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THETA CORRESPONDENCE FOR UNITARY GROUPS

Dipendra Prasad

In this paper we study the theta correspondence for Unitary groups of the same size over local and global fields. This correspondence has been studied in many cases by several authors. We are able to unify and generalise all these known results in terms of two conjectures, one local and the other global. These conjectures are in terms of the parametrisation of irreducible admissible representations of groups over local fields which are formulated by David Vogan refining Langlands parametrization, and which are now called Vogan parameters. In turn, the simple form of the conjecture here, gives support to the importance of Vogan's refinement of Langlands parametrisation.

1. Generalities.

Let K be a quadratic extension of a local field k, and let V and W be two finite dimensional Hermitian spaces over K. Let $\delta \in K^*$ be fixed with $\mathrm{tr}(\delta) = 0$. Multiplication by δ turns the Hermitian space W into a skew-Hermitian space, and therefore we get a symplectic structure on the k-vector space $V \otimes_K W$, making $(U(V), U(W))$ into a dual reductive pair in $Sp(V \otimes_K W)$. In the paper [**HKS**], Harris, Kudla and Sweet have made a detailed study of the $\mathbb{C}^*$-metaplectic covering of $Sp(V \otimes_K W)$ restricted to $U(V) \times U(W)$. They prove that the $\mathbb{C}^*$-metaplectic cover of $Sp(V \otimes_K W)$ splits over $U(V) \times U(W)$ for non-Archimedean local fields, and that the splittings over $U(V) \times U(W)$ can be parametrised by pairs of characters (χ_1, χ_2) of K^* such that $\chi_1|_{k^*} = \omega_{K/k}^{\dim W}$, and $\chi_2|_{k^*} = \omega_{K/k}^{\dim V}$, where $\omega_{K/k}$ is the quadratic character of k^* associated by local class field theory to the quadratic extension K of k. In this paper, we will have $\dim V = \dim W = n$, and we therefore take $\chi_1 = \chi_2 = \chi$, a fixed character of K^* whose restriction to k^* is $\omega_{K/k}$ if n is odd, and is the trivial character of K^* if n is even.

The two-fold metaplectic cover of $Sp(2n^2)$ when restricted to $U(V) \times U(W)$ gives rise to two-fold covers of $U(V)$ and $U(W)$. We will denote these covers by $\bar{U}(V)$ and $\bar{U}(W)$. These covers are split over the special Unitary groups. This one can see by using the result of Harris, Kudla, Sweet that the $\mathbb{C}^*$-metaplectic cover splits over Unitary groups, and therefore over the special Unitary group, and the special Unitary group being its own commutator (when the unitary group is non-compact), the mapping from $H^2(SU(n), \mathbb{Z}/2)$ to $H^2(SU(n), \mathbb{C}^*)$ is injective. Therefore if $\bar{U}(1)$ denotes the inverse image of $U(1)$ inside $\bar{U}(V)$ where $U(1)$ is a subgroup of $U(V)$ constructed using a one dimensional Hermitian subspace of the n-dimensional Hermitian space V, then there is an isomorphism $\bar{U}(V) = \bar{U}(1) \cdot SU(V)$, a semi-direct product with $SU(V)$ as the normal subgroup.

If we fix a character of $\bar{U}(1)$ which is non-trivial on the kernel of the natural mapping to $U(1)$, then multiplication by this character gives a bijection between *genuine* representations of $\bar{U}(V)$ i.e., those representations of $\bar{U}(V)$ on which the $\mathbb{Z}/2$ which is the kernel of the map of $\bar{U}(V)$ to $U(V)$ acts non-trivially, to representations of $U(V)$. This way, questions on representation theory of $\bar{U}(V)$ can be reduced to questions about representation theory of the group $U(V)$. Fixing splitting over $U(V) \times U(W)$ via the choice of the character χ of K^* achieves exactly this.

We fix ψ to be a non-trivial additive character of k, and ψ_K the additive character of K obtained by composing ψ with the trace map from K to k.

The Weil representation of $Sp(V \otimes_K W)$ associated to the character ψ_K thus gives rise to a representation of $U(V) \times U(W)$. When working globally, we take global analogues of ψ and χ.

