IMRN International Mathematics Research Notices
1999, No. 13

Projective Structures on a Riemann Surface, II

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1 Introduction

This is a continuation of an earlier work [BR] (referred to as Part I), where we studied some algebraic-geometric aspects of projective structures on a compact Riemann surface.

For a compact Riemann surface X, let Δ be the diagonal divisor in $X \times X$, and let p_i, $i = 1, 2$, be the projection of $X \times X$ onto the i-th factor. We denote by Δ_n the n-th order infinitesimal neighborhood of Δ in $X \times X$, defined by the nonreduced divisor $(n+1)\Delta$, and denote by $\mathcal{L}$ a square root of the holomorphic tangent bundle T_X of X.

We recall that a *projective structure* on X is an equivalence class of coverings by holomorphic coordinate charts such that all the transition functions are Möbius transformations. The line bundle $\mathcal{M} := p_1^*\mathcal{L} \bigotimes p_2^*\mathcal{L} \bigotimes \mathcal{O}_{X\times X}(-\Delta)$ admits a natural trivialization over Δ_1. In Part I, it was shown that a projective structure on X can be viewed as a choice of an extension of the trivialization of the line bundle $\mathcal{M}$ on Δ_1 to a trivialization on Δ_2.

The question that we address here is the interpretation of the trivializations of $\mathcal{M}$ on higher order infinitesimal neighborhoods of Δ. As in Part I, the motivation for this comes from mathematical physics. As in the earlier case, the results are of independent geometric interest.

In Theorem 3.6 (cf. Remark 3.1), we prove that for $n \geq 3$, the space of all trivializations of $\mathcal{M}$ on Δ_n, which restrict to the canonical trivialization over Δ_1, is canonically identified with the space of all projective structures on X together with a differential of order i for each $i \in [3, n]$.

In Theorem 3.11, we establish that for $n \geq 2$, the above space of trivializations can be identified with a certain natural class of differential operators on X of order n.

Received 5 January 1999.
Communicated by Enrico Arbarello.

The differential operators in question map sections of $\mathcal{L}^{\otimes(n-1)}$ to sections of $\mathcal{L}^{\otimes(-n-1)}$ and have the following form in local coordinates:

$$\frac{d^n}{dx^n} + \sum_{i=2}^{n} f_i \frac{d^{n-i}}{dx^{n-i}},$$

where f_i are local holomorphic functions.

P. Deligne has shown in [D] that a projective structure can also be defined as the extension of a natural embedding of Δ_2 in a certain projective bundle P_{tg} to an embedding of Δ_3. The projective bundle P_{tg} over X coincides with the projectivized jet bundle $\mathbb{P}(J^1(\mathcal{L}))$.

In Theorem 4.9, we show that the above space of trivializations of $\mathcal{M}$ over Δ_n is canonically identified with the space of all embeddings of Δ_{n+1} into $\mathbb{P}(J^1(\mathcal{L}))$,which extend the above mentioned canonical embedding of Δ_2.

In Sections 5 and 6, we consider vector bundles over X. In Section 5, we recall the construction of a natural flat connection on the endomorphism bundle of a semistable vector bundle E of rank r and degree $r(g-1)$, where $g = \text{genus}(X)$, and with $H^0(X, E) = 0$. The main step in the construction of the flat connection on $\text{End}(E)$, for such a vector bundle E, is the existence of a natural section of $p_1^*E \bigotimes p_2^*(E^* \bigotimes K_X) \bigotimes \mathcal{O}_{X\times X}(\Delta)$. Here we give an interpretation of this section as the kernel of the inverse of the Dolbeault operator on E.

A vector bundle E of the above type is known to give a projective structure on X. In Section 6, we make some observations on the spaces of differential operators associated to E, by making use of the flat connection on E and the projective structure on X.

The results of Appendices A and B are used in Sections 2 and 3.

We now describe briefly the relationship of this work with some questions concerning conformal quantum field theory (CQFT) and vertex algebras (see [K]). Those with no interest in such questions may proceed immediately to Section 2.

The present work is part of a continuing study of a certain model quantum field theory on a curve, known as the $b-c$ system, which appears in CQFT and string theory (see [R2] for a review of earlier work). This study is based on the development of a geometric understanding of its *operator product expansion (OPE)* on a compact Riemann surface of arbitrary genus. The concept of an OPE has been made precise in the theory of Vertex Algebras, a mathematically precise formulation of CQFT on the complex plane (see [K]).

The OPE of the $b - c$ system on the complex plane is a Laurent expansion of the product of the two quantum fields $b(z)$ and $c(w)$ in powers of $z - w$. The singular part is a first order pole, while the holomorphic part has as coefficients *normal ordered* products $:\partial^{n-1}bc:(w)$, which are generators of certain algebras. For $n = 1$ and 2, the algebras in question are infinite oscillator and Virasoro algebras, respectively (see [KR]); for $n \geq 2$, it is an infinite W-algebra (see [FKRW]).

The OPE is usually defined only on the complex plane, though recent work of Beilinson and Drinfeld (see [G]) gives it a meaning in the higher genus case as well. Our viewpoint is different: as shown in earlier work (see [R2] and references therein), the singular part of the OPE leads to the consideration of the line bundle $\mathcal{M}$ on $X \times X$. In the present paper, we make the geometric *ansatz* that the sum of the one point function of the first n ($n \geq 2$) terms of the *holomorphic* part of this OPE can be identified with *trivialisations* of the line bundle $\mathcal{M}$ on Δ_n which restrict to a canonical trivialisation on Δ_2. In [R1] and Part I, the cases $n = 1$ and $n = 2$ were studied, respectively. Theorems 3.6, 3.11, and 4.9, stated above, now give three different global geometric interpretations to this sum of terms coming from the OPE of the $b - c$ system. Mathematical physicists can immediately recognise them as describing aspects of so-called "W-geometry." Remarkably, our geometric ansatz correctly captures the geometry of the very complicated algebraic expansion into normal-ordered operators, which is the OPE. Those familiar with the Grassmannian formulation of soliton equations (see [SW]) may note some fascinating parallels between our results and some formulas to be found there (see also [M]).

2 Projective structures on a Riemann surface

A *projective atlas* on a Riemann surface X is a covering $\{U_\alpha, \phi_\alpha\}_{\alpha \in I}$ of X by holomorphic coordinate charts, where ϕ_α is a biholomorphism from U_α to an open set in $\mathbb{C}$, such that any composition of maps $\phi_\beta \circ \phi_\alpha^{-1}$, $\alpha, \beta \in I$, is the restriction of a Möbius transformation to $\phi_\alpha\left(U_\alpha \cap U_\beta\right)$. Such an atlas on X gives a one cocycle on X with values in $PSL(2, \mathbb{C})$, the Möbius group of automorphisms of $\mathbb{CP}^1$. Another projective atlas $\{U_\alpha, \phi_\alpha\}_{\alpha \in I'}$ is called *equivalent* to $\{U_\alpha, \phi_\alpha\}_{\alpha \in I}$ if $\{U_\alpha, \phi_\alpha\}_{\alpha \in I \cup I'}$ is also a projective structure. A *projective structure* on X is an equivalence class of compatible projective atlases, and it determines an element in $H^1(X, PSL(2, \mathbb{C}))$ (see [Gu]).

Let X be a compact Riemann surface equipped with a projective structure $\mathfrak{P}$. Choosing a line bundle $\mathcal{L}$, with $\mathcal{L}^{\otimes 2} = T_X$, is equivalent to choosing a lift of the element corresponding to $\mathfrak{P}$, in $H^1(X, PSL(2, \mathbb{C}))$, to an element in $H^1(X, SL(2, \mathbb{C}))$ (see [T]). The map from $H^1(X, SL(2, \mathbb{C}))$ to $H^1(X, PSL(2, \mathbb{C}))$ in question is the one induced by the natural projection of $SL(2, \mathbb{C})$ onto $PSL(2, \mathbb{C})$.

2(a) A line bundle on the self-product

We first recall an alternative description of a projective structure from Part I.

Let X be a compact connected Riemann surface. Fix a line bundle $\mathcal{L}$ on X, which is a square-root of T_X, along with an isomorphism

$$\Psi\colon \mathcal{L}^{\otimes 2} \longrightarrow T_X. \tag{2.1}$$

Let p_i, $i = 1, 2$, denote the natural projection of $X \times X$ onto the i-th factor.

It was shown in Part I that the space of all projective structures on X has the following description in terms of the line bundle $\mathcal{M} := p_1^*\mathcal{L} \bigotimes p_2^*\mathcal{L} \bigotimes \mathcal{O}_{X\times X}(-\Delta)$ on $X \times X$.

Theorem 2.2. Take any nonzero integer n. The restriction of the line bundle $\mathcal{M}^{\otimes n}$ on $X \times X$ to the (nonreduced) divisor 2Δ has a canonical trivialization. This trivialization is invariant under the involution of $X \times X$, defined by switching coordinates if n is even; and it is anti-invariant if n is odd. To each projective structure on X, there is a naturally associated trivialization of the formal completion of $\mathcal{M}^{\otimes n}$ along Δ, invariant (respectively, anti-invariant) under the involution of $X \times X$ if n is even (respectively, odd), with the property that the trivialization restricts to the natural trivialization of $\mathcal{M}^{\otimes n}$ over 2Δ. Moreover, this association gives a bijective map between the space of all projective structures on X and the space of all trivializations of $\mathcal{M}^{\otimes n}$ over 3Δ that restrict to the natural trivialization over 2Δ. □

Remark 2.3. (1) A trivialization of the formal completion of $\mathcal{M}^{\otimes n}$ along Δ means compatible trivializations of the line bundle $\mathcal{M}^{\otimes n}$ over infinitesimal neighborhoods of Δ in $X \times X$ of every order [Ha, p. 194]. This is equivalent to trivializing $\mathcal{M}^{\otimes n}$ over some analytic neighborhood of Δ.

(2) The restriction of the line bundle $\mathcal{O}_{X\times X}(-n\Delta)$ to Δ is $K_\Delta^{\otimes n}$, the n-th tensor power of the cotangent bundle; and hence the restriction of $\mathcal{M}^{\otimes n}$ to Δ is the trivial line bundle. If n is even, then the trivialization is canonical; if n is odd, then there is a natural trivialization up to sign. This indeterminacy of sign can be removed by using the ordering of factors in the Cartesian product $X \times X$. There is a natural trivialization of $\mathcal{M}^{\otimes n}$ over 2Δ, which is an extension of the trivialization over Δ. A description of this trivialization using coordinate charts is given in (2.4). This trivialization over 2Δ is determined by the condition that when n is even (respectively, odd), then it is invariant (respectively, anti-invariant) under the involution of $X \times X$ defined by switching coordinates. Thus, if the involution and the trivialization are denoted by τ and s, respectively, then s coincides with τ^*s (respectively, $-\tau^*s$) if n is even (respectively, odd).

