A NOTE ON CASCADE THEORY WITH IONISATION LOSS

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Received January 3, 1957

Abstract

The cascade theory of cosmic ray showers including ionisation loss is dealt with on the basis of the new approach suggested by us in an earlier contribution and an explicit Mellin transform solution is obtained for the mean number of particles produced in an infinite thickness of matter.

Some years ago, Bhabha and Chakrabarthy (1943) obtained and discussed the equations of the cascade theory expressing the mean behaviour of the particles when collision loss is also taken into account. If $f(E;t)\,dE$ and $g(E;t)\,dE$ are the mean numbers in the energy interval dE of electrons and photons respectively and $R(E'|E)$ and $\rho(E'|E)$ are the cross-sections for radiation and pair-production respectively, the equations are

$$\frac{\partial f(E;t)}{\partial E} = -f(E;t)\int_0^E R(E'|E)\,dE' + \int_{E}^{\infty} f(E';t)\,R(E|E')\,dE' + 2\int_E^{\infty} g(E';t)\,\rho(E|E')\,dE' + \beta\frac{\partial f(E;t)}{\partial E} \tag{1}$$

$$\frac{\partial g(E;t)}{\partial t} = -g(E;t)\int_0^E \rho(E'|E)\,dE' + \int_E^{\infty} f(E';t)\,R(E'-E|E')\,dE' \tag{2}$$

with the initial conditions

$$f(E;0) = \delta(E-E_0) \tag{3}$$

where δ is the Dirac delta function and

$$g(E;0) = 0 \tag{4}$$

corresponding to a shower excited by a single electron of energy E_0. In conformity with the theory of 'product densities'* (which is usually useful for obtaining higher moments of the distribution) we recognise f and g as

* For a formulation of the theory of product densities, *see* Ramakrishnan (1950).

the product densities of degree one of electrons and photons at t. The collision loss in traversing unit thickness of matter is assumed to be a constant, β. Defining the Mellin transforms of $f(E; t)$ and $g(E; t)$ as

$$p(s; t) = \int_0^\infty f(E; t)\, E^{s-1} dE \tag{5}$$

$$q(s; t) = \int_0^\infty g(E; t)\, E^{s-1} dE \tag{6}$$

according to Bhabha and Chakrabarthy, equations (1) and (2) can be reduced to

$$\frac{\partial p(s; t)}{\partial t} = -A_s p(s; t) + B_s q(s; t) - (s-1)\beta p(s-1; t) \tag{7}$$

$$\frac{\partial q(s; t)}{\partial t} = C_s p(s; t) - D q(s; t) \tag{8}$$

A_s, B_s, C_s and D are given by (*see* Bhabha and Chakrabarthy, 1943)

$$A_s = \left(\frac{4}{3} + a\right)\left\{\frac{d}{ds}\log \lfloor s + \gamma - 1 + \frac{1}{s}\right\} + \tfrac{1}{2} - \frac{1}{s(s+1)}$$

$$B_s = 2\left\{\frac{1}{s} - \left(\frac{4}{3} + a\right)\frac{1}{(s+1)(s+2)}\right\}$$

$$C_s = \frac{1}{s+1} + \left(\frac{4}{3} + a\right)\frac{1}{s(s-1)}$$

$$D = \frac{7}{9} - \frac{1}{6}a \tag{9}$$

We note that (7) is a difference-differential equation where the difference relates to the complex variable s and hence cannot be solved by direct iteration. Moreover the dependence on t makes the solution much more difficult. A simple question that can be asked is: What is the spectrum at $t = \infty$? Unfortunately the product density functions $f(E; t)$ and $g(E; t)$ tend to zero as $t \to \infty$ (for $E > 0$) and thus nothing interesting can be obtained. However on the basis of the new approach suggested by us in a previous paper (1956) we can ask for the mean number of particles produced in the entire shower, each of the particles having an energy greater than E_c at the point of its production. We shall show that explicit Mellin transform solution for the mean number of particles in this case can indeed be obtained using standard mathematical techniques.