We will assume the conjecture of Langlands and its refinement due to Vogan [**Vo**] about parametrization of representations of reductive groups over local fields. Very briefly put, let G be a quasi-split reductive group over a local field k with ${}^LG$ its L-group which is a semi-direct product of $G^\vee$, the dual group of G, with the Weil group of k. Fix a non-degenerate character of the unipotent radical of a Borel subgroup of G. Then according to Vogan, the equivalence classes of pairs (ϕ, μ) under the inner-conjugation action of $G^\vee$ where ϕ is a parameter for G, and μ is an irreducible representation of $\mathcal{S}_\phi$, the group of connected components of the centraliser of ϕ in $G^\vee$, is in one-to-one correspondence with the set of irreducible admissible representations of pure inner forms of G. Besides the Archimedean case treated by Vogan in [**Vo**], the best known result confirming the Vogan parametrization is due to Lusztig [**Lu**] for what are called unipotent representations of p-adic groups. Also, the case of Unitary groups in up to 3 variables is fully understood by the work of Rogawski [**Ro2**].

We will denote the L-group of $U(V)$ by ${}^LU(n)$ which is a semi-direct product of $GL(n, \mathbb{C})$ by the Weil group of k which acts via its quotient $\mathrm{Gal}(K/k)$ by $g \to \Phi_n {}^t g^{-1} \Phi_n^{-1}$ where Φ_n is the $n \times n$ matrix whose only non-zero entries are at the places $(i, n+1-i), 1 \leq i \leq n$ where it takes the value $(-1)^{i+1}$.

For G, the quasi-split unitary group $U(V_0)$, the pure inner forms of $U(V_0)$ are in one-to-one correspondence with the isomorphism classes of Hermitian spaces of dimension $n = \dim(V_0)$. The additive character ψ_δ of the trace zero elements of K defined by $\psi_\delta(x) = \psi(\delta x)$ can be used to fix a non-degenerate character on the unipotent radical of a Borel subgroup of $U(V_0)$. In the case of a non-Archimedean local field, the discriminant of V, denoted disc$V$, which is an element of $k^*/\mathrm{Nm}K^*$ determines the isomorphism class of V. A Vogan parameter (ϕ, μ) corresponds to a representation of $U(V)$ such that

1.1 $$\mu(-Id) = \omega_{K/k}\left(\frac{\mathrm{discV}}{\mathrm{discV_0}}\right)$$

where $-Id$ is the element of $\mathcal{S}_\phi$ represented by the negative of the identity matrix in $GL(n, \mathbb{C})$.

2. Component group for parameters of unitary group.

Let G be a group containing a subgroup H as an index 2 subgroup. Later we will apply this setup to $G = W'_k$, the Weil-Deligne group of k, and $H = W'_K$, the Weil-Deligne group of K. (We take the Weil-Deligne group to be the product of the Weil group with $SL(2, \mathbb{C})$.) Fix an element s in G which does not belong to H. Let ϕ be a homomorphism from G to $GL(n, \mathbb{C}) \rtimes \mathbb{Z}/2$ such that H lands inside $GL(n, \mathbb{C})$, and the element s of G does not land inside $GL(n, \mathbb{C})$. Here the semi-direct product $GL(n, \mathbb{C}) \rtimes \mathbb{Z}/2$ is generated by $GL(n, \mathbb{C})$ and j such that

$$j^2 = 1, \; jgj^{-1} = \Phi_n {}^t g^{-1} \Phi_n^{-1} \quad \text{for all} \;\; g \in GL(n, \mathbb{C}).$$

It is clear that $GL(n, \mathbb{C}) \rtimes \mathbb{Z}/2$ is a quotient of the L-group of $U(n)$, and that a Langlands parameter for $U(n)$ gives rise to such a homomorphism.

Suppose that the element s goes to the element ωj under the homomorphism ϕ. Let $h^s = shs^{-1}$, and for a representation W of H, let W^s denote the representation of H in which $h \in H$ operates via h^s. Then

$$\phi(h^s) = \omega \Phi_n {}^t\phi(h)^{-1} \Phi_n^{-1} \omega^{-1}.$$

Define an inner product on $\mathbb{C}^n$ by

$$\begin{aligned}
\langle v_1, v_2 \rangle &= {}^t v_1 \Phi_n \omega^{-1} v_2 \\
&= {}^t v_2 \, {}^t\omega^{-1} \, {}^t\Phi_n v_1 \\
&= {}^t v_2 \Phi_n \omega^{-1} \omega \Phi_n \, {}^t\omega^{-1} \Phi_n v_1 \\
&= \langle v_2, \omega \Phi_n \, {}^t\omega^{-1} \Phi_n v_1 \rangle \\
&= (-1)^{n-1} \langle v_2, \phi(s^2) v_1 \rangle.
\end{aligned}$$

(We have used that $\Phi_n^2 = (-1)^{n-1}$, and ${}^t\Phi_n = (-1)^{n-1}\Phi_n$.)