Denoting by q_i, $i = 1, 2$, the projection of $\mathbb{CP}^1 \times \mathbb{CP}^1$ onto the i-th factor, the line bundle $q_1^*\mathcal{O}(n) \bigotimes q_2^*\mathcal{O}(n) \bigotimes \mathcal{O}_{\mathbb{CP}^1\times\mathbb{CP}^1}(-n\Delta_0)$, where $\Delta_0 \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is the diagonal divisor, has a canonical trivialization given by the section

$$s(n) := \left(z_1 - z_2\right)^n \left(\frac{\partial}{\partial z_1}\right)^{n/2} \bigotimes \left(\frac{\partial}{\partial z_2}\right)^{n/2}, \tag{2.4}$$

where $\left(z_1, z_2\right)$ is the coordinate function on $\mathbb{CP}^1 \times \mathbb{CP}^1$ obtained from the obvious coordinate function z on $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, and $\left(\frac{\partial}{\partial z}\right)^{1/2}$ denotes a section of $\mathcal{O}(1)$ such that $\left(\left(\frac{\partial}{\partial z}\right)^{1/2}\right)^{\otimes 2}$ coincides with the section $\frac{\partial}{\partial z}$ by the natural isomorphism between $\mathcal{O}_{\mathbb{CP}^1}(2)$ and $T_{\mathbb{CP}^1}$. Note that though there are two choices for the section $\left(\frac{\partial}{\partial z}\right)^{1/2}$, the section in (2.4) does not depend upon the choice.

For any biholomorphism $\phi\colon U \longrightarrow U'$, between open sets of $\mathbb{CP}^1$, the two sections, $s(n)$ and $(\phi, \phi)^* s(n)$, respectively, defined over $U \times U$, actually coincide when they are restricted to $2\Delta_0 \cap (U \times U)$. Thus, $s(n)$ gives a trivialization of $\mathcal{M}^{\otimes n}$ over 2Δ for any Riemann surface X. Indeed, the above property of $s(n)$ ensures that for any two coordinate charts on X, the two pullbacks of $s(n)$ using the two coordinate functions actually coincide on the first order infinitesimal neighborhood of the diagonal.

We noted earlier that for a projective structure $\mathfrak{P}$ on X, the element

$$\rho \in H^1(X, PSL(2, \mathbb{C})),$$

corresponding to $\mathfrak{P}$, lifts to a cohomology class

$$\overline{\rho} \in H^1(X, SL(2, \mathbb{C}))$$

using $\mathcal{L}$. The line bundle on X associated to $\overline{\rho}$, for the natural action of $SL(2, \mathbb{C})$ on $\mathcal{O}_{\mathbb{CP}^1}(1)$, is canonically identified with $\mathcal{L}$.

The diagonal action of $SL(2, \mathbb{C})$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ naturally lifts to the line bundle $q_1^*\mathcal{O}(n) \bigotimes q_2^*\mathcal{O}(n) \bigotimes \mathcal{O}(-n\Delta_0)$, preserving its trivialization.

For a covering of X by coordinate charts compatible with a given projective structure, the pullbacks of $\mathcal{O}_{\mathbb{CP}^1}(1)$ by coordinate maps patch together to produce the line bundle $\mathcal{L}$ on X. The trivialization of $q_1^*\mathcal{O}(n) \bigotimes q_2^*\mathcal{O}(n) \bigotimes \mathcal{O}(-n\Delta_0)$ gives a trivialization of $\mathcal{M}^{\otimes n}$ over an analytic neighborhood of Δ. The association between projective structures and trivializations, which is mentioned in Theorem 2.2, is obtained by restricting this trivialization to the completion of $X \times X$ along Δ.

Theorem 2.2 has been proved in Part I with $n = 2$. The proof is identical to it in any other case.

Restricting $\mathcal{M}^{\otimes n}$ to 2Δ and 3Δ, respectively, and using the trivialization of $\mathcal{M}^{\otimes n}$ over Δ, we have the exact sequence

$$0 \longrightarrow K_\Delta^{\otimes 2} \longrightarrow \left(\mathcal{M}^{\otimes n}\right)\big|_{3\Delta} \longrightarrow \left(\mathcal{M}^{\otimes n}\right)\big|_{2\Delta} \longrightarrow 0.$$

Hence, if $\overline{\mathfrak{P}}$ and $\overline{\mathfrak{P}'}$ denote the trivializations of $(\mathcal{M}^{\otimes n})|_{3\Delta}$ corresponding to any two projective structures $\mathfrak{P}$ and $\mathfrak{P}'$ on X, then

$$\overline{\mathfrak{P}'} - \overline{\mathfrak{P}} \in H^0\left(X, K_X^{\otimes 2}\right),$$

since $\overline{\mathfrak{P}}$ and $\overline{\mathfrak{P}'}$ agree over 2Δ.

On the other hand, the space of all projective structures on X is an affine space for $H^0\left(X, K_X^{\otimes 2}\right)$ (see [Gu]). Thus

$$\mathfrak{P}' - \mathfrak{P} \in H^0\left(X, K_X^{\otimes 2}\right).$$

The relationship between these two observations is provided by the following identity (see Lemma 3.6 of Part I):

$$\overline{\mathfrak{P}'} - \overline{\mathfrak{P}} = \frac{n}{12}\left(\mathfrak{P}' - \mathfrak{P}\right). \tag{2.5}$$

2(b) Decomposition of a differential operator

For a holomorphic vector bundle E on X and a positive integer n, the n-th order *jet bundle* of E, denoted by $J^n(E)$, is defined to be the following direct image on X:

$$J^n(E) := p_{1*}\left(\frac{p_2^* E}{p_2^* E \otimes \mathcal{O}_{X\times X}(-(n+1)\Delta)}\right),$$

where, as before, p_i is the projection of $X \times X$ onto the i-th factor. Since Δ is an effective divisor, $p_2^* E \bigotimes \mathcal{O}_{X\times X}(-(n+1)\Delta)$ is a subsheaf of $p_2^* E \bigotimes \mathcal{O}_{X\times X}(-n\Delta)$. So, there is a natural exact sequence

$$0 \longrightarrow K_X^{\otimes(n+1)} \otimes E \longrightarrow J^{n+1}(E) \longrightarrow J^n(E) \longrightarrow 0.$$

The inclusion $K_X^{\otimes(n+1)} \bigotimes E \longrightarrow J^{n+1}(E)$ is constructed by using the inclusion

$$K_X^{\otimes(n+1)} \longrightarrow J^{n+1}\left(\mathcal{O}_X\right),$$

which is defined at $x \in X$ by $(df)^{\otimes(n+1)} \longmapsto f^{n+1}/(n+1)!$, where f is any function with $f(x) = 0$.

The sheaf of *differential operators* $\mathrm{Diff}_X^n(E, F)$ coincides with $\mathrm{Hom}(J^n(E), F)$. The homomorphism

$$\sigma\colon \mathrm{Diff}_X^n(E, F) \longrightarrow \mathrm{Hom}\left(K_X^{\otimes n} \otimes E, F\right),$$

which is obtained by restricting a homomorphism from $J^n(E)$ to F to the subsheaf $K_X^{\otimes n} \bigotimes E$, is known as the *symbol map*.

Furthermore, for any two integers $r, s \in \mathbb{N}$, the natural projection

$$J^{r+s}(E) \longrightarrow J^s(E)$$

admits a canonical lift

$$\mu: J^{r+s}(E) \longrightarrow J^r\left(J^s(E)\right),$$

which is an injective homomorphism of vector bundles.

Now consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}^{-n-1} = K_X^{\otimes n} \otimes \mathcal{L}^{\otimes(n-1)} & \longrightarrow & J^n\left(\mathcal{L}^{\otimes(n-1)}\right) & \longrightarrow & J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right) & \longrightarrow & 0 \\ & & & & \downarrow\mu & & \| & & \\ 0 & \longrightarrow & K_X \otimes J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right) & \longrightarrow & J^1\left(J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right)\right) & \longrightarrow & J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right) & \longrightarrow & 0. \end{array} \tag{2.6}$$

The differential operator $\mathcal{D}_{\mathfrak{P}}(n)$ constructed in (B.4) of Appendix B gives a homomorphism

$$D_n: J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right) \longrightarrow J^n\left(\mathcal{L}^{\otimes(n-1)}\right), \tag{2.7}$$

which is a splitting of the top exact sequence in (2.6). The composition $\mu \circ D_n$ is a splitting of the bottom exact sequence in (2.6). In other words, $\mu \circ D_n$ is a holomorphic connection on $J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right)$. Thus we have the following lemma (see [B1, Theorem 4.1]).

Lemma 2.8. For a Riemann surface X with a projective structure, for any $n \in \mathbb{N}$, the jet bundle $J^n\left(\mathcal{L}^{\otimes n}\right)$ is equipped with a natural flat connection. □

Using the splitting D_n in (2.7) as the base point, the space of all splittings of the top exact sequence in (2.6) is identified with

$$H^0\left(X, \mathrm{Diff}_X^{n-1}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right).$$

Similarly, the space of all splittings of the bottom exact sequence in (2.6) is identified with

$$H^0\left(X, \mathrm{End}\left(J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right)\right) \otimes K_X\right).$$

The composition of a splitting of the top exact sequence in (2.6) with the inclusion μ (in (2.6)) is a splitting of the bottom exact sequence in (2.6). Thus we have a natural homomorphism

$$\Phi: H^0\left(X, \mathrm{Diff}_X^{n-1}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right) \longrightarrow H^0\left(X, \mathrm{End}\left(J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right)\right) \otimes K_X\right).$$

Using the $GL(n, \mathbb{C})$ invariant polynomials $A \longmapsto \mathrm{trace}\left(A^i\right)$, $1 \leq i \leq n$, on $M(n, \mathbb{C})$ we have a map

$$H: H^0\left(X, \mathrm{End}\left(J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right)\right) \otimes K_X\right) \longrightarrow \bigoplus_{i=1}^{n} H^0\left(X, K_X^{\otimes i}\right), \tag{2.9}$$

which is known as the *Hitchin map* (see [Hi]).

The following decomposition of differential operators, which was constructed in Corollary C of [B3], is very useful for our purpose. However, we wish to give here an alternative construction of the decomposition.

Theorem 2.10. Let X be a compact Riemann surface equipped with a projective structure. The map

$$H \circ \Phi : \; H^0\left(X, \mathrm{Diff}_X^{n-1}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right) \longrightarrow \bigoplus_{i=1}^{n} H^0\left(X, K_X^{\otimes i}\right)$$

is bijective. □

Proof. We first consider the case where the genus of X is at least 2. The remaining cases are treated separately.