Let $F(E; t)\,dEdt$ and $G(E; t)\,dEdt$ be the mean numbers of elcetrons and photons produced between t and $t+dt$ respectively with their energies at the points of their production lying between E and $E+dE$. $F(E; t)$ and $G(E; t)$ are product densities of degree one with respect to the two variables E and t. Using elementary probability arguments as we have done in our previous paper, we obtain

$$F(E; t) = 2\int_E^\infty g(E'; t)\,\rho(E|E')\,dE' \tag{10}$$

$$G(E; t) = \int_E^\infty f(E'; t)\,R(E'-E|E')\,dE' \tag{11}$$

Note that (10) and (11) are valid whether or not we take ionisation loss into account. In fact, the dependence of F and G on ionisation loss enters through f and g.

(10) and (11) are amenable to the standard Mellin transform technique. Defining $P(s; t)$ and $Q(s; t)$ as the Mellin transforms with respect to E, we find

$$P(s; t) = \int_0^\infty F(E; t)\,E^{s-1}dE = B_s\,q(s; t) \tag{12}$$

$$Q(s; t) = \int_0^\infty G(E; t)\,E^{s-1}dE = C_s\,P(s; t). \tag{13}$$

The mean numbers of electrons and photons produced in the shower between 0 and t with the 'primitive' energies (energy at the point of production) lying above E_c is given by

$$\epsilon\{N(E_c; t)\} = \frac{1}{2\pi i}\int_0^t d\tau \int_{E_c}^\infty dE \int_{\sigma-i\infty}^{\sigma+i\infty} P(s; \tau)\,E^{-s}ds \tag{14}$$

$$\epsilon\{M(E_c; t)\} = \frac{1}{2\pi i}\int_0^t d\tau \int_{E_c}^\infty dE \int_{\sigma-i\infty}^{\sigma+i\infty} Q(s; \tau)\,E^{-s}ds \tag{15}$$

That these expressions tend to finite limits as $t\to\infty$ is obvious from a physical point of view since the number of particles with non-zero primitive energy by an initial particle of finite energy must be finite. Inverting the order of integrations we write the limits as

$$\epsilon\{N(E_c; \infty)\} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_c^{-s+1}}{s-1}\,B_s ds \int_0^\infty q(s; \tau)\,d\tau \tag{16}$$

$$\epsilon\{M(E_c;\infty)\} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_c^{-s+1}}{s-1} C_s ds \int_0^\infty p(s;\tau)\,d\tau \tag{17}$$

Thus the problem of obtaining the mean numbers reduces to that of solving for $\kappa_1(s)$ and $\kappa_2(s)$ where

$$\kappa_1(s) = \int_0^\infty p(s;\tau)\,d\tau$$

$$\kappa_2(s) = \int_0^\infty q(s;\tau)\,d\tau \tag{18}$$

We next proceed to show that though we cannot obtain explicit solutions of $p(s;t)$ and $q(s;t)$ from (7) and (8), we can solve for $\kappa_1(s)$ and $\kappa_2(s)$ quite easily. Integrating both sides of equations (7) and (8) with respect to t from 0 to ∞ and observing that $p(s;0) = E_0^{s-1}$ and $q(s;0) = 0$ we obtain

$$-E_0^{s-1} = A_s\kappa_1(s) + B_s\kappa_2(s) - (s-1)\beta\kappa_1(s-1) \tag{19}$$

$$0 = C_s\kappa_1(s) - D\kappa_2(s) \tag{20}$$

Eliminating $\kappa_2(s)$, we have

$$-E_0^{s-1} = -A_s\kappa_1(s) + \frac{B_sC_s}{D}\kappa_1(s) - (s-1)\beta\kappa_1(s-1) \tag{21}$$

i.e.,

$$\kappa_1(s) = \frac{E_0^s + \left(\frac{C_{s+1}B_{s+1}}{D} - A_{s+1}\right)\kappa_1(s+1)}{\beta s} \tag{22}$$

This recurrence relation can be continued and we have

$$\kappa_1(s) = \sum_{r=0}^{m-1} \frac{E_0^{s+r}\left(\frac{C_{s+1}B_{s+1}}{D} - A_{s+1}\right)\left(\frac{C_{s+2}B_{s+2}}{D} - A_{s+2}\right)\cdots\left(\frac{C_{s+r}B_{s+r}}{D} - A_{s+r}\right)}{\beta^r s(s+1)\cdots(s+r)}$$

$$+ \frac{\left(\frac{C_{s+1}B_{s+1}}{D} - A_{s+1}\right)\left(\frac{C_{s+2}B_{s+2}}{D} - A_{s+2}\right)\cdots\left(\frac{C_{s+m}B_{s+m}}{D} - A_{s+m}\right)}{\beta^m s(s+1)\cdots(s+m)} \tag{23}$$