The following lemmas are easy to verify.

Lemma 2.1. $\langle \phi(h) v_1, \phi(h^s) v_2 \rangle = \langle v_1, v_2 \rangle$ *for all* h *in* H *and* v_1, v_2 *in* $\mathbb{C}^n$.

Lemma 2.2. *An element $g \in GL(n,\mathbb{C})$ commutes with $\phi(s) = \omega j$ if and only if*

$$\langle gv_1, gv_2\rangle = \langle v_1, v_2\rangle \quad \text{for all } v_1, v_2 \in \mathbb{C}^n.$$

Lemma 2.3. *Let W be an irreducible representation of H such that W^s is isomorphic to the dual W^* of W. Fix a non-degenerate bilinear form (which is unique up to scaling) $B : W \times W \to \mathbb{C}$ such that $B(hw_1, h^s w_2) = B(w_1, w_2)$. Then there exists a constant $c(W) \in \{\pm 1\}$ independent of s such that the bilinear form B has the property that*

$$B(v, w) = c(W)B(w, s^2 v) \quad \text{for all } v, w \in W.$$

Remark 2.1. The invariant $c(W)$ of Lemma 2.3 was introduced by Rogawski in [**Ro2**, Lemma 15.1.1]. For $H = W'_K$, he proves there that if W has odd dimension then $c(W)$ is 1 or -1 depending on whether the determinant of W which is a character of K^* is trivial on k^* or not. Furthermore, $c(W) = (-1)^{\dim W - 1}$ if and only if the representation W of W'_K can be extended to a parameter of the corresponding unitary group over k, cf. Rogawski [**Ro2**, Lemma 15.1.2], with correction in [**Ro1**].

Let ϕ be a parameter for unitary group $U(V)$ over a local field k defined in terms of a Hermitian form on an n-dimensional vector space V over the quadratic extension K of k. The restriction of ϕ to the Weil-Deligne group W'_K of K defines a group homomorphism $\phi_K : W'_K \to GL(n,\mathbb{C})$. Assume that ϕ_K decomposes as a direct sum of irreducible representations $\phi_K = \sum n_i \phi_i$ where n_i is the multiplicity of the irreducible representation ϕ_i. The representation ϕ_K has the property that $\bar{\phi}_K \cong \phi_K^*$ where ϕ_K^* denotes the dual of ϕ_K, and $\bar{\phi}_K$ is the representation of W'_K obtained from ϕ_K by conjugation by an element of W'_k which does not belong to W'_K. It follows that the set of irreducible representations ϕ_i which appear in the decomposition of ϕ_K is stable under $\phi_i \to \bar{\phi}_i^*$.

Suppose that the representation ϕ_i has the property that $\bar{\phi}_i^* = \phi_i$. Then the bilinear form $\langle,\rangle$ restricted to the ϕ_i isotypic subspace of $\mathbb{C}^n$ is non-degenerate. Let the representation ϕ_i of the Weil-Deligne group W'_K of K be realised on a vector space W_i. Fix a non-degenerate bilinear form $B_i : W_i \times W_i \to \mathbb{C}$ such that $B_i(hw_1, h^s w_2) = B_i(w_1, w_2)$ for all $w_1, w_2 \in W_i$.

Write the ϕ_i isotypic component as $U_i \otimes W_i$ for some vector space U_i of dimension n_i where the action of W_K on $U_i \otimes W_i$ is via its action on W_i. By the uniqueness of the inner product B_i, there exists a unique non-degenerate bilinear form $(,)_i$ on U_i such that

$$\langle u_1 \otimes w_1, u_2 \otimes w_2\rangle \;=\; (u_1, u_2)_i B_i(w_1, w_2).$$

Since

$$\langle u_1 \otimes w_1, u_2 \otimes w_2\rangle \;=\; (-1)^{n-1}\langle u_2 \otimes w_2, u_1 \otimes \phi(s^2) w_1\rangle,$$

and

$$B_i(w_1, w_2) = c(\phi_i)B_i(w_2, \phi(s^2)w_1),$$

we find that

$$(u_1, u_2) = (-1)^{n-1}c(\phi_i)(u_2, u_1).$$

Proposition 2.1. *Let ϕ be a parameter for unitary group $U(V)$ over a local field k defined in terms of a Hermitian form on an n-dimensional vector space V over the quadratic extension K of k. Assume that ϕ_K decomposes as a direct sum of irreducible representations $\phi_K = \sum n_i\phi_i$ where n_i is the multiplicity of the irreducible representation ϕ_i. Then the group of connected components of the centraliser of ϕ in $GL(n, \mathbb{C})$ is a product of $\mathbb{Z}/2$'s, the product being indexed by those irreducible representations ϕ_i which have the property that*

(a) $\phi_i \cong \bar{\phi}_i^*$.
(b) $c(\phi_i) = (-1)^{n-1}$.

