CONDITIONS OF PLANE ORBITS IN CLASSICAL AND RELATIVISTIC FIELDS

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1. The classical equations for the motion of a test-particle in a field of force (X, Y, Z) are

$$\ddot{x} = X, \; \ddot{y} = Y, \; \ddot{z} = Z. \tag{1}$$

What are the restrictions on X, Y, Z as functions of (x, y, z, t) if the orbit of the test-particle is required to be *invariably* plane? This is a simple interesting question and as it does not appear to have been investigated before we consider it here.

If s denotes the distance of the current point, that is, the position of the test-particle on the orbit at time t from some fixed point on it we have

$$\begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds} \\ \frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \\ \frac{d^3x}{ds^3} & \frac{d^3y}{ds^3} & \frac{d^3z}{ds^3} \end{vmatrix} = \kappa^2 \tau, \tag{3}$$

where κ and τ are the local curvature and torsion respectively. It readily follows that if the orbit is to be plane we must have everywhere on it

$$\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} = 0. \tag{3}$$

(1) and (3) lead to

$$\Sigma \dot{x} \left\{ Y \left(\frac{\partial Z}{\partial x} \dot{x} + \frac{\partial Z}{\partial y} \dot{y} + \frac{\partial Z}{\partial y} \dot{z} + \frac{\partial Z}{\partial t} \right) - Z \left(\frac{\partial Y}{\partial x} \dot{x} + \frac{\partial Y}{\partial y} \dot{y} + \frac{\partial Y}{\partial z} \dot{z} + \frac{\partial Y}{\partial t} \right) \right\} = 0. \tag{4}$$

The last equation can be valid at any point of the field for arbitrary values of the velocities only if

$$Y \frac{\partial Z}{\partial t} - Z \frac{\partial Y}{\partial t} = 0, \quad Z \frac{\partial X}{\partial t} - X \frac{\partial Z}{\partial t} = 0, \quad X \frac{\partial Y}{\partial t} - Y \frac{\partial X}{\partial t} = 0, \tag{5}$$

$$Y\frac{\partial Z}{\partial x}-Z\frac{\partial Y}{\partial x}=0,\quad Z\frac{\partial X}{\partial y}-X\frac{\partial Z}{\partial y}=0,\quad X\frac{\partial Y}{\partial z}-Y\frac{\partial X}{\partial z}=0, \tag{6}$$

$$\begin{aligned} Y\frac{\partial Z}{\partial y}-Z\frac{\partial Y}{\partial y}-X\frac{\partial Z}{\partial x}+Z\frac{\partial X}{\partial x}&=0,\\ Z\frac{\partial X}{\partial z}-X\frac{\partial Z}{\partial z}-Y\frac{\partial X}{\partial y}+X\frac{\partial Y}{\partial y}&=0,\\ X\frac{\partial Y}{\partial x}-Y\frac{\partial X}{\partial x}-Z\frac{\partial Y}{\partial z}+Y\frac{\partial Z}{\partial z}&=0, \end{aligned} \tag{7}$$

From (5) and (6) it follows that:

$$\begin{aligned} X/Y &= \theta(x, y),\\ Y/Z &= \phi(y, z),\\ Z/X &= \psi(z, x), \end{aligned} \tag{8}$$

where θ, ϕ, ψ are arbitrary functions. One may safely conclude that X, Y, Z are of the form:

$$X = Pf(x),\ Y = Pg(y),\ Z = Ph(z), \tag{9}$$

where f, g, h are arbitrary functions of the variables concerned and P is an arbitrary function of x, y, z, t. Substituting for X, Y, Z from (9) in (7) we have

$$\frac{\partial f}{\partial x}=\frac{\partial g}{\partial y}=\frac{\partial h}{\partial z} \tag{10}$$

Hence

$$f = \lambda x + \epsilon_1,\ g = \lambda y + \epsilon_2,\ h = \lambda z + \epsilon_3, \tag{11}$$

where λ, ϵ_1, ϵ_2, ϵ_3 are arbitrary constants. Thus we find that the orbits are always plane in the field of force:

$$X = (\lambda x + \epsilon_1)P,\ Y = (\lambda y + \epsilon_2)P,\ Z = (\lambda z + \epsilon_3)P. \tag{12}$$

It is obvious that the forces are not derivable from a potential function unless P is of the form,

$$P = P(\chi, t), \tag{13}$$

where $$\chi = \lambda(x^2 + y^2 + z^2) + 2\epsilon_1 x + 2\epsilon_2 y + 2\epsilon_3 z + \eta, \tag{14}$$

η being an arbitrary constant.

It can be easily shown that if the forces (12) are supplemented by a resisting force the orbit still remains plane.

It is interesting to notice also that the condition for plane orbits leads to what Bergmann[1] calls the classical force law.