Our first step is to establish the injectivity of the map $H \circ \Phi$.

For $D \in H^0(X, \mathrm{Diff}_X^{n-1}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}))$, let $\sigma \in H^0(X, K_X^{\otimes i})$ be the component of $H \circ \Phi(D)$ of lowest degree. In other words, $\sigma \neq 0$, and furthermore, if $j < i$, then the component of $H \circ \Phi(D)$ in $H^0(X, K_X^{\otimes j})$ vanishes identically. It is a simple computation to check that the symbol of the differential operator D coincides with σ. (This implies that D is actually of order $n - i$.) Thus $H \circ \Phi$ must be injective.

We complete the proof by showing that the dimensions of the domain and the target of $H \circ \Phi$ coincide.

Using the long exact sequence of cohomologies for the exact sequence of vector bundles on X,

$$0 \longrightarrow \mathrm{Diff}_X^{i}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right) \longrightarrow \mathrm{Diff}_X^{i+1}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right) \longrightarrow K_X^{\otimes(n-i-1)} \longrightarrow 0,$$

it can be deduced that

$$H^1\left(X, \mathrm{Diff}_X^{j}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right) = 0 \tag{2.11}$$

for all $j \leq n-2$. Indeed, if $n \geq 2$, then

$$H^1\left(X, \mathrm{Diff}_X^{0}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right) = H^1\left(X, K_X^{\otimes n}\right) = 0,$$

since the genus of X is at least two. Now (2.11) follows from the exact sequence

$$H^1\left(X, \mathrm{Diff}_X^{j}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right) \longrightarrow H^1\left(X, \mathrm{Diff}_X^{j+1}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right)$$
$$\longrightarrow H^1\left(X, K_X^{\otimes(n-j-1)}\right) \longrightarrow 0$$

by using induction on j.

Finally, (2.11) implies that

$$\dim H^0\left(X, \mathrm{Diff}_X^j\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right) = \sum_{i=n-j}^{n} \dim H^0\left(X, K_X^{\otimes i}\right)$$

for all $j \leq n-1$. This can be seen by considering the long exact sequence

$$0 \longrightarrow H^0\left(X, \mathrm{Diff}_X^j\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right) \longrightarrow H^0\left(X, \mathrm{Diff}_X^{j+1}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right)$$
$$\longrightarrow H^0\left(X, K_X^{\otimes(n-j-1)}\right) \longrightarrow H^1\left(X, \mathrm{Diff}_X^j\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right)$$

and using induction on j. Since the map $H \circ \Phi$ descends to a map between weighted projective spaces, it must be bijective. This completes the proof of the theorem in the case where the genus of X is at least 2.

If $X = \mathbb{CP}^1$, then the dimension of both the domain and the target is 0. To see that the domain has dimension 0, observe that $J^{n-1}(\mathcal{L}^{\otimes(n-1)})$ is a trivial bundle, as it admits a flat connection ensured by Lemma 2.8.

If X is an elliptic curve, then $\mathrm{Diff}_X^{n-1}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)})$ is actually isomorphic to $\bigoplus_{i=1}^n H^0(X, K_X^{\otimes i})$. This follows from the observation that $J^{n-1}(\mathcal{L}^{\otimes(n-1)})$ is isomorphic to the symmetric power $S^{n-1}(J^1(\mathcal{L}))$ (see [B1]). This completes the proof of the theorem. ■

The isomorphism $H \circ \Phi$ depends upon the projective structure that was used in its construction. The homomorphisms Φ or H are not individually isomorphisms. In fact, if $g > 1$, then

$$\dim H^0\left(X, \mathrm{End}\left(J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right)\right) \otimes K_X\right) > \sum_{i=1}^{n} \dim H^0\left(X, K_X^{\otimes i}\right).$$

Now, any element $D \in H^0(X, \mathrm{Diff}_X^n(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}))$ decomposes uniquely as

$$D = c\mathcal{D}_{\mathfrak{P}}(n) + D_0,$$

where $c \in \mathbb{C}$ and $D_0 \in H^0(X, \mathrm{Diff}_X^{n-1}(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}))$. Hence Theorem 2.10 has the following corollary.

Corollary 2.12. For a Riemann surface X with a projective structure, the decomposition

$$H^0\left(X, \mathrm{Diff}_X^n\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right) = \bigoplus_{i=0}^{n} H^0\left(X, K_X^{\otimes i}\right)$$

is valid. □

The differentials appearing on the right-hand side may be viewed as the *Laguerre-Forsyth invariants* of the differential operator (see [W]).

In the next section, we establish an identification between a certain subspace of the space of differential operators $H^0(X, \mathrm{Diff}_X^n(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}))$ and the space of trivializations of $\mathcal{M}$ over an infinitesimal neighborhood of the diagonal.

3 Differential operators on a Riemann surface

We continue with the notation of the previous section, but in the present section, we do not assume that the Riemann surface X is equipped with a projective structure.

For a nonnegative integer n and an integer $d \geq 3$, let

$$S(n, d) \in H^0\left(X \times X, \frac{\mathcal{M}^{\otimes n}}{\mathcal{M}^{\otimes n} \otimes \mathcal{O}_{X\times X}(-d\Delta)}\right) \tag{3.1}$$

be a trivialization of $\mathcal{M}^{\otimes n}$ over $d\Delta$, such that its restriction to Δ is the constant function 1. Furthermore, denoting the involution of $X \times X$ by τ, the section $S(n, d)$ is required to coincide with $\tau^* S(n, d)$ (respectively, $-\tau^* S(n, d)$) if n is even (respectively, odd). Hence the restriction of $S(n, d)$ to 2Δ is the canonical trivialization referred to in Theorem 2.2.

Consider the restriction of the section $S(n, d)$ to 3Δ. We denote by $\mathfrak{P}_{S(n,d)}$ the natural projective structure on X associated to it by Theorem 2.2. Let

$$S'(n, d) \in H^0\left(X \times X, \frac{\mathcal{M}^{\otimes n}}{\mathcal{M}^{\otimes n} \otimes \mathcal{O}_{X\times X}(-d\Delta)}\right) \tag{3.2}$$

be the trivialization associated to the projective structure $\mathfrak{P}_{S(n,d)}$ using Theorem 2.2.

Let ξ denote the direct image

$$\xi := p_{1*}\left(\frac{S(n, d)}{S'(n, d)}\right) \in H^0\left(X, J^{d-1}\left(\mathcal{O}_X\right)\right)$$

on X of the function over $d\Delta$ given by the quotient of the two trivializations defined in (3.1) and (3.2), respectively. The image of ξ in $J^2(\mathcal{O}_X)$ coincides with the image of the constant function 1.

Setting $d = n + 2$, let

$$M_\xi : J^{n+1}(\mathcal{L}^{\otimes n}) \longrightarrow J^{n+1}(\mathcal{L}^{\otimes n})$$

be the homomorphism defined by multiplication with ξ. The multiplication in question is the natural surjective homomorphism

$$J^n(L) \bigotimes J^n(L') \longrightarrow J^n(L \bigotimes L'),$$

where L and L' are any two line bundles on X, which sends any two local sections s and s' of L and L', respectively, to $s \otimes s'$.

Notation 3.3. For any integer $n \geq 1$, let $\mathcal{T}(n)$ denote the space of all trivializations $S(n, n+2)$ of $\mathcal{M}^{\otimes n}$ over $(n + 2)\Delta$ that restrict to the canonical trivialization over 2Δ.

Note that there is a one-to-one correspondence between $\mathcal{T}(n)$ and the space of all trivializations of $\mathcal{M}$ over $(n+2)\Delta$ that restrict to the canonical trivialization over 2Δ. The map is defined by simply taking the n-th tensor power of a section of $\mathcal{M}$ over $(n + 2)\Delta$.

For any $S(n, n+2) \in \mathcal{T}(n)$, let $\mathcal{D} = \mathcal{D}_{\mathfrak{P}_{S(n,n+2)}}(n+1)$ be the differential operator defined in (B.4) for the projective structure $\mathfrak{P}_{S(n,n+2)}$ constructed earlier. Since $\mathcal{D}$ is a homomorphism from $J^{n+1}(\mathcal{L}^{\otimes n})$ to $\mathcal{L}^{\otimes(-n-2)}$, the composition

$$\mathcal{D}_S := \mathcal{D} \circ M_\xi \tag{3.4}$$

is an element of $H^0(X, \mathrm{Diff}_X^{n+1}(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}))$. The earlier observation that the image of ξ in $J^2(\mathcal{O}_X)$ coincides with the image of the constant function 1, implies that the difference

$$\mathcal{D}_S - \mathcal{D} \in H^0\left(X, \mathrm{Diff}_X^{n-2}\left(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}\right)\right).$$

Let $\mathfrak{P}(X)$ denote the space of all projective structures on the Riemann surface X. The following theorem identifies $H^0(X, \mathrm{Diff}_X^{n-2}(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}))$ with the space of trivializations.

Theorem 3.5. For a compact Riemann surface X, the map

$$W\colon \mathcal{T}(n) \longrightarrow \mathfrak{P}(X) \times H^0\left(X, \mathrm{Diff}_X^{n-2}\left(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}\right)\right),$$

defined by $S(n, n+2) \longmapsto (\mathfrak{P}_{S(n,n+2)}, \mathcal{D}_S - \mathcal{D})$, where $\mathfrak{P}_{S(n,n+2)}$ was constructed earlier and $\mathcal{D}_S$ was defined in (3.4), is bijective. □

Proof. To construct the inverse of W, for any

$$(\mathfrak{P}, D) \in \mathfrak{P}(X) \times H^0\left(X, \mathrm{Diff}_X^{n-2}\left(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}\right)\right),$$

consider the differential operator $\mathcal{D}' := \mathcal{D}_{\mathfrak{P}}(n+1) + D$, where $\mathcal{D}_{\mathfrak{P}}(n+1)$ is the operator constructed in (B.4). Let

$$M\colon J^{n+1}(\mathcal{L}^{\otimes n}) \longrightarrow J^{n+1}(\mathcal{L}^{\otimes n})$$

be the homomorphism defined by the identity $\mathcal{D}' = \mathcal{D}_{\mathfrak{P}}(n+1) \circ M$.

Let $S \in \mathcal{T}(n)$ be the trivialization associated to the projective structure $\mathfrak{P}$ by Theorem 2.2. Since $M \in H^0((n+2)\Delta, \mathcal{O})$, the product $M.S \in \mathcal{T}(n)$.