Proof. The subgroup of $GL(n, \mathbb{C})$ which commutes with the image of W'_k inside ${}^LU(n)$ is precisely the subgroup of $GL(n, \mathbb{C})$ which commutes with the image of W'_K and s. By Lemma 2.2, the elements of $GL(n, \mathbb{C})$ commuting with the image of s is precisely those elements which preserve the bilinear form $\langle , \rangle$ on $\mathbb{C}^n$. The centraliser of H preserves the isotypical components of ϕ_K and in the decomposition of the isotypical component as $U_i \otimes W_i$, it can be taken to be $\mathrm{Aut}(U_i)$ acting on $U_i \otimes W_i$ through the first factor. Therefore the centraliser of the image of W'_k in $GL(n, \mathbb{C})$ is precisely those elements of $\mathrm{Aut}(U_i)$ which preserves the bilinear form $(,)$ on U_i. We have seen above that this bilinear form on U_i is symmetric or skew-symmetric according as $(-1)^{n-1}c(\phi_i) = 1$, or -1. Therefore this is precisely the condition that the part of the centraliser corresponding to the i-th isotypical piece is $\mathbb{Z}/2$ or trivial. It is easy to see that representations ϕ_i which are not isomorphic to $\bar{\phi}_i^*$, do not contribute to the group of connected components. The proposition follows.

Since a non-degenerate bilinear form on an odd dimensional vector space cannot be alternating, we have the following corollary to the proof of Proposition 2.1.

Corollary 2.1. *With the notation as in the proposition, an irreducible representation ϕ_i of W'_K with $\bar{\phi}_i^* \cong \phi_i$ appearing with odd multiplicity in ϕ_K must have $c(\phi_i) = (-1)^{n-1}$.*

3. The character χ_ϕ and the local conjecture.

For representations ϕ_i of W_K with the property $\bar{\phi}_i = \phi_i^*$, $\det \phi_i$ which is a character of K^* is trivial on those elements of k^* which are norms from K^*. Recall that we have fixed $\delta \in K^*$ with $\mathrm{tr}(\delta) = 0$. Since $\delta^2 = -\mathrm{Nm}(\delta)$,

$\det \phi_i(\delta^2) = \det \phi_i(-1)$. We will now be using epsilon factors associated to representations of the Weil-Deligne group for which we refer to the paper of Tate [**Ta**]. Since $\epsilon(\phi_i, \psi_K) = \epsilon(\bar{\phi}_i, \psi_K)$, and $\epsilon(\phi_i, \psi_K) \cdot \epsilon(\phi_i^*, \psi_K) = \det \phi_i(-1)$, we find that

$$\epsilon(\phi_i, \psi_K)^2 = \det \phi_i(-1) = \det \phi_i(\delta^2),$$

and therefore

$$\epsilon(\phi_i, \psi_K) \cdot \det \phi_i(\delta^{-1}) = \pm 1.$$

To a parameter ϕ of the unitary group $U(V)$, we will now define a character χ_ϕ on the group of connected components $\mathcal{S}_\phi$ of the centraliser of ϕ in $GL(n, \mathbb{C})$. For the decomposition of ϕ restricted to K, $\phi_K = \sum n_i \phi_i$, as we have seen in Proposition 2.1, the group of connected components is a product of $\mathbb{Z}/2$, the product being parametrised by certain ϕ_i's. Define χ_ϕ on the $\mathbb{Z}/2$ associated to such a ϕ_i to be the character of $\mathbb{Z}/2$ taking the non-trivial element of $\mathbb{Z}/2$ to

$$\epsilon(\phi_i \otimes \chi, \psi_K) \cdot \det(\phi_i \otimes \chi)(\delta^{-1}).$$

Remark 3.1. From the definition of χ_ϕ given above, it is easy to see that

$$\chi_\phi(-Id) = \epsilon(\phi_K \otimes \chi, \psi_K) \cdot \det(\phi_K \otimes \chi)(\delta^{-1}),$$

where $-Id$ is the element of $\mathcal{S}_\phi$ represented by the negative of the identity matrix in $GL(n, \mathbb{C})$.