From (22), we note $\kappa_1(s+m)$ for very large m is given by

$$\kappa_1(s+m) \simeq \frac{E_0^{s+m}}{\beta s + A_{s+m}} \tag{24}$$

so that the last term in (23) is negligible if m is sufficiently large. Hence we can write formally

$$\kappa_1(s) = \sum_{r=0}^{\infty} \frac{E_0^{s+r} \prod_{i=1}^{r} (C_{s+i}B_{s+i} - DA_{s+i})}{(\beta D)^r s(s+1)\cdots(s+r)} \tag{25}$$

We shall show that the infinite series on the R.H.S. of (25) does converge absolutely so that (25) is the solution for $\kappa_1(s)$.
Writing

$$\kappa_1(s) = \sum_{m=0}^{\infty} \phi(m) \tag{26}$$

we find

$$\frac{\phi(m)}{\phi(m+1)} = \frac{D\beta}{E_0} \cdot \frac{s+m}{C_{s+m}B_{s+m} - DA_{s+m}} \tag{27}$$

For very large values of m, C_{s+m} and B_{s+m} are negligible and hence we write

$$\frac{\phi(m)}{\phi(m+1)} = -\frac{\beta}{E_0} \cdot \frac{s+m}{A_{s+m}} \tag{28}$$

For large m

$$A_{s+m} \simeq \left(\frac{4}{3} + a\right)\left(\frac{d}{dz}\log \lfloor z\right)_{z=s+m} \tag{29}$$

The asymptotic expression for $d/dz \log \lfloor z$ can be derived by standard techniques [*see* for example Jeffreys and Jeffreys (1950), p. 466] as

$$\frac{d}{dz}\log \lfloor z \simeq \log z + \frac{1}{2z} \tag{30}$$

Thus

$$\left|\frac{\phi(m)}{\phi(m+1)}\right| \simeq \frac{\beta}{E_0}\left(\frac{4}{3} + a\right)^{-1} \left|\frac{s+m}{\log(s+m)}\right| \tag{31}$$

so that

$$\left|\frac{\phi(m)}{\phi(m+1)}\right|$$

tends to infinity as $m \to \infty$. Hence the series $\Sigma\,\phi(m)$ converges absolutely.

Substituting (25) into (15) and (16), we find

$$\epsilon\{N(E_c;\infty)\} = \frac{1}{2\pi i} \sum_{m=0}^{\infty} \frac{E_0^{m+1}}{(\beta D)^m} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_0}{E_c}\right)^{s-1} \frac{B_s C_s}{D(s-1)}$$

$$\cdot \frac{\prod_{i=1}^{m} (C_{s+i}B_{s+i} - DA_{s+i})}{s(s+1)\cdots(s+m)} \, ds \tag{32}$$

$$\epsilon \{M(E_c ; \infty)\} = \frac{1}{2\pi i} \sum_{m=0}^{\infty} \frac{E_0^{m+1}}{(\beta D)^m} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_0}{E_c}\right)^{s-1} \frac{B_s}{s-1}$$

$$\cdot \frac{\prod_{i=1}^{m} (C_{s+i}B_{s+i} - DA_{s+i})}{s(s+1)\cdots(s+m)} \, ds \tag{33}$$

We note that the series in (32) and (33) contains inverse powers of β as contrasted with the assumed expansion in terms of positive powers of β obtained by Bhabha and Chakrabarthy (1943, 1948). It is interesting to note that in the case of a simple multiplicative model discussed in connection with the theory of multiple points by us elsewhere, the solution for the mean number is also a series of inverse powers of β.

We are thankful to Dr. V. K. Balachandran for discussions regarding the validity of our iterative procedure.

One of us (S. K. S.) is grateful to the University of Madras for a Government of India Scholarship.

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142-57. Printed at The Bangalore Press, Bangalore City, by C. Vasudeva Rao, Superintendent, and Published by The Indian Academy of Sciences Bangalore.