not have any parabolic subgroup).

Let G be a complex reductive algebraic group and let $Q \subset G$ be a parabolic subgroup. Let $E_Q \subset E_G$ be a reduction to Q of the structure group of a principal G–bundle E_G over M. If E_G is semistable (or polystable), it is not necessary that the principal Q–bundle E_Q over M be semistable (or polystable). However, this is valid for admissible reductions, as shown in the following proposition.

Proposition 2.2. *Let $E_Q \subset E_G$ be an admissible reduction. The G–bundle E_G is semistable if and only if E_Q is semistable. If E_G is polystable, then E_Q is polystable.*

Proof. That E_Q is semistable if and only if E_G is semistable is Lemma 3.5.11(i) in [Ra2, p. 311]. Although in [Ra2] the base M is assumed to be a curve, the proof of [Ra2, Lemma 3.5.11(i)] does not use this assumption on dimension.

The assertion on polystability can be deduced from the combination of Lemma 3.5.11 of [Ra2] with [Ra2, p. 317, Lemma 3.13]. But we will give a more self–contained proof. Assume that E_G is polystable.

Since E_G is polystable, the adjoint vector bundle $\mathrm{ad}(E_G) := (E_G \times \mathfrak{g})/G$ is polystable [RS, p. 29, Theorem 3], where $\mathfrak{g}$ is the Lie algebra of G equipped with the adjoint action of G, and the quotient is for the twisted diagonal action of G on $E_G \times \mathfrak{g}$. From the assumption that $E_Q \subset E_G$ is an admissible reduction it follows that the adjoint vector bundle $\mathrm{ad}(E_Q) := (E_Q \times \mathfrak{q})/Q$ is of degree zero ($\mathfrak{q}$ is the Lie algebra of Q). Indeed, considering the adjoint action of Q on the line $\bigwedge^{\mathrm{top}} \mathfrak{q}$ we see that the corresponding line bundle $(E_Q \times \bigwedge^{\mathrm{top}} \mathfrak{q})/Q$ associated to E_Q, which is identified with $\bigwedge^{\mathrm{top}} \mathrm{ad}(E_Q)$, is of degree zero; the character of Q defined by the action on $\bigwedge^{\mathrm{top}} \mathfrak{q}$ is trivial on Z_0, and hence from the definition of an admissible reduction it follows immediately that the associated line bundle $\bigwedge^{\mathrm{top}} \mathrm{ad}(E_Q)$ is of degree zero.

A subbundle of degree zero of a polystable vector bundle of degree zero is also polystable. Since G is reductive, its Lie algebra $\mathfrak{g}$ has a nondegenerate G–invariant bilinear form. Hence the vector bundle $\mathrm{ad}(E_G)$ is isomorphic to $\mathrm{ad}(E_G)^*$; in particular, $\deg(\mathrm{ad}(E_G)) = 0$. The vector bundle $\mathrm{ad}(E_Q)$, being a subbundle of degree zero of the polystable vector bundle $\mathrm{ad}(E_Q)$ of degree zero, is also polystable.

A quotient vector bundle of degree zero of a polystable vector bundle bundle of degree zero is again polystable. Let $L(Q)$ denote the Levi group of Q and $E_{L(Q)}$ denote the principal $L(Q)$–bundle over M obtained by extending the structure group of E_Q using the projection of Q to $L(Q)$. So the adjoint vector bundle $\mathrm{ad}(E_{L(Q)})$ is a quotient vector bundle of degree zero (the group $L(Q)$ is reductive) of the polystable vector bundle $\mathrm{ad}(E_Q)$ of degree zero. Hence $\mathrm{ad}(E_{L(Q)})$ is polystable.

For a complex reductive group H, a principal H–bundle E_H over M is polystable if its adjoint vector bundle $\text{ad}(E_H)$ is polystable [ABi, p. 224, Corollary 3.8]. As the vector bundle $\text{ad}(E_{L(Q)})$ is polystable, we conclude that the principal $L(Q)$–bundle $E_{L(Q)}$ is polystable. This completes the proof of the proposition. $\square$

Let $\mathfrak{h}$ be the Lie algebra of a complex algebraic group H. Fix an invariant polynomial

$$\psi \in (\text{Sym}^k(\mathfrak{h}^*))^H$$

of degree k on $\mathfrak{h}$ invariant under the adjoint action of H on its Lie algebra. Given a principal H–bundle E_H over M, we have a characteristic class

$$C_\psi(E_H) \in H^{2k}(M, \mathbb{C})$$

of degree k. For the Chern–Weil construction of the cohomology class, choose a C^∞ connection on the principal bundle E_H. Now $C_\psi(E_H)$ is represented, in the de Rham cohomology, by the closed $2k$–form over M obtained by evaluating ψ on the curvature; since the curvature is an $\text{ad}(E_H)$–valued two–form, the evaluation of ψ is a $2k$–form. The cohomology class represented by this closed form is independent of the choice of the connection. See [Ch, p. 115, Corollary 3.1] for the details.

For a homomorphism $\rho : H \longrightarrow H'$, let $E_{H'}$ be the principal H'–bundle obtained by extending the structure group of E_H using ρ. Then any characteristic class of $E_{H'}$ is clearly a characteristic class of E_H. Indeed, any H'–invariant polynomial on the Lie algebra of H' gives an H–invariant polynomial on $\mathfrak{h}$ by using the homomorphism of Lie algebras induced by ρ. In other words, the pullback of an H'–invariant polynomial on the Lie algebra of H' is an H–invariant polynomial on $\mathfrak{h}$.

By *higher characteristic classes* we will mean characteristic classes of degree at least one.

Let $R_n(\mathfrak{h}) \subset \mathfrak{h}$ be the nilpotent radical. In other words, $R_n(\mathfrak{h})$ is the Lie algebra of the unipotent radical of H. So

$$L(\mathfrak{h}) := \frac{\mathfrak{h}}{R_n(\mathfrak{h})},$$

which is the Lie algebra of the Levi factor of H, is the maximal reductive quotient.

Using the natural projection of $\mathfrak{h}$ to $L(\mathfrak{h})$, an $L(H)$–invariant polynomial on $L(\mathfrak{h})$ defines an H–invariant polynomial on $\mathfrak{h}$. All the higher characteristic classes of E_H are obtained this way, namely by the $L(H)$–invariant polynomials on $L(\mathfrak{h})$. The nilpotent part $R_n(\mathfrak{h})$ can have invariant polynomials of positive degree, but they do not define nontrivial characteristic classes. Therefore, to construct all the higher characteristic classes of E_H it suffices to consider the pullback of the $L(H)$–invariant polynomials on $L(\mathfrak{h})$.

A reductive algebraic group is isogenus to a product of copies of $\mathbb{C}^*$ and simple groups. In other words, there is a surjective homomorphism to the reductive group from some product group of this type such that the kernel is a finite group. For a $\mathbb{C}^*$–bundle, the algebra of characteristic classes is generated by the degree one polynomial that identifies the Lie algebra with $\mathbb{C}$.

For a simple Lie group, in [Ko, p. 381, Theorem 7] a natural set of invariant polynomials are described that generate the algebra of invariant polynomials.