Conjecture 1. *If an irreducible admissible representation $\pi_V \otimes \pi_W$ of $U(V) \times U(W)$ appears as a quotient in the Weil representation of $\mathrm{Sp}(V \otimes_K W)$ restricted to $U(V) \times U(W)$, then:*

(i) *The Langlands parameters associated to π_V and π_W are the same; call it ϕ.*

(ii) *The theta correspondence between $U(\mathcal{V})$ and $U(\mathcal{W})$ as $\mathcal{V}$ and $\mathcal{W}$ vary over the isomorphism classes of Hermitian spaces of dimension n defines a bijection between the irreducible admissible representations of pure inner forms of $U(V)$ (represented by $U(\mathcal{V})$ as $\mathcal{V}$ varies over the isomorphism classes of Hermitian spaces of dimension n) belonging to one Vogan L-packet to itself.*

(iii) *The characters χ_V and χ_W associated to the representatins π_V and π_W are related by*

$$\chi_V = \chi_W \cdot \chi_\phi$$

where χ_ϕ is the character of the component group $\mathcal{S}_\phi$ defined earlier.

Remark 3.2. If for an irreducible admissible representation π of $U(V)$, the theta lift of π to $U(W)$ is non-zero for a Hermitian space W with $\dim V = \dim W$, then from the above conjecture together with Remark 3.1 above, and Equation 1.1, the discriminants of V and W are related by

$$\omega_{K/k}(\mathrm{disc} V) = \epsilon(\phi_K \otimes \chi, \psi_K) \det(\phi_K \otimes \chi)(\delta^{-1}) \omega_{K/k}(\mathrm{disc} W),$$

where ϕ_K is the parameter of π base changed to K.

Remark 3.3. Conjecture 1 has been motivated by the work of Rogawski [**Ro1**] for $U(1)$, the work of Harris-Kudla-Sweet [**HKS**], and the work of J.-S. Li [**Li1**]. Indeed, parts (i) and (ii) of Conjecture 1 already appear in the work of Harris-Kudla-Sweet. The Langlands parameter of representations associated by theta correspondence was studied by Adams in [**Ad**] in the Archimedean case.

4. Theta lifting for unitary groups: The Archimedean case.

We verify Conjecture 1 for the discrete series representations of the unitary group $U(p,q)$ here. The discrete series representations of $U(p,q)$ are parametrized by $p+q$ tuples of distinct numbers $(\lambda_1, \lambda_2, \cdots, \lambda_{p+q})$ where all the λ_i are integers if $p+q$ is odd, and all the λ_i are half-integers (i.e., belong to $\frac{1}{2}\mathbb{Z}$ but not to $\mathbb{Z}$) if n is even. We can further assume that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_p, \qquad \text{and}$$
$$\lambda_{p+1} > \lambda_{p+2} > \cdots > \lambda_{p+q}.$$

Assume that

$$\lambda_1 > \lambda_2 > \cdots \lambda_a \geq 0 > \lambda_{a+1} > \cdots > \lambda_p,$$
$$\lambda_{p+1} > \cdots > \lambda_{p+b} \geq 0 > \lambda_{p+b+1} > \cdots \lambda_{p+q},$$

then it follows from the results of J.-S. Li [**Li1**] that the local theta lift of π_λ to $U(r,s)$ for $r+s = p+q$, is non-zero if

$$r = a - b + q,$$
$$s = b - a + p,$$

and in this case the local theta lift is again a discrete series whose Harish-Chandra parameter is

$$(\lambda_1, \ldots, \lambda_a, \lambda_{p+b+1}, \ldots, \lambda_{p+q}, \lambda_{p+1}, \ldots, \lambda_{p+b}, \lambda_{a+1}, \ldots, \lambda_p).$$

Actually as Li is concerned with local theta lifts of discrete series representations of $U(p,q)$ to general unitary groups, he expresses his results in terms of Vogan-Zuckerman $A_q(\lambda)$ which even in the case of same size unitary groups that we are considering, seem much more complicated because of various shifts involved.

This result of J.-S. Li was completed by A. Paul [**Pa**] who proved that the local theta lift of π_λ to $U(r,s)$ for $r+s = p+q$, is non-zero if and only if

$$r = a - b + q,$$
$$s = b - a + p.$$

Given this result of J.-S. Li and A.Paul, Conjecture 1 can now be easily checked for discrete series representations of $U(p,q)$, but before we can do that, we need to review the Vogan parametrization of representations of real reductive groups.