Associating $M.S$ to $(\mathfrak{P}, D)$, the inverse of the map W is obtained. It is rather straightforward to check that this map is indeed the inverse of W. ■

From Theorem 2.10, together with the fact used in the proof of Theorem 2.10 that for any

$$D' \in H^0\left(X, \mathrm{Diff}_X^{n-1}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right),$$

the lowest order term of $H \circ \Phi(D')$ is the symbol of D', it immediately follows that if X is equipped with a projective structure, then

$$H^0\left(X, \mathrm{Diff}_X^{n-2}\left(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}\right)\right) = \bigoplus_{i=3}^{n+1} H^0\left(X, K_X^{\otimes i}\right).$$

Now the following theorem is a consequence of Theorems 2.10 and 3.5.

Theorem 3.6. For a compact Riemann surface X, the space of trivializations $\mathcal{T}(n)$ is identified, in a canonical fashion, with the Cartesian product

$$\mathfrak{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0\left(X, K_X^{\otimes i}\right) \right). \qquad \square$$

Let

$$\Sigma(n+1) \subset H^0\left(X, \mathrm{Diff}_X^{n+1}\left(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}\right)\right) \tag{3.7}$$

be the subset consisting of all differential operators with symbol 1. The vector space

$$\mathcal{V} := H^0\left(X, \mathrm{Diff}_X^{n-1}\left(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}\right)\right)$$

acts on $\Sigma(n+1)$ simply by addition.

It is somewhat surprising (at least to the authors) that the action of $\mathcal{V}$ on $\Sigma(n+1)$ has a distinguished orbit. This is the orbit containing the differential operator $\mathcal{D}_{\mathfrak{P}}(n+1)$, defined in (B.4), where $\mathfrak{P}$ is a projective structure on X. The observation noted down in (B.6) that the difference of two such operators differ by an operator of degree $n-1$, implies that the orbit defined above does not depend upon the choice of the projective structure.

Let

$$\Sigma_0(n+1) \subset \Sigma(n+1) \tag{3.8}$$

denote the distinguished orbit for the action of $\mathcal{V}$.

Remark 3.9. Using the exact sequence of coherent sheaves,

$$0 \longrightarrow K_X^{\otimes(i+1)} \otimes \mathcal{L}^{\otimes(n-1)} \longrightarrow J^{i+1}(\mathcal{L}^{\otimes(n-1)}) \longrightarrow J^i(\mathcal{L}^{\otimes(n-1)}) \longrightarrow 0,$$

and the isomorphism between $\mathcal{L}^{\otimes 2}$ and T_X, it readily follows that

$$\bigwedge^n J^{n-1}(\mathcal{L}^{\otimes(n-1)}) = \mathcal{O}_X. \tag{3.10}$$

Hence, the top exterior product $\bigwedge^n J^{n-1}(\mathcal{L}^{\otimes(n-1)})$ has a canonical trivialization.

We describe a property of the class of differential operators $\Sigma_0(n)$ defined in (3.8), in terms of the trivializations above. For a $D \in \Sigma_0(n)$, let ∇ be the holomorphic connection on $J^{n-1}(\mathcal{L}^{\otimes(n-1)})$ constructed using the map μ in (2.6) and the splitting given by D. It is straightforward to check that the connection on $\bigwedge^n J^{n-1}(\mathcal{L}^{\otimes(n-1)})$ induced by ∇ coincides with the natural connection on $\mathcal{O}_X$ using the isomorphism (3.10).

Consider the map

$$\mathfrak{P}(X) \times H^0\left(X, \mathrm{Diff}_X^{n-2}\left(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}\right)\right) \longrightarrow \Sigma_0(n+1),$$

defined by $(\mathfrak{P}, D) \longmapsto \mathcal{D}_{\mathfrak{P}}(n+1) + D$. Lemma B.9 implies that this map is an isomorphism.

Now the following theorem is easily derived from Theorem 3.5.

Theorem 3.11. For any compact Riemann surface X, the space of trivializations $\mathcal{T}(n)$ is canonically identified with the space of differential operators $\Sigma_0(n+1)$. □

For any trivialization $t \in \mathcal{T}(n)$, its restriction to 3Δ gives a projective structure $\mathfrak{P}$ on X. If D is the differential operator that corresponds to t by Theorem 3.11, then $\mathfrak{P}$ is the unique projective structure such that

$$D - \mathcal{D}_{\mathfrak{P}}(n+1) \in H^0\left(X, \mathrm{Diff}_X^{n-2}\left(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}\right)\right).$$

Remark 3.12. Following Notation 3.3, we noted that the space $\mathcal{T}(n)$ is identified in a canonical fashion with the space of all trivializations of $\mathcal{M}$ over $(n+2)\Delta$, which restrict to the canonical trivialization of $\mathcal{M}$ over 2Δ. Thus, denoting the space of such trivializations of $\mathcal{M}$ by $\mathcal{T}_n$, Theorems 3.6 and 3.11 remain valid if $\mathcal{T}(n)$ is replaced with $\mathcal{T}_n$ in their statements.

Let

$$\mathbb{P}_X(n) := \mathbb{P}\left(J^{n+1}\left(\mathcal{L}^{\otimes n}\right)\right) \longrightarrow X$$

denote the projective bundle over X of dimension $n+1$ consisting of all lines in the vector bundle $J^{n+1}(\mathcal{L}^{\otimes n})$. Let

$$\mathbb{P}_X := \mathbb{P}\left(J^{n}\left(\mathcal{L}^{\otimes n}\right)\right) \subset \mathbb{P}_X(n)$$

denote the projective bundle over X for the natural surjection of $J^{n+1}(\mathcal{L}^{\otimes n})$ onto $J^{n}(\mathcal{L}^{\otimes n})$.

A differential operator $D \in H^0(X, \mathrm{Diff}_X^{n+1}(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}))$, with the constant function 1 as the symbol, naturally gives a section of the fiber bundle

$$\mathcal{A}_X(n) := \mathbb{P}_X(n) - \mathbb{P}_X$$

by simply considering D as a homomorphism of $J^{n+1}(\mathcal{L}^{\otimes n})$ onto $\mathcal{L}^{\otimes(-n-2)}$.

Conversely, for any section of $\mathcal{A}_X(n)$, there is a differential operator such that the section arises from it in the above fashion. Indeed, any section s of $\mathbb{P}_X(n)$ is given by a surjective homomorphism, say, H, of $J^{n+1}(\mathcal{L}^{\otimes n})$ onto a line bundle ζ over X. If s is a section of $\mathcal{A}_X(n)$, then the restriction of H to $\mathcal{L}^{\otimes(-n-2)}$ is nowhere zero. In other words, ζ is isomorphic to $\mathcal{L}^{\otimes(-n-2)}$ by H.

Thus we have a one-to-one bijective correspondence between the space of sections of $\mathcal{A}_X(n)$ and a subspace of the space of differential operators

$$H^0\left(X, \mathrm{Diff}_X^{n+1}\left(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes(-n-2)}\right)\right).$$

The subspace in question consists of all operators with symbol 1.

Using the above one-to-one correspondence, the class of operators $\Sigma_0(n+1)$ defined in (3.8) gives a class of sections of $\mathcal{A}_X(n)$. Let $\mathcal{S}$ denote this class of sections. It is interesting to be able to characterize geometrically the class of sections $\mathcal{S}$. Also, it is interesting to be able to directly find the element in $\mathcal{T}(n)$, given by Theorem 3.11, which corresponds to any given element of $\mathcal{S}$.

4 Embeddings of the higher order infinitesimal neighborhoods of the diagonal in a projective bundle

The structure sheaf of the subscheme $n\Delta \subset X \times X$ is isomorphic to the ringed space

$$(X, p_{1*}\mathcal{O}_{n\Delta}),$$

where p_1 is the projection of $X \times X$ onto the first factor. The $\mathcal{O}_X$-algebra $p_{1*}\mathcal{O}_{n\Delta}$ on X coincides with the jet bundle $J^{n-1}(\mathcal{O}_X)$ with its natural algebra structure.

Another natural embedding of X in a fiber bundle over X is obtained from the following exact sequence of coherent sheaves:

$$0 \longrightarrow K_X \otimes \mathcal{L} \longrightarrow J^1(\mathcal{L}) \longrightarrow \mathcal{L} \longrightarrow 0.$$

In this case, X is embedded in the projective bundle $\mathbb{P}(X) := \mathbb{P}(J^1(\mathcal{L}))$. Denoting the image of X in $\mathbb{P}(X)$ by $\overline{X}$, the structure sheaf of $n\overline{X}$ is the ringed space

$$(X, q_*\mathcal{O}_{n\overline{X}}), \tag{4.1}$$

where q is the projection of $\mathbb{P}(X)$ onto X.

Lemma 4.2. For any $n \geq 0$, the vector bundle $q_*\mathcal{O}_{(n+1)\overline{X}}$ on X, defined in (4.1), is canonically isomorphic to

$$\mathcal{O}_X \oplus J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \otimes \mathcal{L}^{\otimes(-n-1)}. \qquad \square$$

Proof. For $x \in X$, let $\mathbb{P}_x := q^{-1}(x)$ be the projective line. The $(n-1)$-th order infinitesimal neighborhood of the point

$$\overline{x} := \mathbb{P}_x \cap \overline{X}$$

in $\mathbb{P}_x$ coincides with the fiber $(q_* \mathcal{O}_{n\overline{X}})_{\overline{x}}$.

In order to determine the infinitesimal neighborhood of $\overline{x}$, consider the following exact sequence of coherent sheaves on $\mathbb{P}_x$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_x}(-(n+1)\overline{x}) \longrightarrow \mathcal{O}_{\mathbb{P}_x} \longrightarrow J^n(\mathcal{O}_{\mathbb{P}_x})_{\overline{x}} \longrightarrow 0.$$

Since $H^1(\mathbb{P}_x, \mathcal{O}_{\mathbb{P}_x}) = 0$, the n-th order infinitesimal neighborhood of $\overline{x}$, namely, $J^n(\mathcal{O}_{\mathbb{P}_x})_{\overline{x}}$, coincides with

$$\mathbb{C} \oplus H^1\left(\mathbb{P}_x, \mathcal{O}_{\mathbb{P}_x}\left(-(n+1)\overline{x}\right)\right). \tag{4.3}$$

Using Serre duality, and the canonical isomorphisms $\mathcal{O}_{\mathbb{P}_x}(-\overline{x})_{\overline{x}} = (K_X)_x$ and $\mathcal{L}^{\otimes 2} = T_X$, and the natural self-duality of $H^0(\mathbb{P}_x, \mathcal{O}_{\mathbb{P}_x}(n-1))$ induced by the natural trivialization of the line $\bigwedge^2 J^1(\mathcal{L})_x$, the vector space in (4.3) is identified with the fiber at x of the vector bundle

$$\mathcal{O}_X \bigoplus \left(J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \otimes \mathcal{L}^{\otimes(-n-1)}\right).$$

The details of the last part of the argument can be found in Section 3 of [B1]. This completes the proof of the lemma. ■

Given a projective structure on X, for any $n \in \mathbb{N}$, there is a natural *compatible embedding* of the n-th order infinitesimal neighborhood of Δ into the projective bundle $\mathbb{P}(X)$ (see [D]). The compatibility of embeddings means that the embedding of the $(n-1)$-th order infinitesimal neighborhood is obtained by restricting the embedding of the n-th order infinitesimal neighborhood. To obtain these embeddings, first observe that if X is $\mathbb{CP}^1$, then $\mathbb{P}(X)$ is $X \times X$, and so the embedding is the (tautological) diagonal embedding. Then observe that this tautological embedding commutes with the diagonal action of $\mathrm{Aut}(\mathbb{CP}^1)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$. This immediately implies the earlier statement on the existence of canonical compatible embeddings for any Riemann surface with a projective structure.