3. Connections on polystable bundles

Let E_G be a principal G–bundle over M, where G is a connected complex linear algebraic group with Lie algebra $\mathfrak{g}$. Let $\mathrm{At}(E_G)$ be the algebraic vector bundle over M defined by the sheaf of all G–invariant vector fields on the total space of E_G (see [At]). So for a sufficiently small contractible analytic open subset U of M, the space of holomorphic sections of $\mathrm{At}(E_G)$ over U is identified with the space of all G–invariant holomorphic vector fields on $\phi^{-1}(U)$, where ϕ is the projection of E_G to M. So $\mathrm{At}(E_G)$ has a subbundle defined by the sheaf of all G–invariant vertical vector fields on the total space of E_G. It is easy to see that this subbundle is identified with the adjoint bundle $\mathrm{ad}(E_G) := (E_G \times \mathfrak{g})/G$, and the quotient bundle $\mathrm{At}(E_G)/\mathrm{ad}(E_G)$ is identified with the holomorphic tangent bundle TM; the projection of $\mathrm{At}(E_G)$ to TM is defined by the differential of the projection ϕ (see [At] for the details). Therefore, we have an exact sequence of vector bundles over M

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G) \longrightarrow TM \longrightarrow 0, \tag{3.1}$$

which is known as the *Atiyah exact sequence*. A *holomorphic connection* on the principal G–bundle E_G is a holomorphic splitting of the Atiyah exact sequence, that is, a holomorphic homomorphism of vector bundles $D : TM \longrightarrow \mathrm{At}(E_G)$ such that its composition with the projection in (3.1) is the identity map of TM.

The sheaf of sections of $\mathrm{At}(E_G)$, TM and $\mathrm{ad}(E_G)$ are equipped with Lie algebra structures defined by the Lie bracket operation for a pair of vector fields. Note that this Lie algebra structure on the sheaf of sections of $\mathrm{ad}(E_G)$ coincides with the one defined by the Lie algebra structure of $\mathfrak{g}$, and the homomorphisms in (3.1) are compatible with the Lie algebra structures. Given a holomorphic homomorphism $D : TM \longrightarrow \mathrm{At}(E_G)$ that splits the exact sequence (3.1) (that is, D is a holomorphic connection on E_G), the obstruction of D to be compatible with the Lie algebra structures of the sheaf of sections of TM and $\mathrm{At}(E_G)$ is a holomorphic section

$$K(D) \in H^0(M, \Omega^2_M \otimes \mathrm{ad}(E_G))$$

which is the *curvature* of the holomorphic connection D [At]. The holomorphic connection D is called *flat* if $K(D) = 0$.

A *complex connection* on E_G is a C^∞ splitting of the Atiyah exact sequence. In other words, a complex connection is a C^∞ homomorphism $D_c : TM \longrightarrow \mathrm{At}(E_G)$ such that its composition with the projection in (3.1) is the identity map of TM. Thus a holomorphic connection on E_G defines a complex connection on E_G. On the other hand, a flat complex connection is a flat holomorphic connection. In other words, a flat holomorphic connection is the same as a flat complex connection.

We will explain the interrelation between the two notions of connection using the example of vector bundles.

Let V be a holomorphic vector bundle of rank n over M. So V defines a principal $\mathrm{GL}(n,\mathbb{C})$–bundle over M. Then the Atiyah exact sequence becomes

$$0 \longrightarrow \mathrm{End}(V) \longrightarrow \mathrm{At}(V) \longrightarrow TM \longrightarrow 0\,, \tag{3.2}$$

where $\mathrm{At}(V) \subset \mathrm{Diff}^1_M(V\,,V)$ is the subbundle with symbol contained in $\mathrm{Id}_V \otimes TM$, the projection to TM is the symbol map, and the inclusion of $\mathrm{End}(V)$ in $\mathrm{At}(V)$ is the inclusion of the sheaf of differential operators of order zero in the sheaf of differential operators of order one (see [At]). A holomorphic splitting of (3.2) defines a first order holomorphic differential operator

$$\partial_V \,:\, V \,\longrightarrow\, E\otimes\Omega^1_M$$

between the holomorphic vector bundles satisfying the holomorphic Leibniz identity which says that

$$\partial_V(fs) \,=\, s\otimes\partial f + f\partial_V(s)\,,$$

where f is a locally defined holomorphic function, and s is a locally defined holomorphic section of V.

A complex connection on V is a first order C^∞ differential operator

$$\nabla^V \,:\, C^\infty(V) \,\longrightarrow\, C^\infty(V\otimes T^*_{\mathbb{C}}) \,=\, C^\infty(V\otimes(\Omega^{1,0}\oplus\Omega^{0,1}))$$

satisfying the Leibniz identity which says that

$$\nabla^V(fs) \,=\, s\otimes \mathrm{d}f + f\nabla^V(s)\,,$$

where f is a smooth function on M and s is a smooth section of V. This implies that the $(0\,,1)$ part $(\nabla^V)^{0,1}$ of ∇^V, defined using the decomposition $T^*_{\mathbb{C}} \,=\, \Omega^{1,0}\oplus\Omega^{0,1}$, coincides with the Dolbeault operator $\overline{\partial}_V$ defining the holomorphic structure of V. Consequently, to give a complex connection ∇^V on V is equivalent to giving $(\nabla^V)^{1,0}$. Giving $(\nabla^V)^{1,0}$ is equivalent to giving a C^∞ splitting of the exact sequence (3.2). In particular, a holomorphic connection on V defines a complex connection on V.

If the curvature $(\nabla^V)^2$ of a complex connection ∇^V on V vanishes identically, then the component $(\nabla^V)^{1,0}$ is a holomorphic operator. In other words, $(\nabla^V)^{1,0}$ defines a holomorphic connection on V. The vanishing of $(\nabla^V)^2$ further implies that the holomorphic connection is actually flat. Conversely, if ∂_V is a flat holomorphic connection on V, then $\partial_V + \overline{\partial}_V$ is a flat complex connection on V, where $\overline{\partial}_V$ is the Dolbeault operator that defines the holomorphic structure of V.

Fix a point $x\in M$. Fix a point in E_G over x; this amounts to identifying the fiber of E_G over x with G as spaces with left G action—the action of G on itself is by translations. Then the monodromy of a flat connection D is a homomorphism from the fundamental group $\pi_1(M,x)$ to G. Note that $\pi_1(M,x)$ and $\pi_1(M,x')$ are identified up to an inner conjugation. For a different choice of the point x or of the point in E_G over it, the two monodromy representations for D are identified up to an inner conjugation. Therefore, a condition on D—the monodromy is contained in a compact subgroup of G—does not depend on the choice of the base points in M or E_G.

If G is a reductive linear algebraic group and E_G a polystable G–bundle over M with vanishing higher characteristic classes, then E_G admits a flat holomorphic connection whose monodromy is contained in a maximal compact subgroup of G [RS, p. 24, Theorem 1]. In other words, E_G is given by a representation of $\pi_1(M)$ in the maximal compact subgroup of G. In fact, it is enough if all the characteristic classes of degrees one and two vanish. Moreover the connection is unique. Conversely, any principal G–bundle over M admitting a flat connection whose monodromy is contained in a maximal compact subgroup of G must be polystable with vanishing higher characteristic classes [RS].

Let P be a connected complex linear algebraic group. Let $R_u(P)$ denote the unipotent radical of P. Define $L(P) := P/R_u(P)$. Let

$$q : P \longrightarrow L(P) \tag{3.3}$$

be the quotient map.

Let E_P be a principal P–bundle over M. By a *unitary flat* connection on E_P we will mean a flat holomorphic connection on E_P whose monodromy has the property that its image in $L(P)$ (by the map q in (3.3)) is contained in a compact subgroup of $L(P)$. Therefore, a principal P–bundle over M admitting a unitary flat connection is given by a homomorphism

$$\tau : \pi_1(M) \longrightarrow P \tag{3.4}$$

with the property that image$(q \circ \tau) \subset L(P)$ is contained in a compact subgroup of $L(P)$.