Let G be a connected real reductive group, and ${}^LG$ its L-group which is a semi-direct product of its dual group $G^\vee$ by the Weil group $W_\mathbb{R}$ of $\mathbb{R}$. Let $\phi : W_\mathbb{R} \to {}^LG$ be a discrete series parameter. The centraliser $\mathcal{S}_\phi$ of ϕ in $G^\vee$ is a finite abelian group isomorphic to $(\mathbb{Z}/2)^r$ where r is the rank of G over $\mathbb{C}$. The pure inner forms of $G = U(p,q)$ are $U(p+q,0), U(p+q-1,1), \ldots, U(0,p+q)$. We will fix a quasi split pure inner form of G together with a non-degenerate character on the unipotent radical of a Borel subgroup. According to Vogan, given this data, every character of $(\mathbb{Z}/2)^{p+q}$ determines in a bijective manner, a pure inner form of G and a discrete series representation on it with parameter ϕ. Starting with a pure inner form and a discrete series representation on it, one constructs the character of $(\mathbb{Z}/2)^{p+q}$ as follows. Let T be a compact Cartan subgroup of G, and $\chi : T \to S^1$ a regular character, i.e., $\langle \chi, \alpha \rangle \neq 0$ for every root α of G. This defines a system of positive roots by declaring a root to be positive if and only if $\langle \chi, \alpha \rangle > 0$. Define a root to be compact if the corresponding root space is contained in the unique maximal compact subgroup of G containing T, and non-compact otherwise.

The torus T in G with associated positive root system gives rise to a torus $T^\vee$, and a Borel subgroup $B^\vee$ in $G^\vee$, and for each root α of T, we have the coroot $\alpha^\vee : \mathbf{G}_m \to T^\vee$. A coroot $\alpha^\vee$ is called compact if and only if the root α is compact. Now given a discrete series parameter $\phi : W_\mathbb{R} \to {}^LG$, we can assume that its restriction $\phi_{\mathbb{C}^*}$ to $\mathbb{C}^*$ lands inside $T^\vee$. The discrete series representations with Harish-Chandra parameter $\chi : T \to S^1$, has $\phi_{\mathbb{C}^*}$ given by $\phi_{\mathbb{C}^*}(z) = (z/\bar{z})^\chi$ where χ belongs to $X^*(T) \cong X_*(T^\vee)$.

The centraliser of $\phi : W_\mathbb{R} \to {}^LG$ in $G^\vee$ is precisely the elements of order ≤ 2, $T^\vee[2]$, in $T^\vee(\mathbb{C})$. For each positive simple root α, we have $\alpha^\vee : \mathbf{G}_m \to T^\vee$. Let $a_\alpha = \alpha^\vee(-1) \in T^\vee[2]$. Corresponding to the discrete series on G with Harish-Chandra parameter χ, define the character $\chi : T^\vee[2] \to \mathbb{C}^*$ by demanding $\chi(a_\alpha) = -1$ if α is a compact simple root. In the case of unitary groups, the a_α's generate a subgroup of index 2 of $T^\vee[2]$, so one needs one bit more of information to fix χ (which distinguishes $U(p,q)$ from $U(q,p)$), but which we don't describe here.

We will not verify Conjecture 1 for general discrete series representations, but we will do it in a particular case; the general case is very similar. We will take the discrete series representation on $U(p,q)$ with Harish-Chandra parameter

$$\Lambda = \lambda_1 > \cdots > \lambda_p > \lambda_{p+1} > \cdots > \lambda_{p+a} \geq 0 > \lambda_{p+a+1} > \cdots > \lambda_{p+q}.$$

According to Vogan, this representation of $U(p,q)$ defines a character χ_A on the component group $(\mathbb{Z}/2)^{p+q}$ which for the standard basis $\{e_i\}$ of

$(\mathbb{Z}/2)^{p+q}$ has the property that:

$$\chi_A(e_ie_{i+1}) = -1 \qquad \text{for } i = 1, \dots, p+q-1, i \neq p.$$

By the result of J.-S. Li and A. Paul recalled above, the lift of this discrete series representation on $U(p,q)$ is non-zero for $U(p+q-a,a)$, and the Harish-Chandra parameter for the theta lift is:

$$\lambda_1 > \lambda_2 > \cdots > \lambda_p > \lambda_{p+a+1} > \cdots > \lambda_{p+q},$$
$$\lambda_{p+1} > \lambda_{p+2} > \cdots > \lambda_{p+a}.$$