The embedding of the second order infinitesimal neighborhood 3Δ of Δ into $\mathbb{P}(X)$ actually does not depend upon the choice of the projective structure. A direct construction of this embedding of the second order infinitesimal neighborhood of Δ into $\mathbb{P}(X)$ is given below.

In view of Lemma 4.2, giving an identification of the second order infinitesimal neighborhoods of the above type is equivalent to giving an isomorphism

$$J^2_0(\mathcal{O}_X) \longrightarrow J^1(\mathcal{L}) \otimes \mathcal{L}^{\otimes -3}, \tag{4.4}$$

where the fiber $J_0^2(\mathcal{O}_X)_x \subset J^2(\mathcal{O}_X)_x$ at any $x \in X$ is given by the jets of functions vanishing at x. Since the map defined by $f \longmapsto df$ identifies $J_0^2(\mathcal{O}_X)$ with $J^1(K_X)$, we have a natural isomorphism of the type (4.4) as soon as we have established the following statement.

For any nonzero integer i, *there is a nondegenerate bilinear pairing*

$$J^1(T_x) \otimes J^1(\mathcal{L}^{\otimes i}) \longrightarrow J^1(\mathcal{L}^{\otimes i})$$

between $J^1(T_X)$ *and* $J^1(\mathcal{L}^{\otimes i})$ *with values in* $\mathcal{L}^{\otimes i}$.

Now, the operation of taking the Lie derivative with respect to a vector field is exactly such a pairing. Indeed, the Lie derivative

$$(V, s) \longmapsto L_V s,$$

where s (respectively, V) is a local section of $\mathcal{L}^{\otimes i}$ (respectively, T_X), defines an isomorphism of $J^1(T_X)$ with $J^1(\mathcal{L}^{\otimes i})^* \bigotimes \mathcal{L}^{\otimes i}$. The Lie derivative of sections of any tensor power of $\mathcal{L}$ is defined using the isomorphism Ψ in (2.1), together with the Leibniz identity

$$L_V(s \otimes t) = (L_V s) \otimes t + s \otimes L_V t.$$

The canonical isomorphism of the type (4.4) that we are seeking is obtained by setting $i = 1$ and -2.

To equip X *with a projective structure is equivalent to extending the above identification*, *of the second order infinitesimal neighborhoods of* Δ *and* $\overline{X}$, *to an identification of their third order infinitesimal neighborhoods* (see [D, Definition 5.6 bis]).

The following lemma is needed in our study of embeddings of the higher order infinitesimal neighborhoods of Δ.

Lemma 4.5. For a Riemann surface with a projective structure, the space of all differential operators of order $n-1$ from $\mathcal{L}^{\otimes(n-1)}$ to $\mathcal{L}^{\otimes(1-n)}$ admits a natural decomposition of the following form:

$$H^0\left(X, \mathrm{Hom}\left(J^{n-1}(\mathcal{L}^{\otimes(n-1)}), \mathcal{L}^{\otimes(-n+1)}\right)\right) = \bigoplus_{i=0}^{n-1} H^0\left(X, K_X^{\otimes i}\right). \qquad \square$$

Proof. Let $V/\mathbb{C}$ be a vector space of dimension 2 equipped with a symplectic form ω. Denoting by L the line bundle $\mathcal{O}_{\mathbb{P}}(1)$ on the projective line $\mathbb{P} = \mathbb{P}(V)$, the vector bundle $J^n(L^{\otimes k})$ on $\mathbb{P}$ is shown to be canonically isomorphic to $J^n(L^{\otimes n}) \bigotimes L^{\otimes(k-n)}$ in the following two cases: (1) when $k < 0$; (2) when $k \geq n$.

The first step in constructing the isomorphism is to consider the long exact sequence of cohomologies for the exact sequence

$$0 \longrightarrow L^{\otimes k} \otimes \mathcal{O}_{\mathbb{P}}(-(n+1)x) \longrightarrow L^{\otimes k} \longrightarrow J^n(L^{\otimes k})_x \longrightarrow 0,$$

where $x \in \mathbb{P}$. Note that in the first case, $H^0(\mathbb{P}, L^{\otimes k}) = 0$; and in the second case, $H^1\left(\mathbb{P}, L^{\otimes k} \otimes \mathcal{O}_{\mathbb{P}}(-(n+1)x)\right) = 0$. A choice of $0 \neq w \in V$ representing x provides an isomorphism between $\mathcal{O}_{\mathbb{P}}(x)$ and L. Now after noting that $H^0(\mathbb{P}, L^{\otimes i}) = S^i(V^*) = S^i(V)$ (using ω), and also using Serre duality when $k < 0$, the key point in the construction is that for any $0 \neq v \in V$, the following two vector spaces are both canonically isomorphic to $S^n(V)$. The first vector space is the kernel of the homomorphism $S^{n-1-k}(V) \longrightarrow S^{-k-2}(V)$ defined using the contraction with $(v^*)^{\otimes(n+1)}$, where $k < 0$ and $v^* \in V^*$, is the dual of v with respect to ω. The second vector space in question is the cokernel of the homomorphism $S^{k-n-1}(V) \longrightarrow S^k(V)$ defined using the multiplication with $v^{\otimes(n+1)}$, where $k \geq n$. Indeed, in the first case, the isomorphism is defined using the multiplication with $v^{\otimes(-k-1)}$ on $S^n(V)$; and in the second case, it is defined using the contraction with $(v^*)^{\otimes(k-n)}$.

Since the isomorphism between $J^n(L^{\otimes k})$ and $J^n(L^{\otimes n}) \otimes L^{\otimes(k-n)}$ is equivariant for the actions of $SL(V)$ on $\mathbb{P}$ and L, it induces an isomorphism

$$\psi(n,k): \; J^n(\mathcal{L}^{\otimes k}) \longrightarrow J^n(\mathcal{L}^{\otimes n}) \otimes \mathcal{L}^{\otimes(k-n)}$$

over X. Here, X is a Riemann surface equipped with a projective structure together with a choice of a square root of the canonical bundle, whenever n and k satisfy either of the above two conditions. (The choice of a *theta characteristic* provides a lift of the transition functions of a projective atlas to $SL(V)$.)

Now consider

$$\begin{aligned} \psi(i, n-1) \otimes \psi(i, -2(n-1-i)) : \; & J^i\left(\mathcal{L}^{\otimes(n-1)}\right) \otimes J^i\left(K_X^{\otimes(n-1-i)}\right) \\ & \longrightarrow J^i(\mathcal{L}^{\otimes i}) \otimes J^i(\mathcal{L}^{\otimes i}) \otimes \mathcal{L}^{\otimes(-n+1)}, \end{aligned} \tag{4.6}$$

where $i \leq n-1$. The symplectic form ω on V induces a nondegenerate form on any $S^i(V)$ (which is symmetric if i is even and skew-symmetric if i is odd), and hence on $J^i(L^{\otimes i})$ over $\mathbb{P}$. This is because any fiber of $J^i(L^{\otimes i})$ is canonically isomorphic to $S^i(V)$ (see [B1]). Since this form on $J^i(L^{\otimes i})$ is equivariant under the action of $SL(V)$, we have a nondegenerate form on $J^i(\mathcal{L}^{\otimes i})$ using the coordinate charts on X compatible with the projective structure. Thus (4.6) gives the pairing

$$\overline{\psi}(i, n-1): \; J^i\left(\mathcal{L}^{\otimes(n-1)}\right) \otimes J^i\left(K_X^{\otimes(n-1-i)}\right) \longrightarrow \mathcal{L}^{\otimes(-n+1)}.$$

Now for a section $s \in H^0(X, K_X^{\otimes(n-1-i)})$, consider the homomorphism $\overline{\psi}(i, n-1)(-, s)$, which sends any vector $v \in J^i(\mathcal{L}^{\otimes(n-1)})_y$ to $\overline{\psi}(i, n-1)(y)(v, s(y))$. The differential operator

$$\overline{s} \in H^0\left(X, \mathrm{Diff}^i_X\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n+1)}\right)\right)$$

defined by this homomorphism has $(-1)^i s \in H^0(X, K_X^{\otimes(n-1-i)})$ itself as its symbol. This lifting of the symbol of a differential operator that maps any s to $(-1)^i \overline{s}$ immediately yields the decomposition (4.5) of differential operators. This completes the proof of the lemma. ■

We note that the decomposition in Lemma 4.5 depends on the choice of projective structure.

For an integer $n \geq 3$, let

$$\rho : J_0^n(\mathcal{O}_X) \longrightarrow J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right) \otimes \mathcal{L}^{\otimes(-n-1)}$$

be an isomorphism of algebra bundles giving an embedding of the n-th order infinitesimal neighborhood of Δ into $\mathbb{P}(X)$ such that the restriction of ρ to the second order infinitesimal neighborhood 3Δ of Δ is the canonical one defined above. Restricting ρ to the third order infinitesimal neighborhood 4Δ of Δ, we get a projective structure $\mathfrak{P}$ on X (see [D]).

Let

$$\rho(\mathfrak{P}) : J_0^n(\mathcal{O}_X) \longrightarrow J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right) \otimes \mathcal{L}^{\otimes(-n-1)}$$

be the isomorphism giving the embedding corresponding to the projective structure $\mathfrak{P}$.