Theorem 3.1. *The P–bundle E_P admits a unitary flat connection if and only if E_P is polystable, and all the characteristic classes of E_P of degrees one and two vanish.*

Proof. Let D be a holomorphic connection on E_P. Since E_P admits a holomorphic connection, all higher characteristic classes of E_P vanish [At, p. 192, Theorem 4].

Let $E_{L(P)} := (E_P \times L(P))/P$ be the principal $L(P)$–bundle obtained by extending the structure group of E_P using the homomorphism q in (3.3).

We recall that a connection on a principal bundle induces connections on each of its associated bundles, in particular, on any principal bundle obtained by extending the structure group. Let D' denote the induced connection on $E_{L(P)}$ induced by D.

If D is unitary flat, then the connection D' on $E_{L(P)}$ is flat and its monodromy is contained in a compact subgroup of $L(P)$. Therefore, from the main theorem of [RS] mentioned above it follows immediately that $E_{L(P)}$ is polystable. In other words, E_P is polystable with vanishing higher characteristic classes.

Now assume that E_P is polystable and all the characteristic classes of degrees one and two of E_P vanish.

To prove that E_P admits a unitary flat connection, we will construct a flat connection on a vector bundle associated to E_P by a faithful representation of P, and then we will show that this connection on the vector bundle comes from a unitary flat connection on E_P.

We start by establishing a group theoretic result.

Lemma 3.2. *There is an injective homomorphism of algebraic groups*

$$\rho : P \hookrightarrow \mathrm{GL}(V)\,,$$

where V is a finite dimensional complex vector space, and a closed subgroup $Q \subset \mathrm{GL}(V)$ with $\rho(P) \subseteq Q$ such that either Q is a parabolic subgroup of $\mathrm{GL}(V)$, or $Q = \mathrm{GL}(V)$, and the following two conditions hold:

1. $\rho(R_u(P)) \subset R_u(Q)$*;*
2. *the induced homomorphism*

$$\phi : L(P) \longrightarrow L(Q) := \frac{Q}{R_u(Q)} \tag{3.5}$$

induced by ρ is injective.

(Since $\rho(R_u(P)) \subset R_u(Q)$, the homomorphism ρ induces a homomorphism of the quotients.)

Proof. Fix a faithful representation

$$\rho : P \hookrightarrow \mathrm{GL}(V)$$

of P with V being a finite dimensional complex vector space. Consider the action of the unipotent radical $R_u(P)$ on V.

Let

$$V_1 := \{v \in V \mid \rho(g)v = v, \forall g \in R_u(P)\}$$

be the subspace on which $R_u(P)$ acts trivially. As $R_u(P)$ is unipotent, we have $V_1 \neq 0$ unless $V = 0$, in which case $P = e$, the trivial group. So $R_u(P)$ acts on the quotient V/V_1, as it preserves V_1. As before, let

$$V_2' := \{v \in V/V_1 \mid \rho(g)v = v, \forall g \in R_u(P)\}$$

and set $V_2 \subset V$ to be the inverse image of V_2' for the natural projection $V \longrightarrow V/V_1$. Proceeding inductively we have filtration

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k = V \tag{3.6}$$

of subspaces of V with the property that V_{i+1}/V_i, $i \in [0\,, k-1]$, is the maximal subspace of V/V_i on which $R_u(P)$ acts trivially.

We will now show that the filtration of subspaces in (3.6) is preserved by the action of P on V.

To prove this, recall that the unipotent radical $R_u(P)$ is a normal subgroup of P. Consequently, for any $g \in P$ and $v \in V_1$ we have

$$\rho(g_u) \circ \rho(g)v = \rho(g) \circ \rho(g^{-1}) \circ \rho(g_u) \circ \rho(g)v = \rho(g) \circ \rho(g^{-1}g_u g)v = \rho(g)v$$

for each $g_u \in R_u(P)$, as $g^{-1}g_u g \in R_u(P)$ and $R_u(P)$ fixes V_1 pointwise. In other words, the subspace $V_1 \subset V$ is preserved by the action of P. Now consider the action of P on the quotient V/V_1 and repeat the argument. This would give that V_2/V_1 is preserved by the action of P on V/V_1. Next consider the action of P on the quotient V/V_2 and repeat the argument. Proceeding inductively, we conclude that the filtration in (3.6) is preserved by the action of P on V.

Let

$$Q \subset \mathrm{GL}(V)$$

be the subgroup preserving the filtration in (3.6). So for any $g \in \mathrm{GL}(V)$ we have $g \in Q$ if and only if $gV_i \subset V_i$ for all $i \in [1\,, k]$. From the definition Q it is immediate that $\rho(P) \subseteq Q$. Note that if $V_1 = V$, then $Q = \mathrm{GL}(V)$, and otherwise Q is a parabolic subgroup of $\mathrm{GL}(V)$.

We recall that $R_u(Q)$ consists of all $g \in Q$ with the property that g acts on each quotient V_{i+1}/V_i, $i \in [0\,, k-1]$, as the identity map. So, from the construction of the filtration (3.6) we have $\rho(R_u(P)) \subset R_u(Q)$.

Consequently, we have an induced homomorphism

$$\phi : L(P) := \frac{P}{R_u(P)} \longrightarrow \frac{Q}{R_u(Q)} =: L(Q)$$

that sends a coset $gR_u(P)$ to the coset $\rho(g)R_u(Q)$. To complete the proof we need to show that ϕ is injective. Since $\rho^{-1}(R_u(Q))$ is a normal unipotent subgroup of P and $R_u(P) \subset \rho^{-1}(R_u(Q))$, it follows from the definition of $R_u(P)$ that $R_u(P) =$

$\rho^{-1}(R_u(Q))$. This implies that the induced homomorphism ϕ is injective. This completes the proof of the lemma. □

Let

$$E := \frac{E_P \times V}{P} \tag{3.7}$$

be the vector bundle over M associated to E_P for the left P–module V obtained in Lemma 3.2. The quotient in (3.7) is with respect to the twisted diagonal action of P with P acting on V through ρ. (The action of any $g \in P$ sends a point $(y, v) \in E_P \times V$ to $(y, \rho(g^{-1})v)$.)

Proposition 3.3. *The vector bundle E is semistable with $c_i(E) = 0$ for all $i \geq 1$.*

Proof. Let $E_Q := (E_P \times Q)/P$ be the principal Q–bundle over M obtained by extending the structure group of E_P using the homomorphism ρ of P to Q obtained in Lemma 3.2. Therefore, the vector bundle

$$\frac{E_Q \times V}{Q}$$

associated to E_Q by the standard action of $Q \subset \mathrm{GL}(V)$ on V is identified with the vector bundle E defined in (3.7). Since Q preserves the filtration (3.6) of V, it follows that the vector bundle E gets a filtration of subbundles

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{k-1} \subset E_k = E, \tag{3.8}$$

where $E_i := (E_P \times V_i)/P$ is the vector bundle associated to E_P for the left P–module V_i defined in (3.6).

Now the proposition follows immediately from the following stronger version.

Lemma 3.4. *Each quotient E_{j+1}/E_j, $j \in [0, k-1]$, is a polystable vector bundle over M. Furthermore,*

$$c_i(E_{j+1}/E_j) = 0$$

for all $i \geq 1$ and $j \in [0, k-1]$.

Proof. For the action of P on V, defined using ρ in Lemma 3.2, the restriction of the action to the subgroup $R_u(P)$ on the graded vector space

$$\overline{V} := \bigoplus_{j=0}^{k-1} \frac{V_{j+1}}{V_j} \tag{3.9}$$

for the filtration in (3.6) is the trivial action. Therefore, the Levi group $L(P) = P/R_u(P)$ acts on each quotient V_{j+1}/V_j.