The corresponding character χ_B on the component group $(\mathbb{Z}/2)^{p+q}$ has the property that:

$$\chi_B(e_ie_{i+1}) = -1, \qquad \text{for } i = 1, \dots, p+q-1, i \neq p, i \neq p+a.$$

For $m \in \frac{1}{2}\mathbb{Z}$, let χ_m be the character of $\mathbb{C}^*$ given by $z \to (z/\bar{z})^m$, or $re^{i\theta} \to e^{2im\theta}$. We have [**Ta**, 3.2.5]

$$\epsilon(\chi_m) = i^{-2m}, \qquad \text{if } \ m > 0, \qquad \text{and}$$
$$\epsilon(\chi_m) = i^{2m}, \qquad \text{if } \ m < 0.$$

It follows that for $m \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}$,

$$\chi_m(i)^{-1}\epsilon(\chi_m) = -1 \qquad \text{if } \ m > 0$$

and

$$\chi_m(i)^{-1}\epsilon(\chi_m) = 1 \qquad \text{if } \ m < 0.$$

If n is odd, then the character χ fixed at the beginning of this section (to split $\mathrm{Sp}(V \otimes_K W)$ over $U(V) \times U(W)$) can be taken to be $\chi_{\frac{1}{2}}$, and if n is even, then χ is taken to be the trivial character. It follows that for any value of n, $\Lambda \otimes \chi$ is a tuple consisting of elements in $\frac{1}{2}\mathbb{Z} - \mathbb{Z}$. Therefore the character χ_Λ associated to the parameter Λ is

$$\chi_\Lambda(e_i) = 1, \qquad \text{if } \ i \leq p+a,$$
$$\chi_\Lambda(e_i) = -1, \qquad \text{if } \ i > p+a.$$

Clearly, $\chi_B = \chi_A \cdot \chi_\Lambda$ on the index 2 subgroup of $(\mathbb{Z}/2)^{p+q}$ generated by e_ie_{i+1}. If n is odd, then $-I = (1, \cdots, 1) \in (\mathbb{Z}/2)^{p+q}$ gives a complement to this index 2 subgroup. Since χ_A corresponds to a representation of $U(p,q)$, and χ_B to a representation of $U(p+q-a,a)$, from 1.1, $\chi_B(-I) = \chi_A(-I)(-1)^{q+a}$, verifying $\chi_B = \chi_A\chi_\Lambda$. If n is even, our result has this ambiguity by 2.

Remark 4.1. One needs to keep in mind the role played by the character χ which is used to fix the splitting of the metaplectic cover of $Sp(V \otimes_K W)$ over $U(V) \times U(W)$ in defining the correspondence between representations of $U(V)$ and $U(W)$. In particular, the particular form of the theorem of Li and Paul given here depends on the choice of χ made here, and if we changed this χ, the form of the theorem will also change.

Remark 4.2. Annegret Paul has calculated the theta correspondence for Unitary groups of the same size over $\mathbb{R}$ explicitly for all representations in [**Pa**]. She informs that her results are in conformity with the conjectures here.

5. Global theta correspondence for unitary groups.

Conjecture 2. *Let V be an n-dimensional Hermitian space over a quadratic extension K of a number field k, and let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $U(V)$. Then:*

(i) *There exists an n-dimensional Hermitian space W over K such that the local theta lift is non-zero at each place of k if and only if the global epsilon factor $\epsilon(BC(\pi) \otimes \chi, \psi_K) = 1$, where $BC(\pi)$ denotes the base change of π to $GL(n, K)$.*

(ii) *Suppose that W is an n-dimensional Hermitian space over K such that for each place v of k, the theta lift, call it $\theta(\pi_v)$, of π_v is non-zero. Then the global representation $\otimes_v \theta(\pi_v)$ is a cuspidal automorphic representation of $U(W)$.*

(iii) *If W is as in (ii), then the global theta lift of π to $U(W)$ is non-zero if and only if $L(BC(\pi) \otimes \chi, \frac{1}{2}) \neq 0$.*

Remark 5.1. Part (i) of the above conjecture follows from the local conjecture, specially the Remark 3.2 following from Conjecture 1. When the number field K splits at a place v of k, then the corresponding local unitary group is just $GL(n, k_v)$, and we are granting ourselves that the duality correspondence for the pair $(GL(n, k_v), GL(n, k_v))$ is just the map taking a representation to its contragredient. One also needs to use the classification theorem of Hermitian forms over a global field due to Landherr (or, the Hasse principle in modern language).