It can be checked that any automorphism of $J^{n-1}(\mathcal{L}^{\otimes(n-1)})$, which as an automorphism of $J^{n-1}(\mathcal{L}^{\otimes(n-1)}) \otimes \mathcal{L}^{\otimes(-n-1)}$ preserves its algebra structure, is actually conjugate to a unique automorphism $J^1(\mathcal{L}^{\otimes(n-1)})$ of the form $c.\mathrm{Id} + A$, where $c \in \mathbb{C}^*$ and $A \in H^0(X, \mathrm{Hom}(J^{n-1}(\mathcal{L}^{\otimes(n-1)}),$
$\mathcal{L}^{\otimes(-n+1)}))$. The inclusion map

$$\mathcal{L}^{\otimes(n-1)} \otimes K_X^{\otimes(n-1)} = \mathcal{L}^{\otimes(-n+1)} \longrightarrow J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right)$$

enables $H^0(X, \mathrm{Hom}(J^{n-1}(\mathcal{L}^{\otimes(n-1)}), \mathcal{L}^{\otimes(-n+1)}))$ to be realized as a subspace of the space of endomorphisms of $J^{n-1}(\mathcal{L}^{\otimes(n-1)})$.

Now consider the automorphism

$$\rho \circ (\rho(\mathfrak{P}))^{-1} : J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right) \longrightarrow J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right),$$

where ρ and $(\rho(\mathfrak{P}))^{-1}$ are defined above. Let

$$\overline{\rho} \in H^0\left(X, \mathrm{Hom}\left(J^{n-1}\left(\mathcal{L}^{\otimes(n-1)}\right), \mathcal{L}^{\otimes(-n+1)}\right)\right)$$

correspond to $\rho \circ (\rho(\mathfrak{P}))^{-1}$ by the above decomposition.

Consider the decomposition of the differential operator $\overline{\rho}$ according to Lemma 4.5 for the projective structure $\mathcal{P}$ on X. Since, by definition, $\rho(\mathfrak{P})$ and ρ agree on the third order infinitesimal neighborhood of Δ, it follows immediately that the components of $\overline{\rho}$ in $H^0(X, K_X^{\otimes i})$ vanish for all $i \leq 2$.

Let

$$F(\rho) \in \bigoplus_{i=3}^{n-1} H^0\left(X, K_X^{\otimes i}\right) \tag{4.7}$$

be the element corresponding to $\overline{\rho}$.

Let $\mathcal{G}(n)$ denote the space of all embeddings of the n-th order infinitesimal neighborhood of Δ into $\mathbb{P}(X)$ whose restriction to the second order infinitesimal neighborhood of Δ is the canonical one. Let $\mathfrak{P}(X)$ denote the space of all equivalence classes of projective structures on X.

Lemma 4.8. The map from the space of embeddings $\mathcal{G}(n)$ to

$$\mathfrak{P}(X) \times \bigoplus_{i=3}^{n-1} H^0\left(X, K_X^{\otimes i}\right),$$

which sends any $\rho \in \mathcal{G}(n)$ to the pair $(\mathfrak{P}, F(\rho))$, constructed in (4.7), is a bijective identification. □

Proof. The above lemma follows from Theorem 2.10. We omit the details. ■

The following theorem is an immediate consequence of Theorem 3.6.

Theorem 4.9. For any compact Riemann surface X, the space of trivializations $\mathcal{T}(n)$, defined in (3.3), is canonically identified with $\mathcal{G}(n+2)$, the space of embeddings of the $(n+2)$-th order infinitesimal neighborhood of Δ into $\mathbb{P}(X)$ restricting to the canonical embedding on the second order infinitesimal neighborhood 3Δ. □

In the following two sections, we consider vector bundles over a Riemann surface. Given a holomorphic vector bundle E over X, with $H^0(X, E) = 0 = H^1(X, E)$, the kernel of the Dolbeault operator for E is studied.

5 The kernel function of the inverse of a Dolbeault operator

Let X be a compact connected Riemann surface of genus g. Let $\mathcal{M}_X(r)$ denote the moduli space of semistable vector bundles of rank r and degree $r(g-1)$ over X. Denote by Θ the reduced divisor on $\mathcal{M}_X(r)$ defined by all E with $H^0(X, E) \neq 0$. In other words, Θ is the *generalized theta divisor* on $\mathcal{M}_X(r)$.

For a vector bundle $E \in \mathcal{M}_X(r)$, a vector bundle $\mathcal{V}_E$ is constructed over $X \times X$ in the following way:

$$\mathcal{V}_E := p_1^* E \otimes p_2^*(E^* \otimes K_X) \otimes \mathcal{O}_{X \times X}(\Delta), \tag{5.1}$$

where K_X is the canonical line bundle over X.

We recall some results of [B2] and, in order to be somewhat self-contained, some proofs are also recalled.

Proposition 5.2. For a vector bundle $E \in \mathcal{M}_X(r) - \Theta$, the restriction to the diagonal

$$H^0\left(X \times X, \mathcal{V}_E\right) \longrightarrow H^0\left(\Delta, \mathcal{V}_E|_\Delta\right) = H^0(X, \mathrm{End}(E))$$

is an isomorphism. □

Proof. Since the restriction of the line bundle $\mathcal{O}_{X\times X}(\Delta)$ to Δ coincides with the normal bundle of Δ in $X \times X$, it follows immediately that with respect to the natural identification of Δ with X, the restriction of $\mathcal{V}_E$ to Δ is naturally isomorphic to $\mathrm{End}(E)$.

Tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}_{X\times X}(-\Delta) \longrightarrow \mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

with $\mathcal{V}_E$, the exact sequence

$$0 \longrightarrow p_1^*E \otimes p_2^*(E^* \otimes K_X) \longrightarrow \mathcal{V}_E \longrightarrow \mathrm{End}(E) \longrightarrow 0$$

is obtained.

Given that $H^0(X, E) = 0 = H^1(X, E)$, and invoking the Serre duality, the Künneth formula asserts that

$$H^0\left(X \times X, p_1^*E \otimes p_2^*\left(E^* \otimes K_X\right)\right) = 0 = H^1\left(X \times X, p_1^*E \otimes p_2^*\left(E^* \otimes K_X\right)\right).$$

Finally, the proof of the proposition is completed by considering the long exact sequence of cohomologies for the above exact sequence. ■

Take any $E \in \mathcal{M}_X(r) - \Theta$. Let

$$\phi_E \in H^0\left(X \times X, \mathcal{V}_E\right) \tag{5.3}$$

be the section that Proposition 5.2 associates to the identity endomorphism of E.

Let

$$\zeta \in H^0\left(2\Delta, \left(p_1^*K_X \otimes p_2^*K_X \otimes \mathcal{O}_{X\times X}(2\Delta)\right)\big|_{2\Delta}\right)$$

denote the invariant section in Theorem 2.2; in a coordinate chart, ζ coincides with $s(-2)$ defined in (2.4).

Finally, consider the section

$$\Phi_E := \left(\phi_E \otimes \sigma^*\phi_E\right)\big|_{2\Delta} \otimes \zeta^* \in H^0\left(2\Delta, \left(p_1^*\mathrm{End}(E) \otimes p_2^*\mathrm{End}(E)\right)\big|_{2\Delta}\right).$$

Since $\phi_E|_\Delta = \mathrm{Id}_E$, we have $\Phi_E|_\Delta = \mathrm{Id}_{\mathrm{End}(E)}$. Therefore, the restriction of the above section Φ_E to 2Δ defines a holomorphic connection on $\mathrm{End}(E)$ (see [D]). However, any holomorphic connection on a Riemann surface is flat.

Let ∇^E *denote the above obtained flat connection on the vector bundle* End(E).

Using the flat connection ∇^E, the section Φ_E naturally extends to 3Δ. On the other hand, $\phi_E \bigotimes \sigma^*\phi_E$ is also defined over 3Δ. So, using the homomorphism $\mathrm{End}(E) \longrightarrow \mathcal{O}_X$ defined by $A \longmapsto \mathrm{trace}(A)/r$, we get a section

$$\zeta_E \in H^0\left(3\Delta, \left(p_1^*K_X \otimes p_2^*K_X \otimes \mathcal{O}_{X\times X}(2\Delta)\right)\big|_{3\Delta}\right),$$

which restricts to the section ζ over 2Δ. From Theorem 2.2, the section ζ_E defines a projective structure on X. This projective structure on X is denoted by $\mathcal{P}_E$.

The above arguments are presented with more details in [B2].

The space of global smooth (p, q)-forms with values in E is denoted by $\Omega^{p,q}(E)$. As we have $H^0(X, E) = 0 = H^1(X, E)$, the Dolbeault operator

$$\overline{\partial}_E\colon \Omega^0(E) \longrightarrow \Omega^{0,1}(E),$$

which defines the holomorphic structure of E, is invertible.

Let $\mathcal{K}_E$ denote the kernel of the pseudo-differential operator $(\overline{\partial}_E)^{-1}$, which is of order -1. In other words, $\mathcal{K}_E$ is a smooth section of $p_1^*E \bigotimes p_2^*(E^* \bigotimes K_X)$ over $X \times X - \Delta$, and for every $f \in \Omega^{0,1}(E)$, the identity

$$(\overline{\partial}_E)^{-1}(f) = \int_X \langle \mathcal{K}_E, f\rangle \tag{5.4}$$

is valid. The pairing $\langle \mathcal{K}_E, f\rangle$ is defined using the contraction of E with E^*.

Lemma 5.5. The following equality is valid:

$$\pi \cdot \mathcal{K}_E = -\phi_E,$$

where ϕ_E is defined in (5.3). □

Proof. For any $f \in \Omega^{0,1}(E)$, we have

$$\overline{\partial}_E \int_X \phi_E f = \int_X \overline{\partial}_E(\phi_E) f, \tag{5.6}$$

as $\overline{\partial}_E f = 0$. For any $x_0 \in X$, let $\overline{E^*_{x_0} \bigotimes K_{x_0}}$ denote the trivial vector bundle over X with $E^*_{x_0} \bigotimes K_{x_0}$ as the typical fiber. Note that $\overline{\partial}_E(\phi_E(-, x_0))$ is a distributional section of the vector bundle $(E \bigotimes (T^{0,1})^*) \bigotimes \overline{E^*_{x_0} \bigotimes K_{x_0}}$ over X. Here $\phi_E(-, x_0)$ denotes the restriction of ϕ_E to $X \times x_0$. Since the section ϕ_E is holomorphic outside Δ, $\overline{\partial}_E(\phi_E(-, x_0))$ is supported at x_0.

For a compactly supported C^∞ function f on $\mathbb{C}$, the identity

$$f \cdot \overline{\partial}\left(\frac{dz}{z}\right) = -\frac{f(0)\delta(0)}{2\pi\sqrt{-1}}$$

follows immediately from Stokes's theorem, where $\delta(0)$ is the Dirac delta function supported at 0, and (dz/z) is considered as a distributional section of K on a neighborhood of zero.