Let

$$E_{L(P)} := \frac{E_P \times L(P)}{P}$$

be the principal $L(P)$–bundle over M obtained by extending the structure group of E_P using the natural projection $P \longrightarrow L(P)$. The above quotient is for the twisted diagonal action of P with P acting on $L(P)$ as left translations.

The vector bundle

$$E_{\overline{V}} := \frac{E_{L(P)} \times \overline{V}}{P} \tag{3.10}$$

associated to the principal $L(P)$–bundle $E_{L(P)}$ for the left $L(P)$–module $\overline{V}$ is naturally identified with the graded vector bundle for the filtration in (3.8), that is,

$$\bigoplus_{j=0}^{k-1} \frac{E_{j+1}}{E_j} = E_{\overline{V}}. \tag{3.11}$$

This follows from the fact that $R_u(P)$ acts trivially on $\overline{V}$.

Recall the assumption that E_P is polystable with vanishing characteristic classes of degrees one and two. In other words, $E_{L(P)}$ is polystable with vanishing characteristic classes of degrees one and two. Consequently, the principal $L(P)$–bundle $E_{L(P)}$ admits a (unique) flat holomorphic connection whose monodromy is contained in a maximal compact subgroup of $L(P)$ [RS, p. 24, Theorem 1]. Let D denote this flat holomorphic connection $E_{L(P)}$.

Let $D_{\overline{V}}$ be the flat holomorphic connection on the associated vector bundle $E_{\overline{V}}$ (associated to the $L(P)$–bundle $E_{L(P)}$) induced by the connection D on $E_{L(P)}$. Since the monodromy of D is contained in a compact subgroup of $L(P)$, the monodromy of $D_{\overline{V}}$ is also contained in a compact subgroup of $\mathrm{Aut}(\overline{V})$, where $\mathrm{Aut}(\overline{V}) \subset \mathrm{GL}(\overline{V})$ is the group of all isomorphisms of $\overline{V}$ preserving the decomposition in (3.9).

Therefore, the connection $D_{\overline{V}}$ on $E_{\overline{V}}$ induces a unitary flat connection on each direct summand E_{j+1}/E_j of $E_{\overline{V}}$ in the decomposition in (3.11). Consequently, each quotient E_{j+1}/E_j, $j \in [0\,, k-1]$, is polystable, and $c_i(E_{j+1}/E_j) = 0$ for all $i \geq 1$. This completes the proof of Lemma 3.4. □

It was already noted that Lemma 3.4 completes the proof of Proposition 3.3. Hence the proof of Proposition 3.3 is complete. □

Now we will recall a result of [Si] which will be crucially used here. Fix, once and for all, a Kähler form ω on M representing the de Rham cohomology class $c_1(\zeta)$, where ζ is the polarization on M.

Let W be a semistable vector bundle over M such that $c_1(W) = 0 = c_2(W)$ (here c_i is a rational Chern class). Then W admits a filtration

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{m-1} \subset W_m = W \tag{3.12}$$

such that each W_i, $i \in [0\,, m-1]$, is a subbundle of W and the quotient W_{i+1}/W_i is a polystable vector bundle with $c_j(W_{i+1}/W_i) = 0$ for all $j \geq 1$ [Si, p. 39, Theorem 2]. Set the Higgs field in [Si, Theorem 2] to be zero to get the above formulation. Note that W can have more than one filtration satisfying all the conditions for the above filtration $\{W_i\}$.

The vector bundle W has a natural flat holomorphic connection [Si, p. 40, Corollary 3.10]. (As before, set the Higgs field to be zero in [Si, Corollary 3.10].) This connection on W will be denoted by ∇^W. Since this connection is very important for our purpose, some comments on it will be made. In [Si, p. 40, Corollary 3.10], an equivalence between the category of all flat vector bundles over M and the category of all semistable Higgs bundles $(F\,, \theta)$ over M, with $c_1(F) = 0 = c_2(F)$, are identified. The flat vector bundle $(W\,, \nabla^W)$ corresponds to the Higgs bundle $(W\,, 0)$ by this identification. It may be noted that in [Si, p. 40, Corollary 3.10], if the Higgs bundle $(F\,, \theta)$ corresponds to the flat vector bundle $(F'\,, \nabla)$, then the two holomorphic vector bundles, namely F and F', are not always isomorphic.

However, in our case the Higgs field is zero, or equivalently, W has a filtration as in (3.12) of subbundles such that each subsequent quotient is a polystable vector bundle with vanishing higher Chern classes. This implies that the holomorphic vector bundle W is holomorphically identified with the holomorphic vector bundle underlying the flat vector bundle that corresponds to the Higgs bundle $(W\,,0)$. (See the final paragraph of the subsection "Examples" in the middle of page 37 of [Si, Section 3]; see also the paragraph in [Si, p. 40] following Corollary 3.10.)

For convenience, the flat connection ∇^W constructed in [Si, p. 40, Corollary 3.10] on a semistable vector bundle over M with vanishing first and second Chern classes will henceforth be called the *canonical connection*.

The canonical connection ∇^W has the property that it is compatible with any filtration of W by subbundles of the type described in (3.12) (that is, each subsequent quotient is polystable with all its higher Chern classes being zero). In other words, the connection ∇^W preserves the subbundle W_i, where $i \in [1\,,m-1]$, with W_i as in (3.12). To prove that ∇^W preserves W_i, note that W_i is semistable with vanishing higher Chern classes; this is an immediate consequence of the fact that each quotient W_{j+1}/W_j, $j \in [0\,,m-1]$, is polystable with vanishing higher Chern classes. Consider the homomorphism of Higgs bundles

$$(W_i\,,0) \longrightarrow (W\,,0)$$

induced by the inclusion map of W_i in W; both of the vector bundles are equipped with the zero Higgs field. Now, Lemma 3.5 of [Si, p. 36] says that the above homomorphism of Higgs bundles induces a homomorphism of the flat bundles (local systems) corresponding to the two Higgs bundles. In [Si, Lemma 3.5], Simpson shows that the category of flat vector bundles is equivalent to the category of Higgs bundles that are extensions of stable Higgs bundles with vanishing higher Chern classes (that is, Higgs bundles admitting a filtration of Higgs bundles such that each subsequent quotient is a stable Higgs bundle of vanishing higher Chern classes). Note that since a polystable vector bundle is a direct sum of stable vector bundles, both $(W_i\,,0)$ and $(W\,,0)$ are Higgs bundles of the type considered in [Si, Lemma 3.5]. Therefore, using the equivalence of categories, the above homomorphism of Higgs bundles gives a homomorphism from the flat bundle corresponding to the Higgs bundle $(W_i\,,0)$ to the flat bundle corresponding to the $(W\,,0)$. It was noted earlier that the underlying holomorphic vector bundle for the flat bundle corresponding to $(W_i\,,0)$ (respectively, $(W\,,0)$) is identified with W_i (respectively, W); this follows from the vanishing of the Higgs fields (see the third paragraph in page 37 of [Si]). The homomorphism of the local systems induced by the above homomorphism $(W_i\,,0) \longrightarrow (W\,,0)$ gives the inclusion map of W_i in W for the underlying holomorphic vector bundles. This immediately implies the following:

(1) the connection ∇^W on W preserves the subbundle W_i, and
(2) the connection on W_i induced by ∇^W coincides with the canonical connection on W_i.