Remark 5.2. For $n = 1$, part (i) of the above conjecture is a theorem due to Rogawski, cf [**Ro1**]; part (ii) is a tautology; part (iii) is due to Gelbart and Rogawski [**GR**, Cor.5.2.2]. For $n = 2$, part (iii) of the above conjecture is proved by M. Harris [**Ha**, Thm. 4.5] in most cases, but part (ii) seems open. Since the unitary group in 2 variables is closely connected to $GL(2)$, part (ii) of the above conjecture should be a consequence of Jacquet-Langlands correspondence together with the multiplicity formula of Labesse-Langlands, but which this author has not verified.

Remark 5.3. The reader will of course not have failed to notice the analogy of Conjecture 2 to the theorem of Waldspurger [**Wa**] regarding theta lifting between $PGL(2)$ and $\bar{SL}(2)$. We would like to point out that in Waldspurger's theorem, there was no local condition as the local theta lift from $PGL(2)$ to $\overline{SL}(2)$ is always non-zero.

Remark 5.4. In the general theory of automorphic forms, there is a conjecture due to J. Arthur [**Ar**] building on the work of Labesse-Langlands in the context of $SL(2)$, which answers when a representation of the adele group which is associated to a parameter ϕ is automorphic. This criterion depends on, again conjectural, pairing

$$\langle , \rangle : \ \mathcal{S}_\phi \times \Pi \rightarrow \mathbb{Z}/2,$$

where $\mathcal{S}_\phi$ is the group of connected components of a global parameter ϕ, and Π is the set of representations of the adele group $G(\mathbb{A})$ whose all the local components belong to the L-packet determined by ϕ (which for simplicity we take to be a tempered parameter). For each representation τ in Π, $\chi_\tau(s) = \langle s, \tau\rangle$ defines a character on $\mathcal{S}_\phi$. Suppose that π is a cuspidal automorphic representation belonging to Π which has a Whittaker model. In such a circumstance, by Vogan there are local pairings

$$\langle , \rangle_v : \ \mathcal{S}_{\phi_v} \times \Pi_v \rightarrow \mathbb{Z}/2.$$

It seems natural to expect that these local pairings proposed by Vogan give rise to the global pairing in the sense that for $\tau = \otimes_v \tau_v$, a representation of $G(\mathbb{A})$, $\langle s, \tau\rangle = \prod_v \langle s_v, \tau_v\rangle_v$ for $s \in \mathcal{S}_\phi$ and s_v its image in $\mathcal{S}_{\phi_v}$. The conjecture of Arthur (which in the case of cuspidal tempered representations that we are considering is due to Kottwitz) is that a global representation τ of $G(\mathbb{A})$ is automorphic if and only if the character χ_τ is trivial. Now we will show how this conjecture implies part (ii) of Conjecture 2. So, we start with a tempered, cuspidal automorphic representation $\pi_V = \otimes_v \pi_v$ of $U(V)$, such that $\pi_W = \otimes_v \theta(\pi_v)$ is a non-zero representation of $U(W)(\mathbb{A})$. By Arthur, we are given that the character χ_{π_V} is the trivial character, and we need to verify that χ_{π_W} is also the trivial character. Since the global component group $\mathcal{S}_\phi$ is $\mathbb{Z}/2$ generated by $\pm\mathrm{Id}$ in $GL(n, \mathbb{C})$, from Remark 3.1, this will be so if and only if $\epsilon(BC(\pi) \otimes \chi, \psi_K) = 1$. But since π_V has a non-zero theta lift to $U(W)$, by part (i) of the Conjecture 2, $\epsilon(BC(\pi) \otimes \chi, \psi_K)$ is indeed 1.

Remark 5.5. The situation of theta correspondence for unitary groups of similar size described here should be compared with the situation of theta correspondence between orthogonal and symplectic groups in [**Pr**]. In the case of dual pairs consisting of orthogonal and symplectic groups, the theta correspondence is completely determined (conjecturally) in terms of a simple mapping on the L-group, the induced mapping of which on the component groups determines the Vogan parameters. On the other hand, in the case of dual pair consisting of unitary groups, the mapping on the L-groups is identity, however, theta correspondence gives a non-trivial correspondence, calculated in terms of epsilon factors, on the 'internal structure' of the L-packet.

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Mehta Research Institute
Allahabad-211 019, India
E-mail address: dprasad@mri.ernet.in

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