Now, comparing (5.4) and (5.6), the lemma follows from the fact that the residue of ϕ_E, considered as a meromorphic section of $p_1^*E \bigotimes p_2^*(E^* \bigotimes K_X)$, along the diagonal Δ, is the identity endomorphism of E. ■

For any $E \in \mathcal{M}_X(r)$ with $H^0(X, E) \neq 0$, the Dolbeault operator $\overline{\partial}_E$ is not invertible. So the kernel function of the inverse of $\overline{\partial}_E$ defined on the image of $\overline{\partial}_E$, with the orthogonal complement of kernel$(\overline{\partial}_E)$ as the target, is not given by a meromorphic section.

The holomorphic section ϕ_E is well behaved with respect to taking the direct image. This is explained next.

Let $\pi\colon X \longrightarrow Y$ be a covering map, possibly ramified, between two Riemann surfaces. For a holomorphic vector bundle E over X, the natural isomorphism

$$H^i(X, E) = H^i(Y\pi_* E)$$

is valid for any $i \geq 0$. Therefore, the condition $E \in \mathcal{M}_X(r) - \Theta$ ensures that π_*E is in the complement of the generalized theta divisor on $\mathcal{M}_Y(rd)$, where d is the degree of the covering π. We show that for every $E \in \mathcal{M}_X(r) - \Theta$, the section ϕ_{π_*E} on $Y \times Y$ is obtained by taking the direct image of ϕ_E. For that, we need the following lemma.

Lemma 5.7. For any $E \in \mathcal{M}_X(r)$, the vector bundle $\mathcal{V}_E$ over $X \times X$, defined in (5.1), has the property that the direct image $(\pi \times \pi)_* \mathcal{V}_E$ is naturally isomorphic to the vector bundle $\mathcal{V}_{\pi_*E}$ over $Y \times Y$. □

Proof. From the definition of a direct image of a coherent sheaf, it is easy to construct a natural homomorphism from $\mathcal{V}_{\pi_*E}$ to $(\pi \times \pi)_* \mathcal{V}_E$. The point to observe is that the meromorphic form (dz/z) on $\mathbb{C}$ pulls back to $\mathbb{C}$ as $k \cdot (dz/z)$ by the mapping $f(z) = z^k$.

If π is ramified over $D \subset Y$, then the above homomorphism is an isomorphism over the complement of $(D \times Y) \bigcup (Y \times D)$ in $Y \times Y$. Using the Grothendieck-Riemann-Roch theorem, it can be checked that

$$c_1((\pi \times \pi)_* \mathcal{V}_E) = c_1(\mathcal{V}_{\pi_*E}).$$

Therefore, the above homomorphism must be an isomorphism. ■

The following lemma shows how the section ϕ_E, which was constructed in (5.3), behaves with respect to a direct image.

Lemma 5.8. Let $E \in \mathcal{M}_X(r) - \Theta$. Using the isomorphism between $(\pi \times \pi)_* \mathcal{V}_E$ and $\mathcal{V}_{\pi_*E}$ obtained in Lemma 5.7, the section $(\pi \times \pi)_* \phi_E \in H^0(Y \times Y, (\pi \times \pi)_* \mathcal{V}_E)$ coincides with $\phi_{\pi_*E} \in H^0(Y \times Y, \mathcal{V}_{\pi_*E})$. □

Proof. We already noted that the given condition $E \in \mathcal{M}_X(r) - \Theta$ implies that $\pi_* E$ is contained in the complement of the generalized theta divisor on $\mathcal{M}_Y(rd)$.

It is straightforward to check that the restriction of $(\pi \times \pi)_* \phi_E$ to the reduced diagonal in $(Y - D) \times (Y - D)$ coincides with the identity endomorphism of $(\pi_* E)\,|_{Y-D}$, where $D \subset Y$, as before, is the divisor over which the map π is ramified. Now, the lemma follows from Proposition 5.2, which says that the restriction to the diagonal is an isomorphism. ■

In the following section, some natural differential operators on vector bundles associated to $E \in \mathcal{M}_X(r) - \Theta$ are constructed.

6 Differential operators on a vector bundle outside the theta divisor

We start with the following lemma.

Lemma 6.1. Let $E \in \mathcal{M}_X(r) - \Theta$. For any $n \geq 0$, the jet bundle $J^n(\mathrm{End}(E) \bigotimes \xi)$ is canonically isomorphic to $\mathrm{End}(E) \bigotimes J^n(\xi)$. □

Proof. This is an immediate consequence of the flat connection ∇^E on End(E) constructed in the previous section. Indeed, any local section s of $\mathrm{End}(E) \bigotimes \xi$ can be uniquely expressed as

$$s = \sum_{j=1}^{r^2} e_j \otimes s_j,$$

where $\{e_j\}$ is a fixed basis of the local system defined by the flat connection, and s_j are local sections of ξ. This decomposition gives an injective homomorphism from $J^n(\mathrm{End}(E) \bigotimes \xi)$ to $\mathrm{End}(E) \bigotimes J^n(\xi)$, which is evidently an isomorphism. ■

Fix once and for all a line bundle $\mathcal{L}$ over X of degree $1 - g$ such that $\mathcal{L}^{\otimes 2} = T_X$.

Given a projective structure on X, using the choice of the square root $\mathcal{L}$ of T_X, the one cocycle with values in $PSL(2, \mathbb{C})$ defined by the transition functions admits a natural lift to a one cocycle with values in $SL(2, \mathbb{C})$ (see [Gu], [T]).

If X is equipped with a projective structure, then we have the following two decompositions of differential operators.

Proposition 6.2. Let X be equipped with a projective structure. Let $k, l \in \mathbb{Z}$, and $n \in \mathbb{N}$ be such that $k \notin [-n+1, 0]$, and $l - k - j \notin \{0, 1\}$ for any integer $j \in [1, n]$. Then the space of global differential operators of order n from $\mathcal{L}^{-k}$ to $\mathcal{L}^{-l}$, namely, $H^0(X, \mathrm{Diff}^n_X(\mathcal{L}^{-k}, \mathcal{L}^{-l}))$, is canonically isomorphic to

$$\bigoplus_{i=0}^{n} H^0\left(X, \mathcal{L}^{k-l} \otimes T^i_X\right),$$

with the property that the image of $H^0(X, \mathcal{L}^{k-l+2j})$ by this isomorphism is contained in the subspace $H^0(X, \mathrm{Diff}^j_X(\mathcal{L}^{-k}, \mathcal{L}^{-l})) \subseteq H^0(X, \mathrm{Diff}^n_X(\mathcal{L}^{-k}, \mathcal{L}^{-l}))$. □

The above proposition is Theorem B of [B3]. This proposition gives a canonical splitting (semisimplification) of the natural filtration of $H^0(X, \mathrm{Diff}^n_X(\mathcal{L}^{-k}, \mathcal{L}^{-l}))$ given by its subspaces $H^0(X, \mathrm{Diff}^j_X(\mathcal{L}^{-k}, \mathcal{L}^{-l}))$ defined by the lower order differential operators, where $0 \leq j \leq n$.

In Section 5, a projective structure $\mathcal{P}_E$ was constructed from any $E \in \mathcal{M}_X(r) - \Theta$. Now using Lemma 6.1 and the fact that $\mathrm{End}(E)^* = \mathrm{End}(E)$, Proposition 6.2 has the following consequence.

Proposition 6.3. Let $E \in \mathcal{M}_X(r) - \Theta$. Let $k, l \in \mathbb{Z}$, and $n \in \mathbb{N}$ be such that $k \notin [-n+1, 0]$, and $l - k - j \notin \{0, 1\}$ for any integer $j \in [1, n]$. Then the space of global differential operators of order n from $\mathrm{End}(E) \bigotimes \mathcal{L}^{-k}$ to $\mathcal{L}^{-l}$, namely, $H^0(X, \mathrm{Diff}^n_X(\mathrm{End}(E) \bigotimes \mathcal{L}^{-k}, \mathcal{L}^{-l}))$, is canonically isomorphic to the direct sum

$$\bigoplus_{i=0}^{n} H^0\left(X, \mathrm{End}(E) \otimes \mathcal{L}^{k-l} \otimes T^i_X\right),$$

with the property that the image of $H^0(X, \mathrm{End}(E) \bigotimes \mathcal{L}^{k-l+2j})$ is contained in the subspace consisting of operators of order j, namely, $H^0(X, \mathrm{Diff}^j_X(\mathrm{End}(E) \bigotimes \mathcal{L}^{-k}, \mathcal{L}^{-l}))$, of $H^0(X, \mathrm{Diff}^n_X(\mathrm{End}(E) \bigotimes \mathcal{L}^{-k}, \mathcal{L}^{-l}))$. □

Similarly, Corollary 2.12 has the following consequence.

Proposition 6.4. Let $E \in \mathcal{M}_X(r) - \Theta$. The space of global differential operators of order n from $\mathrm{End}(E) \bigotimes \mathcal{L}^n$ to $\mathcal{L}^{-n-2}$ admits the following natural decomposition:

$$H^0\left(X, \mathrm{Diff}^{n+1}_X\left(\mathrm{End}(E) \otimes \mathcal{L}^n, \mathcal{L}^{-n-2}\right)\right) = \bigoplus_{i=0}^{n+1} H^0\left(X, \mathrm{End}(E) \otimes K^i_X\right). \quad \square$$

The differential operator of order $n+1$,

$$\mathcal{D}_{n+1}(E) \in H^0\left(X, \mathrm{Diff}^{n+1}_X\left(\mathrm{End}(E) \otimes \mathcal{L}^n, \mathcal{L}^{-n-2}\right)\right), \tag{6.5}$$

corresponding to the section of $\mathrm{End}(E)$ defined by the identity endomorphism, is a generalization of the Bol's operator to the context of the vector bundles.

Let f and g be two local sections of $\mathrm{End}(E) \bigotimes \mathcal{L}^n$ over $U \subset X$. Then

$$\langle \mathcal{D}_{n+1}(E)f, g\rangle - \langle \mathcal{D}_{n+1}(E)g, f\rangle$$

is a section of $\mathrm{End}(E) \bigotimes K_X$ over U, where $\langle -, - \rangle$ denotes the contraction of $\mathcal{L}^n$ with $\mathcal{L}^{-n}$. In other words, the above operation defines a $\mathbb{C}$-linear skew-symmetric pairing

$$\underline{\mathrm{End}(E) \otimes \mathcal{L}^n} \bigotimes_{\mathbb{C}} \underline{\mathrm{End}(E) \otimes \mathcal{L}^n} \longrightarrow \underline{\mathrm{End}(E) \otimes K_X} \tag{6.6}$$

on the coherent sheaf $\underline{\mathrm{End}(E) \bigotimes \mathcal{L}^n}$ associated to the vector bundle $\mathrm{End}(E) \otimes L^n$; the sheaf $\underline{\mathrm{End}(E) \otimes K_X}$ is similarly the coherent sheaf associated to the vector bundle $\mathrm{End}(E) \otimes K_X$. The above pairing is evidently nondegenerate. Indeed, it is an immediate consequence of the fact that the symbol of $\mathcal{D}_{n+1}(E)$ is the section of End(E) defined by the identity endomorphism of E.