Therefore, there is an induced connection on each quotient W_{i+1}/W_i, $i \in [0\,,m-1]$, induced by ∇^W. Note that since the vector bundle W_{i+1}/W_i is polystable with vanishing higher Chern classes, it has a unique flat Hermitian–Yang–Mills connection [Do2], [Si]. The flat connection on W_{i+1}/W_i induced by ∇^W coincides with the Hermitian–Yang–Mills connection on W_{i+1}/W_i [Si, p. 40, Corollary 3.10] (see also the paragraph following Corollary 3.10 in [Si]). We remark that from these properties of the connection ∇^W constructed in [Si] it is easy to see that if we set P in Theorem 3.1 to be a parabolic subgroup of $\mathrm{GL}(n,\mathbb{C})$, then the connection ∇^W on the associated rank n vector bundle (associated to E_P for the standard representation of $\mathrm{GL}(n,\mathbb{C})$) satisfies all the conditions asserted in Theorem 3.1.

The canonical connection is natural in the sense that it is compatible with the direct sum, tensor product and dualization operations. To explain this, note that the dual vector bundle W^* is also semistable with $c_1(W^*) = 0 = c_2(W^*)$. If ∇^{W^*} denotes the canonical connection on W^*, then ∇^{W^*} coincides with the connection on W^* induced by the canonical connection ∇^W on W. Let W' be another semistable vector bundle over M with $c_1(W') = 0 = c_2(W')$. Let $\nabla^{W'}$ denote the canonical connection on W'. Now note that both the vector bundles $W \oplus W'$ and $W \otimes W'$ are semistable with vanishing first and second Chern classes. Let $\nabla^{W\oplus W'}$ and $\nabla^{W\otimes W'}$ be the canonical connections on $W \oplus W'$ and $W \otimes W'$, respectively. Then

$$\nabla^{W\oplus W'} = \nabla^W \oplus \nabla^{W'}$$

and

$$\nabla^{W\otimes W'} = (\nabla^W \otimes \mathrm{Id}_{W'}) + (\mathrm{Id}_W \otimes \nabla^{W'}) .$$

In other words, $\nabla^{W\oplus W'}$ and $\nabla^{W\otimes W'}$ coincide with the connections induced by ∇^W and $\nabla^{W'}$ on $W \oplus W'$ and $W \otimes W'$, respectively. This compatibility of the canonical connection with the above standard operations on vector bundles follows directly from the construction of the canonical connection in [Si]. This also follows from the second part of [Si, p. 40, Corollary 3.10]. See [Si, p. 41 – p. 43] for the details.

It should be mentioned that the construction of the canonical connection in [Si] completely breaks down in the absence of the assumption that the first two Chern classes vanish. Also, it is not known whether the above–mentioned Theorem 2 of [Si, p. 39] remains valid if M is compact Kähler. The proof in [Si] does not work for Kähler manifolds, as it crucially uses a restriction theorem in [MR] which is valid only for projective manifolds.

To have some understanding of the canonical connection, it will be described for the special case of two–step filtrations, which is much simpler than the general case. A reader not interested in this digression from the proof of Theorem 3.1 should go directly to the paragraph that starts as "Continuing with the proof of the theorem".

Let

$$0 \longrightarrow F_1 \longrightarrow W \longrightarrow F_2 := W/F_1 \longrightarrow 0$$

be an exact sequence of vector bundles over M such that both F_1 and F_2 are polystable vector bundles and $c_i(F_1) = 0 = c_i(F_2)$ for $i = 1, 2$.

Let ∇_1 (respectively, ∇_2) denote the (unique) flat Hermitian connection (the Hermitian–Yang–Mills connection) on F_1 (respectively, F_2). So the $(0\,,1)$–part of ∇_1 (respectively, ∇_2) is the Dolbeault operator on F_1 (respectively, F_2) and the $(1\,,0)$–part is a flat holomorphic connection on F_1 (respectively, F_2). Using these two connections, we will now construct a connection on W.

Let

$$c \in H^1(M, \mathrm{Hom}(F_2\,, F_1)) \tag{3.13}$$

be the extension class for the above exact sequence. The connections ∇_1 and ∇_2 together induce a unitary flat connection ∇' on $\mathrm{Hom}(F_2\,, F_1)$.

Let

$$\theta \in C^\infty(M, \Omega^{0,1}_M(\mathrm{Hom}(F_2\,, F_1))) \tag{3.14}$$

be the harmonic representative of c (for the Dolbeault resolution) corresponding to the unitary flat connection ∇' and the (fixed) Kähler form ω on M. Note that $\mathrm{Hom}(F_2\,, F_1) \subset \mathrm{End}(F_1 \oplus F_2)$ in an obvious way. So θ in (3.14) is a $\mathrm{End}(F_1 \oplus F_2)$–valued $(0\,,1)$–form on M.

Finally, consider the connection

$$\nabla \;:=\; \nabla_1 \oplus \nabla_2 + \theta \tag{3.15}$$

on $F_1 \oplus F_2$, where θ is defined in (3.14). In other words, if s is a locally defined smooth section of F_1 (respectively, F_2), then $\nabla(s)$ coincides with $\nabla_1(s)$ (respectively, $\nabla_2(s) + \theta(s)$). Since θ is harmonic, the connection ∇ is flat.

Since θ represents the cohomology class c, the holomorphic vector bundle over M defined by the Dolbeault operator $\nabla^{0,1}$ (the $(0\,,1)$–part of ∇) is isomorphic to W. Note that the holomorphic isomorphism between W and the one defined on $F_1 \oplus F_2$ by $\nabla^{0,1}$ can be so chosen that it takes the subbundle F_1 of W to the C^∞ direct summand F_1 of $F_1 \oplus F_2$, and the automorphism of the graded bundle $F_1 \oplus F_2$ (for W) induced by this isomorphism is the identity automorphism. Transport the connection ∇ on $F_1 \oplus F_2$ to W using such an isomorphism. So we have constructed a flat connection on W. This flat connection on W coincides with the canonical connection on W.

Continuing with the proof of the theorem, the vector bundle E defined in (3.7) is semistable with $c_1(E) \;=\; 0 \;=\; c_2(E)$ (see Proposition 3.3). Let ∇^E be the canonical connection on E. From Lemma 3.4 we know that the filtration of E in (3.8) by subbundles has the property that each E_{j+1}/E_j, $j \,\in\, [0\,,k-1]$, is a polystable vector bundle with $c_i(E_{j+1}/E_j) \;=\; 0$ for all $i \geq 1$. As we noted above, this implies that the canonical connection ∇^E preserves each subbundle E_j. In other words, ∇^E induces a flat holomorphic connection on each quotient E_{j+1}/E_j, $j \,\in\, [0\,,k-1]$. Since each E_{j+1}/E_j is polystable with $c_i(E_{j+1}/E_j) \;=\; 0$ for all $i \,\geq\, 1$, the vector bundle E_{j+1}/E_j admits a unique unitary flat connection (the Hermitian–Yang–Mills connection) [Do1], [MR]. We also noted that this unitary flat connection on E_{j+1}/E_j coincides with the one induced on E_{j+1}/E_j by ∇^E.

Let $E_Q \;:=\; (E_P \times Q)/P$ be the principal Q–bundle over M obtained by extending the structure group of E_P using the homomorphism $\rho \,:\, P \,\longrightarrow\, Q$ in Lemma 3.2 (the Q–bundle defined in the proof of Proposition 3.3). Since the connection ∇^E preserves the filtration of E in (3.8), it follows immediately that the connection ∇^E on E naturally induces a flat connection, which we will denote by ∇^Q, on the principal Q–bundle E_Q. Indeed, since the vector bundle associated to E_Q for the standard action of Q on V is identified with E, any connection on Q induces a connection on the associated vector bundle E. This identifies connections on the principal Q–bundle E_Q with the space of all connections on the vector bundle E that preserve that filtration in (3.8).

