CONGRUENCE PROPERTIES OF RAMANUJAN'S FUNCTION $\tau(n)$

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Ramanujan's multiplicative arithmetic function $\tau(n)$* is defined by

$$\sum_{n=1}^{\infty} \tau(n)\, x^n = x\,[(1-x)(1-x^2)\ldots\ldots]^{24} \tag{1}$$

He stated, some congruence properties of $\tau(n)$ for the moduli 5, 7 and 23. These were later proved by L. J. Mordell.[1] In this paper I prove congruences to moduli which are divisors of 24, *viz.*,

(*a*) $\tau(n)$ is odd if and only if n is the square of an odd number.
(*b*) $\tau(3n-1) \equiv 0 \pmod 3$.
(*c*) $\tau(4n-1) \equiv 0 \pmod 4$
(*d*) $\tau(6n-1) \equiv 0 \pmod 6$
(*e*) $\tau(8n-1) \equiv 0 \pmod 8$.

These imply in turn

(*f*) $\tau(12n-1) \equiv 0 \pmod{12}$
(*g*) $\tau(24n-1) \equiv 0 \pmod{24}$.

I think, these congruences have not been stated before [except (*a*) and (*e*) proved by Mr. Hansraj Gupta]. These resemble the congruences to $\sigma(n)$ proved by me namely,

If $\sigma(n)$ is the sum of the divisors of the positive integer n then

$$\sigma(kn-1) \equiv 0 \pmod k$$

for $k = 3, 4, 6, 8, 12$ and 24 and $n \geqslant 1$.

2. We shall now prove the congruences of $\tau(n)$. It is well known[1] that

$$[(1-x)(1-x^2)\ldots\ldots]^3 = 1 - 3x + 5x^3 - 7x^6 + \ldots\ldots$$

$$= \sum_{n=1}^{\infty} (-1)^n (2n+1)\, x^{\frac{n\cdot n+1}{2}} \tag{2}$$

* It might be pointed out that Hardy's statement that $\tau(23n) \equiv 0 \pmod{23}$ given on p. 165 is false since $\tau(23) = 18643272 \not\equiv 0 \pmod{23}$.

so that $$\sum_{1}^{\infty} \tau(n)\, x^n = x\,[1 - 3x + 5x^3 - 7x^6 + \ldots.]^8 \qquad (3)$$

But if $f(x)$ is any rational integral function of x then

$$[f(x)]^8 \equiv f(x^8) \pmod 2. \qquad (4)$$

Therefore

$$\sum_{1}^{\infty} \tau(n)\, x^n \equiv x \sum_{0}^{\infty} (-1)^n (2n+1)\, x^{4n^2+4n} \pmod 2.$$

$$\equiv \sum_{0}^{\infty} (-1)^n (2n+1)\, x^{(2n+1)^2} \pmod 2$$

which shows that: $\tau(n)$ is odd if and only if n is the square of an odd number. (5)

3. Differentiating logarithmically both sides of (1) we obtain

$$\sum_{1}^{\infty} n\, \tau(n)\, x^n = \mathrm{P}(x) \sum_{1}^{\infty} \tau(n)\, x^n \qquad (6)$$

where $\mathrm{P}(x) = 1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \ldots\right)$

$$= 1 - 24 \sum_{1}^{\infty} \sigma(n)\, x^n \qquad (7)$$

where $\sigma(n)$ is the sum of the divisors of n.

Equating coefficients of like powers of x we get

$$(1 - m)\,\tau(m) = 24 \sum_{r=1}^{m-1} \sigma(r)\,\tau(m-r) \qquad (8)$$

which shows that since $\tau(n)$ and $\sigma(n)$ are integers.

$$(m - 1)\,\tau(m) \equiv 0 \pmod{24} \quad m \geqslant 1 \qquad (9)$$

Putting in succession $m = 2n, 3n, 3n-1, 4n, 4n-1$ we get

$$\tau(2n) \equiv 0 \pmod 8 \qquad (10)$$

$$\tau(3n) \equiv \tau(3n-1) \equiv 0 \pmod 3 \qquad (11)$$

$$\tau(4n) \equiv \tau(4n-1) \equiv \tau(4n-2) \equiv 0 \pmod 4 \qquad (12)$$

As regards $\tau(6n-1)$ we see that, since -1 is a quadratic non-residue of 6, $6n-1$ cannot be a square and by (5) $\tau(6n-1)$ is even.

Also by (11) $\tau(6n-1)$ is divisible by 3. Thus:

$$\tau(6n-1) \equiv 0 \pmod 6 \qquad (13)$$

For $1 < r \leqslant 8n - 1$ we have

$$\tau(8n - r)\,\sigma(r-1) \equiv 0 \pmod 2. \qquad (14)$$

Because if $r \equiv 7 \pmod 8$ then $r - 1 \equiv 6 \pmod 8$, *i.e.*, $-1 \pmod 4$ so that $\sigma(r-1) \equiv 0 \pmod 2$. For all other r's we see that (14) holds in virtue of 10, (11) and (12) and the congruences of $\sigma(n)$ stated by me.[3]

$$\therefore 2(1-4n)\,\tau(8n-1) = 24 \sum_{r=2}^{8n-1} \sigma(r-1)\,\tau(8n-r)$$

$$\equiv 48 \pmod 8$$

$$\therefore \tau(8n-1) \equiv 0 \pmod 8 \qquad (15)$$

(11) and (12) imply $\tau(12n-1) \equiv 0 \pmod{12}$ and
(11) and (15) imply $\tau(24n-1) \equiv 0 \pmod{24}$.

REFERENCES

1. Hardy, G. H. .. *Ramanujan*, Cambridge, 1940, Lecture X, pp. 165–69.
2. Hansraj Gupta .. *Abstracts of Papers presented to the Thirteenth Conference of Ind. Math. Soc.*, 1943, p. 6.
3. Ramanathan, K. G. .. *Ibid.*, 1943, p. 6.