2. The above discussion of a classical field of force suggests an interesting mathematical problem, which appears rather artificial, but which is based all the same on the established practice in general relativity.[2] For a relatvistic line-element of the form,

$$ds^2 = -e^{\lambda}\, dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) + e^{\nu}\, dt^2, \tag{15}$$

$$\lambda = \lambda(r, t), \quad \nu = \nu(r, t), \tag{16}$$

the geodesies provide three differential equations in r, θ, ϕ, t and the first and second order derivatives of r, θ, ϕ with respect to t. If we now assume that these equations represent the motion of a test-particle in the flat space,

$$d\sigma^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{17}$$

the time t being regarded as Newtonian, the question naturally arises as to whether the orbit so obtained is invariably plane. As regards the assumption on which the question is based it may be pointed out that the same assumption underlies the usual calculation of the motion of the perihelion of Mercury in general relativity. Without discussing the justification for such an assumption, which is sanctioned by current usage, we only examine whether it leads to plane orbits invariably. The vanishing of the left-hand side of (2) is now equivalent to

$$\begin{vmatrix} \lambda^1 & \lambda^2 & \lambda^3 \\ \frac{\delta\lambda^1}{\delta\sigma} & \frac{\delta\lambda^2}{\delta\sigma} & \frac{\delta\lambda^3}{\delta\sigma} \\ \frac{\delta^2\lambda^1}{\delta\sigma^2} & \frac{\delta^2\lambda^2}{\delta\sigma^2} & \frac{\delta^2\lambda^3}{\delta\sigma^2} \end{vmatrix} = 0, \tag{18}$$

where $\lambda^1, \lambda^2, \lambda^3$ stand for $dr/d\sigma$, $d\theta/d\sigma$, $d\phi/d\sigma$ respectively and $\delta\lambda^i/\delta\sigma$, $\delta^2\lambda^i/\delta\sigma^2$ are the first and second covariant[3] derivatives of λ^i with respect to σ. From the equations of the geodesies we get

$$\ddot{r} - P\dot{r} + \tfrac{1}{2}\lambda' \dot{r}^2 - re^{-\lambda}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + e^{\nu-\lambda}\nu' + \dot{\lambda}\dot{r} = 0, \tag{19}$$

$$\ddot{\theta} - P\dot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\, \dot{r}^2 = 0, \tag{20}$$

$$\ddot{\phi} - P\dot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0, \tag{21}$$

where $P = \tfrac{1}{2}e^{\lambda-\nu}\dot{\lambda}\dot{r}^2 + \nu'\dot{r} + \tfrac{1}{2}\dot{\nu}$, (22)

it being understood that $\lambda' = \partial\lambda/\partial r$, $\dot{\lambda} = \partial\lambda/\partial t$ here. Thus we find

$$\lambda^1 = \dot{r}/\dot{\sigma}, \quad \lambda^2 = \theta\,\dot{\theta}/\dot{\sigma}, \quad \lambda^3 = \dot{\phi}/\dot{\sigma}; \tag{23}$$

$$\frac{\delta\lambda^1}{\delta\sigma} = \frac{d^2r}{d\sigma^2} - r\left(\frac{d\theta}{d\sigma}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{d\sigma}\right)^2 \tag{24}$$

$$= Q + R\dot{r}, \tag{25}$$

where

$$Q = \frac{1}{\dot{\sigma}^2}[-\tfrac{1}{2}\lambda'\dot{r}^2 + r(e^{-\lambda}-1)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + e^{\nu-\lambda}\nu' + \dot{\lambda}\dot{r} \tag{26}$$

$$R = P - \ddot{\sigma}/\dot{\sigma}^3. \tag{27}$$

In obtaining (25) the independent variable is changed from σ to t and (19) is used. Similarly on using (20) and (21) one obtains

$$\frac{\delta\lambda^2}{\delta\sigma} = R\,\dot{\theta},\quad \frac{\delta\lambda^3}{\delta\sigma} = R\,\dot{\phi}. \tag{28}$$

When (23), (25) and (28) are used in (18) we find that the condition for plane orbits reduces to

$$\frac{Q}{\dot{\sigma}}\begin{vmatrix} \dot{\theta} & \dot{\phi} \\ \frac{\delta^2\lambda^2}{\delta\sigma^2} & \frac{\delta^2\lambda^3}{\delta\sigma^2} \end{vmatrix} = 0, \tag{29}$$

Calculations show that

$$\frac{\delta^2\lambda^2}{\delta\sigma^2} = S\,\dot{\theta},\ \frac{\delta^2\lambda^3}{\delta\sigma^2} = S\,\dot{\phi} \tag{30}$$

where

$$S = \frac{1}{\dot{\sigma}}\left[PR + \frac{1}{r}Q + \dot{R}\right]\cdot \tag{31}$$

Thus (29) is identically satisfied and all the orbits are invariably plane.

3. We have tried to extend the above analysis to the similar question arising out of the most general line-element,

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu. \tag{32}$$

The three equations of motion[4] with $x^4 \equiv t$ as the independent variable are

$$\ddot{x}^i + \Gamma^i_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta - \dot{x}^i\,\Gamma^4_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta = 0,\ i = 1, 2, 3 \tag{33}$$

in the notation of general relativity. Here $\dot{x}^4$ means 1. These equations of motion may be considered, as was once suggested by Rosen,[5] with reference to any Euclidean (three-dimensional) metric,

$$d\sigma^2 = h_{ij}\,dx^i\,dx^j.$$

The condition for plane orbits turns out to be extremely complicated, and as no particular cases of interest arise we do not report here our formal result. The case treated in the second section covers the usual gravitational questions of interest in general relativity, including that of Schwarzschild's external solution. None of the investigators responsible for discussing the motion of the perihelion of Mercury seems to have been aware that once the relativist commits himself to the use of Riemannian co-ordinates the orbit of a test-particle, or of a small planet like Mercury, can be plane only in a special artificial sense. It is in the light of this special sense that we have carried out the discussion of plane orbits in relativity fields.

Summary

We have discovered the most general classical field of force for which the orbit of a test-particle is invariably plane. If the relativistic equations of a test-particle are interpreted in the classical sense, a general result is obtained, of which a particular case accounts for the orbit of Mercury being plane even according to relativists.

References

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