The final step in the proof of the theorem would be to show that the connection ∇^Q on the Q–bundle E_Q induces a connection on E_P.

Consider the closed subgroup $\rho(P)$ of $\mathrm{GL}(V)$, where ρ is defined in Lemma 3.2. A theorem of C. Chevalley (see [Hu, p. 80]) says that there is a finite–dimensional left $\mathrm{GL}(V)$–module W and a line $l \,\subset\, W$ such that $\rho(P)$ is exactly the isotropy subgroup, of the point in $P(W)$ representing the line l, for the action of $\mathrm{GL}(V)$ on the projective space $P(W)$ of lines in W.

Let $E_{\mathrm{GL}(V)} \;:=\; (E_P \times \mathrm{GL}(V))/P$ be the principal $\mathrm{GL}(V)$–bundle obtained by extending the structure group of the principal P–bundle E_P using the homomorphism

ρ in Lemma 3.2. Therefore, the vector bundle $(E_{\mathrm{GL}(V)} \times V)/\mathrm{GL}(V)$ associated to $E_{\mathrm{GL}(V)}$ for the standard action of $\mathrm{GL}(V)$ on V is identified with the vector bundle E.

Let

$$E_W := \frac{E_{\mathrm{GL}(V)} \times W}{\mathrm{GL}(V)}$$

be the vector bundle associated to $E_{\mathrm{GL}(V)}$ for the above left $\mathrm{GL}(V)$–module W obtained from Chevalley's theorem. Since the action of P on W fixes the line l, the line l defines a line subbundle of the vector bundle $(E_P \times W)/P$ associated to E_P for the action of P on W. But this vector bundle is canonically identified with E_W (as the action of P on W is the restriction of the action of $\mathrm{GL}(V)$, and $E_{\mathrm{GL}(V)}$ is the extension of E_P).

Let $L \subset E_W$ be the line subbundle defined by the line l left invariant by the action of P on W.

The canonical connection ∇^E on E induces a connection of the associated vector bundle E_W. (The connection ∇^E induces a connection on $E_{\mathrm{GL}(V)}$.) This connection on E_W will be denoted by ∇^W. The connection ∇^W is flat since ∇^E is flat.

Since P is the isotropy subgroup of the point in $P(W)$ represented by the line l for the action of $\mathrm{GL}(V)$ on $P(W)$, the connection ∇^W induces a connection on the P–bundle E_P if and only if ∇^W preserves the above–defined line subbundle L of E_W. To explain this assertion, let

$$E_{\mathrm{GL}(W)} := \frac{E_{\mathrm{GL}(V)} \times \mathrm{GL}(W)}{\mathrm{GL}(V)}$$

be the principal $\mathrm{GL}(W)$–bundle obtained by extending the structure group of $E_{\mathrm{GL}(V)}$ using the homomorphism

$$\mathrm{GL}(V) \longrightarrow \mathrm{GL}(W) \tag{3.16}$$

defined by the action of $\mathrm{GL}(V)$ on W. The connection ∇^W defines a connection on $E_{\mathrm{GL}(W)}$, which is a $\mathrm{gl}(W)$–valued one–form Ω on the total space of $E_{\mathrm{GL}(W)}$; here $\mathrm{gl}(W)$ is the Lie algebra of $\mathrm{GL}(W)$. Pull back the form Ω to the total space of E_P (since $E_{\mathrm{GL}(W)}$ is obtained by extending the structure group of E_P, there is a natural map of the total space of E_P to the total space of $E_{\mathrm{GL}(W)}$). Denote by Ω' this pulled back $\mathrm{gl}(W)$–valued form on the total space of E_P.

Since the connection ∇^W is induced by a connection on $E_{\mathrm{GL}(V)}$, the form Ω' is actually $\mathrm{gl}(V)$–valued in the sense that there is a $\mathrm{gl}(V)$–valued form on the total space of E_P such that the $\mathrm{gl}(W)$–valued form obtained from it by using the Lie algebra homomorphism $\mathrm{gl}(V) \longrightarrow \mathrm{gl}(W)$ coincides with Ω' (the homomorphism of Lie algebras is obtained from (3.16)).

If the line subbundle L of E_W is preserved by the connection ∇^W on E_W, then the $\mathrm{gl}(V)$–valued form Ω' actually takes values in the Lie algebra of the isotropy subgroup of the point in $P(W)$ represented by l for the action of $\mathrm{GL}(V)$ on $P(W)$. However, that Lie algebra is the Lie algebra of P, as $\rho(P)$ is precisely the isotropy subgroup.

Hence we conclude that if the connection ∇^W preserves L, then there is a connection ∇^P on E_P with the property that the connection induced on the associated vector bundle $E_{\mathrm{GL}(W)}$ (associated to E_P) by the connection ∇^P coincides with ∇^W.

We will first prove that L is of degree zero.

Proposition 3.5. *The first Chern class $c_1(L) \in H^2(M, \mathbb{C})$ vanishes.*

Proof. The line l in W is preserved by the action of P. Therefore, we have a character $\chi : P \longrightarrow \mathbb{C}^*$ defined by the identity $g \circ v = \chi(g)v$ for all $g \in P$ and $v \in l$. The line bundle L is identified with the one associated to E_P for the character χ of P. By assumption, all the characteristic classes of degree one for the principal P–bundle E_P vanish. Since L is associated to E_P by a character, the Chern class $c_1(L)$ is a characteristic class of degree one for E_P. Hence we have $c_1(L) = 0$, proving the proposition. □

We will now show that the vector bundle E_W is semistable with vanishing higher Chern classes, and the connection ∇^W on E_W coincides with the canonical connection on E_W. Before proving this let us see how this assertion actually equips E_P with a flat connection.

If E_W is semistable with $c_j(E_W) = 0$ for all $j \geq 1$, then the fact that the line subbundle L of E_W has $c_1(L) = 0$ (see Proposition 3.5) immediately implies that the quotient vector bundle E_W/L is semistable and all the higher Chern classes of E_W/L vanish. Let

$$0 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_{m-1} \subset U_m = \frac{E_W}{L}$$

be a filtration such that each U_i is a subbundle of E_W/L and each subsequent quotient U_{i+1}/U_i, $i \in [0\,, m-1]$, is polystable with $c_j(U_{i+1}/U_i) = 0$ for all $j \geq 1$. We noted earlier that the existence of such a filtration by subbundles is ensured by [Si, p. 39, Theorem 2]. Let

$$q : E_W \longrightarrow \frac{E_W}{L}$$

be the natural projection. Note that the filtration

$$0 = U'_0 \subset U'_1 := L \subset U'_2 := q^{-1}(U_1) \subset \cdots \subset U'_{i+1} := q^{-1}(U_i) \subset \cdots \subset U'_{m+1} := E_W$$

of E_W has the property that

(1) each U'_i is a subbundle of E_W;
(2) each quotient U'_{i+1}/U'_i, $i \in [0\,, m]$, is polystable;
(3) $c_j(U'_{i+1}/U'_i) = 0$ for all $j \geq 1$ and $i \in [0\,, m]$.

(Clearly, $U'_1/U'_0 = L$ and $U'_{i+2}/U'_{i+1} = U_{i+1}/U_i$.) So the canonical connection on E_W preserves the subbundle L of E_W. Therefore, if the canonical connection coincides with the connection ∇^W on E_W constructed earlier, it follows immediately that ∇^W preserves the line subbundle L of E_W. As was shown earlier, this implies that ∇^W induces a connection on E_P.

Consequently, the following proposition implies that ∇^W induces a connection on the P–bundle E_P.

Proposition 3.6. *The vector bundle E_W is semistable and $c_i(E_W) = 0$ for all $i \geq 1$. The connection ∇^W on E_W coincides with the canonical connection on E_W.*

Proof. Let V_1 be a finite–dimensional faithful complex left G_1–module, where G_1 is a connected reductive complex linear algebraic group. Proposition 3.1(a) of [De, p. 40] says that if W_1 is another finite–dimensional complex left G_1–module, then W_1 is a direct summand of a direct sum of G_1–modules of the form

$$\bigoplus_{i=1}^{k} V_1^{\otimes m_i} \otimes (V_1^{\otimes n_i})^* .$$

Since V is a faithful $\mathrm{GL}(V)$–module and W is a $\mathrm{GL}(V)$–module, we conclude that W is a direct summand of a left $\mathrm{GL}(V)$–module

$$\bigoplus_{i=1}^{k} V^{\otimes m_i} \otimes (V^{\otimes n_i})^*$$

for some k, m_i and n_i.

This immediately implies that the vector bundle E_W is a direct summand of the vector bundle

$$\mathcal{V} := \bigoplus_{i=1}^{k} E^{\otimes m_i} \otimes (E^{\otimes n_i})^* \tag{3.17}$$

(as E (respectively, E^*) is the vector bundle associated to $E_{\mathrm{GL}(V)}$ by the standard action of $\mathrm{GL}(V)$ on V (respectively, V^*)).

Now, a tensor product of semistable vector bundles over M is again semistable [Si, p. 38, Corollary 3.8]. Consequently, from Proposition 3.3 it follows immediately that each $E^{\otimes m_i} \otimes (E^{\otimes n_i})^*$ is semistable with vanishing higher Chern classes.

Therefore, the vector bundle $\mathcal{V}$ defined in (3.17) is semistable with vanishing higher Chern classes. Since E_W is a direct summand of $\mathcal{V}$, we conclude that E_W is semistable.

The vector bundle E_W has a flat holomorphic connection, namely ∇^W. Therefore, all the higher Chern classes of E_W vanish [At, p. 192, Theorem 4].

Recall that the canonical connection is well–behaved with respect to the direct sum, tensor product and dualization operations. So, the connection on the vector bundle $\mathcal{V}$ (defined in (3.17)) induced by the canonical connection ∇^E on E coincides with the canonical connection of $\mathcal{V}$. Note that since $\mathcal{V}$ is associated to E, any connection on E induces a connection on $\mathcal{V}$.

Since the canonical connection is well–behaved with respect to the direct sum operation, the canonical connection on $\mathcal{V}$ preserves the direct summand E_W, and moreover, the connection on E_W induced by the canonical connection on $\mathcal{V}$ coincides with the canonical connection of E_W. This immediately implies that the connection ∇^W on E_W induced by the connection ∇^E coincides with the canonical connection of E_W. This completes the proof of the proposition. $\square$

It was noted prior to Proposition 3.6 that Proposition 3.6 implies that ∇^W induces a connection on the P–bundle E_P. Let ∇^P denote this induced connection on E_P. Since ∇^W is flat, the connection ∇^P is also flat.

Note that the Levi group of Q is the product

$$L(Q) \cong \prod_{j=1}^{k-1} \mathrm{GL}(V_{j+1}/V_j)\,,$$

where V_i are as in (3.6).

Let $E_{L(Q)}$ be the principal $L(Q)$–bundle over M obtained by extending the structure group of E_Q using the natural projection of Q to its Levi group $L(Q)$. Let $\nabla^{L(Q)}$ denote the connection on $E_{L(Q)}$ obtained from the connection ∇^Q on Q.

The connection on the quotient vector bundle E_{j+1}/E_j in (3.11) induced by the connection ∇^E on E has the property that it coincides with the unique flat Hermitian connection on E_{j+1}/E_j. Consequently, the monodromy of the flat connection $\nabla^{L(Q)}$ is contained in a compact subgroup of $L(Q)$.

The image of the monodromy of ∇^Q in $L(Q)$ is contained in a maximal compact subgroup of $L(Q)$, as the monodromy of $\nabla^{L(Q)}$ is contained in a compact subgroup of $L(Q)$. Therefore, the image of the monodromy of ∇^P in $L(Q)$, using the homomorphism

$$\phi \circ q \,:\, P \longrightarrow L(Q)\,,$$

where ϕ and q are defined in (3.5) and (3.3), respectively, is contained in a compact subgroup of $L(Q)$. Since ϕ is injective, this implies that the image of the monodromy of ∇^P in $L(P)$ is contained in a compact subgroup of $L(P)$. Therefore, the connection ∇^P is unitary flat. This completes the proof of Theorem 3.1. □

Remark 3.7. Let E_G be a polystable G–bundle over M with vanishing higher characteristic classes, where G is a reductive linear algebraic group. Then E_G admits a unique flat connection whose monodromy is contained in a compact subgroup of G [RS, p. 24, Theorem 1]. However, for a general group P, as opposed to the case of reductive groups, the condition that the image of the monodromy of the flat connection lies in a compact subgroup of $L(P)$ does not produce at most one connection on a given principal P–bundle. More precisely, if E_P is polystable with vanishing higher characteristic classes, then there may be more than one flat holomorphic connection on E_P that induces the unitary flat connection $E_{L(P)}$ given by [RS]. To give an example, take M to be a compact Riemann surface of genus at least two and P to be the parabolic subgroup of $\mathrm{SL}(2,\mathbb{C})$ (the group of all upper triangular matrices in $\mathrm{SL}(2,\mathbb{C})$). So $L(P) = \mathbb{C}^* \times \mathbb{C}^*$. A rank two vector bundle E that fits in an exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow L^* \longrightarrow 0\,, \tag{3.18}$$

where L is a holomorphic line bundle of degree zero, is a polystable principal P–bundle over M. The corresponding $L(P)$–bundle is $L \oplus L^*$. Let

$$\theta \,\in\, H^0(M,\, L^{\otimes 2} \otimes K_X) \setminus \{0\}$$

be a nonzero section, where K_M is the holomorphic cotangent bundle of M. So θ defines a homomorphism

$$\theta' \,\in\, H^0(M,\, K_M \otimes \mathrm{End}(E))\,,$$

as $L^{\otimes 2} \,=\, \mathrm{Hom}(L^*, L) \,\subset\, \mathrm{End}(E)$. Note that if ∇ is a flat connection on the vector bundle E, then $\nabla + \theta'$ is also a flat connection on E. Now, if ∇^E is a flat connection on the principal P–bundle E that induces the flat Hermitian–Yang–Mills connection on the principal $L(P)$–bundle $L \oplus L^*$, then the flat connection $\nabla^E + \theta'$ on the P–bundle E also induces the Hermitian–Yang–Mills connection on $L \oplus L^*$.

It is perhaps natural to ask what happens to Theorem 3.1 if we look for holomorphic flat connections on E_P whose monodromy lies in a compact subgroup of P instead of the image of monodromy lying in a compact subgroup of $L(P)$. If the exact sequence (3.18) is nontrivial in the sense that it does not split, then E is not polystable as a principal $\mathrm{SL}(2,\mathbb{C})$–bundle. So the principal P–bundle E does not admit any flat holomorphic connection whose monodromy lies in a compact subgroup of P.

Let G be a connected unipotent linear algebraic group over $\mathbb{C}$. So, $R_u(G) = G$. Consequently, any G–bundle over M is stable. As was noted earlier, there are no characteristic classes of higher degree for unipotent groups. Therefore, Theorem 3.1 has the following corollary:

Corollary 3.8. *Any G–bundle over M, where G is unipotent, admits a flat holomorphic connection.*

Let G be a connected solvable linear algebraic group over $\mathbb{C}$. It was noted in Section 2 that the Levi group of G is a product of copies of $\mathbb{C}^*$, and any G–bundle is stable. Therefore, Theorem 3.1 has the following corollary:

Corollary 3.9. *Let E_G be a principal G–bundle over M, where G is a connected solvable linear algebraic group over $\mathbb{C}$. The G–bundle E_G admits a flat holomorphic connection if and only if for every character χ of G, the associated line bundle $(E_G \times \mathbb{C})/G$ over M is of degree zero.*

Let E_G be a semistable principal G–bundle over M, where G is a connected reductive linear algebraic group. Assume that all the (rational) characteristic classes of E_G of degree one vanish. Take a finite–dimensional complex left G–module W. We will show that the associated vector bundle

$$E_W := \frac{E_G \times W}{G}$$

is semistable. For this, let

$$W \cong \bigoplus_{i=1}^{m} W_i$$

be a decomposition into irreducible left G–modules. Since the G–module W_i is irreducible, the center of G acts on W_i as scalar multiplications. Therefore, the associated vector bundle

$$E_{W_i} := \frac{E_G \times W_i}{G}$$

is semistable [RS, p. 29, Theorem 3].

Since all the characteristic classes of E_G of degree one vanish, we have $\deg(E_{W_i}) = 0$. So, as E_{W_i} is semistable, the direct sum

$$E_W \cong \bigoplus_{i=1}^{m} E_{W_i}$$

is also semistable.

In view of this observation, the proof of Theorem 3.1 gives the following proposition.

Proposition 3.10. *Let P be any complex connected linear algebraic group and let E_P be a semistable principal P–bundle over M with the property that all the characteristic classes of E_P of degrees one and two vanish. Then E_P admits a flat holomorphic connection.*

Since the vector bundle E constructed in (3.7) is semistable and $c_1(E) = 0 = c_2(E)$, it has a canonical connection ∇^E. As in the proof of Theorem 3.1, this connection ∇^E induces a flat connection on E_P.

In the final section, an application of Proposition 3.10 will be given for principal bundles over abelian varieties.

4. Principal bundles on an abelian variety

Let M be an abelian variety. For any point $x \in M$, let

$$T_x : M \longrightarrow M$$

be the translation automorphism using x. So, $T_x(y) = x + y$.

Proposition 4.1. *Let P be any complex linear algebraic group. A principal P–bundle E_P on the abelian variety M admits a flat connection if and only if the pullback $T_x^* E_P$ is isomorphic to E_P for all $x \in M$.*

Proof. Since the isomorphism T_x is homotopic to the identity map of M, if E_P is given by some representation of the fundamental group $\pi_1(M)$ in P (that is, E_P admits a flat connection), then $T_x^* E_P$ is isomorphic to E_P for all $x \in M$.

Assume that $T_x^* E_P$ is isomorphic to E_P for all $x \in M$. So if

$$\chi : P \longrightarrow \mathbb{C}^*$$

is a character, then the associated line bundle $E_P(\chi) := (E_P \times \mathbb{C})/P$ also has the property that $T_x^* E_P(\chi) \cong E_P(\chi)$. Now, if ξ is a holomorphic line bundle over M with the property that $T_x^* \xi \cong \xi$ for all $x \in M$, then

$$c_1(\xi) = 0 \tag{4.1}$$

[Mu, p. 74, Definition; p. 86]. Since $c_1(E_P(\chi)) = 0$, we conclude that all characteristic classes of E_P of degree one vanish.

Let $E_{L(P)}$ be the principal $L(P)$–bundle for E_P, where $L(P)$, as before, is the Levi group of P. If the adjoint bundle vector $\mathrm{ad}(E_{L(P)})$ is not semistable, then consider the first term

$$W \subset \mathrm{ad}(E_{L(P)})$$

in the Harder–Narasimhan filtration of $\mathrm{ad}(E_{L(P)})$ (the maximal semistable subsheaf of $\mathrm{ad}(E_{L(P)})$). From the uniqueness of the Harder–Narasimhan filtration it follows immediately that the condition $T_x^*\mathrm{ad}(E_{L(P)}) \cong \mathrm{ad}(E_{L(P)})$ implies that $T_x^* W \cong W$. So, from (4.1) we have $c_1(W) = c_1(\det W) = 0$. Since $c_1(\mathrm{ad}(E_{L(P)})) = 0$ (as $L(P)$ is reductive), we conclude that $\mathrm{ad}(E_{L(P)})$ is semistable. Therefore, $E_{L(P)}$, and hence the P–bundle E_P, is semistable.

In view of Proposition 3.10, to show that E_P admits a flat connection all we need to show is that if $c \in H^{2,2}(M)$ is a (rational) characteristic class of E_P of degree two, then $c = 0$.

For this, let

$$C \in \mathrm{CH}^2(M) \tag{4.2}$$

be the characteristic class, in the Chow group of codimension two cycles, corresponding to the same invariant polynomial on the Lie algebra $\mathfrak{p}$ of P to which c corresponds (see Chapter 3 of [Fu] for characteristic classes with values in Chow groups). We may need to take an integral multiple of c as it is rational.

Consider the map

$$f : M \longrightarrow J^2(M) := \frac{H^3(M,\, \mathbb{C})}{F^2H^3(M,\, \mathbb{C}) + H^3(M,\, (2\pi\sqrt{-1})^2\mathbb{Z})}$$

defined by $x \longmapsto \mathrm{AJ}_M(T_x(C) - C)$, where AJ_M is the Abel–Jacobi map (see [Gr, p. 22]), and C is the cycle in (4.2). This map f is holomorphic.

The holomorphic cotangent space Ω^1_e of M at the identity element $e \in M$ will be denoted by V. So $H^{i,j}(M) = (\bigwedge^i V) \otimes (\bigwedge^j \overline{V})$ using the structure of cohomology of an abelian variety.

Consider the differential

$$\mathrm{d}f(e) : V^* \longrightarrow T_0 J^2(M) = (V \otimes \wedge^2 \overline{V}) \oplus \wedge^3 \overline{V} \tag{4.3}$$

at the point $e \in M$ of the map f constructed above; here $0 \in J^2(M)$ is the zero element. Let

$$\varphi(C) \in H^{2,2}(M) \cong (\wedge^2 V) \otimes (\wedge^2 \overline{V})$$

be the cycle class of C. The homomorphism $\mathrm{d}f(e)$ in (4.3) is the contraction of $\varphi(C)$. In other words,

$$\mathrm{d}f(e)(v) = \langle v\,, \varphi(C)\rangle \in V \otimes \wedge^2 \overline{V} \subset T_0 J^2(M)$$

for all $w \in V^*$, where $\langle -\,, -\rangle$ denotes the contraction of V^* with V. That $\mathrm{d}f(e)$ coincides with the contraction homomorphism follows from the description of the differential of the Abel–Jacobi map (see [Gr, p. 28]).

From the assumption that $T_x^* E_P \cong E_P$ for all $x \in M$ it follows that the map f is identically zero. Since the homomorphism $\mathrm{d}f(e)$ in (4.3) is identically zero, we have $\varphi(C) = 0$ from the above description of $\mathrm{d}f(e)$. (If the contraction with V^* of an element $\alpha \in (\bigwedge^2 V) \otimes (\bigwedge^2 \overline{V})$ vanishes identically, then $\alpha = 0$.)

Since $\varphi(C) = nc$, where n is some positive integer, we conclude that all characteristic classes of E_p of degree two vanish. So by Proposition 3.10 the principal P–bundle E_P admits a flat connection. This completes the proof of the proposition. □

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