In [B1, Theorem 4.1], it was shown that a projective structure on X induces a flat connection on the jet bundle $J^n(\mathcal{L}^n)$. For any $E \in \mathcal{M}_X(r) - \Theta$, the jet bundle

$$J^n(\mathrm{End}(E) \otimes \mathcal{L}^n) = \mathrm{End}(E) \otimes J^n(\mathcal{L}^n)$$

has a natural flat connection induced by a flat connection on $J^n(\mathcal{L}^n)$ for the projective structure $\mathcal{P}_E$ together with ∇^E on End(E).

In [CMZ], a natural isomorphism

$$\phi_X : \; J^{n-1}(T_X^n) \oplus \mathcal{O}_X \longrightarrow \mathrm{Diff}_X^n(\mathcal{O}_X, \mathcal{O}_X)$$

has been constructed for any Riemann surface X equipped with a projective structure. (An alternative description of this isomorphism was later given in [B1, p. 465].)

For any $E \in \mathcal{M}_X(r) - \Theta$, using Lemma 6.1 and the projective structure $\mathcal{P}_E$, the above isomorphism ϕ_X induces an isomorphism

$$\phi_X^E : \; J^{n-1}(T_X^n) \otimes \mathrm{End}(E) \oplus \mathrm{End}(E) \longrightarrow \mathrm{Diff}_X^n(\mathrm{End}(E), \mathcal{O}_X). \tag{6.7}$$

If genus(X) ≥ 2, then $T_X^i \bigotimes \mathrm{End}(E)$, where $i \geq 1$, does not have any global nonzero section, since $T_X^i \bigotimes \mathrm{End}(E)$ is semistable of strictly negative degree. Now, using the long exact sequence of cohomologies for the exact sequence (2.5), and using induction on n, it follows easily that

$$H^0\left(X, J^{n-1}(T_X^n) \otimes \mathrm{End}(E)\right) = 0.$$

Therefore, the isomorphism ϕ_X^E in (6.7) implies that the natural inclusion

$$H^0\left(X, \mathrm{End}(E)\right) = H^0\left(X, \mathrm{Diff}_X^0\left(\mathrm{End}(E), \mathcal{O}_X\right)\right) \longrightarrow H^0\left(X, \mathrm{Diff}_X^n\left(\mathrm{End}(E), \mathcal{O}_X\right)\right)$$

is actually an isomorphism.

In the rest of the paper, we investigate some natural differential operators on Riemann surfaces with a projective structure.

Appendix A: A differential operator

For a holomorphic function g on an open set of $\mathbb{C}$, its n-th derivative, namely, $\frac{\partial^n g}{\partial z^n}$, is denoted by $g^{(n)}$. Also, often g' is used instead of $g^{(1)}$. Let

$$\gamma : U \longrightarrow U_1 \tag{A.1}$$

be a biholomorphism between two simply connected analytic open sets of $\mathbb{C}$. Fix on U a square root $(\gamma')^{1/2}$ of the function $\gamma^{(1)}$. For an integer $n \geq 1$ and a holomorphic function $f \in \Gamma(U_1, \mathcal{O})$ on U_1, define $D(\gamma, n)f \in \Gamma(U, \mathcal{O})$ by

$$(D(\gamma, n)f)(z) = (f^{(n)} \circ \gamma)(z) \left(\gamma'(z)\right)^{(n+1)/2} - \left(\frac{f \circ \gamma}{(\gamma')^{(n-1)/2}}\right)^{(n)}(z), \tag{A.2}$$

where $z \in U$.

Evidently, the map $f \longmapsto D(\gamma, n)f$ is a differential operator of order n from $\Gamma(U_1, \mathcal{O})$ to $\Gamma(U, \mathcal{O})$. Alternatively, the $\mathbb{C}$-linear homomorphism

$$\psi \longmapsto D(\gamma, n)(\psi \circ \gamma^{-1}), \tag{A.3}$$

where $\psi \in \Gamma(U, \mathcal{O})$, is a section over U of the sheaf of differential operators of order n on the trivial line bundle. Since the tangent bundle TU has a natural trivialization, the symbol of a differential operator of order i from $\Gamma(U, \mathcal{O})$ to $\Gamma(U, \mathcal{O})$ is an element of $\Gamma(U, \mathcal{O})$. It is a simple calculation to check that the symbol of this order n differential operator $D(\gamma, n)$ vanishes identically. It involves a slightly more nontrivial computation to check that the symbol of the order $n-1$ differential operator also vanishes identically. The conclusion is that the operator $D(\gamma, n)$ is actually of order $n-2$.

Another simple computation yields that if M is a Möbius transformation, which means that

$$M(z) = \frac{az + b}{cz + d},$$

where $ad - bc = 1$, then

$$D(M, n) = 0 \tag{A.4}$$

for all $n \geq 1$. In fact, more generally, for a Möbius transformation M as above, denoting the function $w \longmapsto (cw+d)^k$ by $M^{[k]}$, where $k \in \mathbb{Z}$, the following identity is valid:

$$D(M \circ \gamma, n)(f)(z) = D(\gamma, n)(M^{[n-1]} f \circ M)(z), \tag{A.5}$$

where $f \in \Gamma(M(U_1), \mathcal{O})$ and $z \in U$. Since $D(\mathrm{Id}, k) = 0$ for any $k \geq 1$, the equality (A.5) immediately gives (A.4). Similarly, the identity

$$D(\gamma \circ M, n)(f)(z) = D(\gamma, n)(f)(M(z)) M^{[-n-1]}(z), \tag{A.6}$$

where $z \in M^{-1}(U)$, is valid.

Clearly, $D(\gamma, 1) = 0$. To calculate $D(\gamma, 2)$, for a holomorphic function g on an open subset of $\mathbb{C}$, let

$$\mathcal{SD}(g) := \frac{2g'(z)g'''(z) - 3\left(g''(z)\right)^2}{2(g'\left(z\right))^2}$$

denote the *Schwarzian derivative* of g (p. 164 of [Gu]). The identity

$$(D(\gamma, 2)f)(z) = \frac{f(\gamma(z))\mathcal{SD}(\gamma)(z)}{2(\gamma')^{1/2}(z)}, \tag{A.7}$$

which relates $D(\gamma, 2)$ with $\mathcal{SD}(\gamma)$, is valid. To calculate the symbol of the higher order operators, let

$$\overline{D(\gamma, n)} \in \Gamma(U, \mathrm{Diff}_U^{n-2}(\mathcal{O}, \mathcal{O}))$$

be the differential operator of order $n-2$ defined in (A.3), which operates on the sheaf $\Gamma(U, \mathcal{O})$. The symbol of this operator $\overline{D(\gamma, n)}$ is calculated to be

$$\mathrm{symbol}(\overline{D(\gamma, n)}) = \frac{n(n^2-1)(\gamma')^{(n-3)/2}}{12} \mathcal{SD}(\gamma). \tag{A.8}$$

The equality (A.7) follows from (A.8).

For any simply connected open set $W \subseteq \mathbb{C}$, let (L_W, Ψ_W) be a pair consisting of a line bundle L_W on W and a section

$$\Psi_W \in \Gamma(W, L_W).$$

Furthermore, there is a given isomorphism $A_W \colon L_W^{\otimes 2} \longrightarrow T_W$, such that $A^{\otimes 2}(\Psi_W^{\otimes 2})$ coincides with the section $\frac{\partial}{\partial z}$, of T_W.

Let

$$\mathcal{D}_W(n) \in \Gamma\left(W, \mathrm{Diff}_W^n\left(L_W^{\otimes(n-1)}, L_W^{\otimes(-n-1)}\right)\right) \tag{A.9}$$

be the differential operator of order n on W defined by

$$\mathcal{D}_W(n)(f\Psi_W^{\otimes(n-1)}) = \frac{\partial^n f}{\partial z^n}\Psi_W^{\otimes(-n-1)},$$

where f is a holomorphic function on W. Note that the operator $\mathcal{D}_W(n)$ remains unchanged if the isomorphism Ψ_W is replaced by $-\Psi_W$.

Let γ be as in (A.1), and let (L_U, Ψ_U) and (L_{U_1}, Ψ_{U_1}) be as defined above. Fix an isomorphism

$$F_\gamma : \gamma^* L_{U_1} \longrightarrow L_U$$

such that $(A_U^{\otimes 2}) \circ F_\gamma^{\otimes 2} \circ (A_{U_1}^{\otimes 2})^{-1}(z) = d\gamma^{-1}(\gamma(z))$ for all $z \in U$, where A_U and A_{U_1} were defined before (A.9); the homomorphism $d\gamma$ is the differential of γ mapping T_U to $\gamma^* T_{U_1}$. The above condition determines F_γ up to the sign. This ambiguity of sign is removed by imposing the condition that

$$(\gamma')^{1/2} F_\gamma(\gamma^* \Psi_{U_1}) = \Psi_U.$$

The isomorphism between $(\gamma^* L_{U_1})^{\otimes i}$ and $L_U^{\otimes i}$, induced by F_γ, is denoted by $F_\gamma^{\otimes i}$. Note, using the identity

$$(\gamma')^{1/2} F_\gamma(\Psi_{U_1}) = \Psi_U,$$

that F_γ automatically fixes a square root $(\gamma')^{1/2}$.

The following lemma follows from a simple computation.

Lemma A.10. Let γ be as in (A.1), and let (L_U, Ψ_U) and (L_{U_1}, Ψ_{U_1}) be as defined above. Then for any holomorphic function $f \in \Gamma(U_1, \mathcal{O})$, the identity

$$F_\gamma^{\otimes(-n-1)} \mathcal{D}_{U_1}(n)\left(f\Psi_{U_1}^{\otimes(n-1)}\right) - \mathcal{D}_U(n)\left(\left(F_\gamma\left(\Psi_{U_1}\right)\right)^{\otimes(n-1)} f \circ \gamma\right) = \Psi_U^{\otimes(-n-1)} D(\gamma, n) f$$

is valid, where $D(\gamma, n)f$ is as in (A.2). □

Note that the differential operator, which maps $f\Psi_{U_1}^{\otimes(n-1)}$ to $\Psi_U^{\otimes(-n-1)} D(\gamma, n)f$, does not depend upon the choice of the square root $(\gamma')^{1/2}$. Indeed, if $(\gamma')^{1/2}$ is replaced by $-(\gamma')^{1/2}$, then F_γ changes to $-F_\gamma$.

In Appendix B, we derive some implications of the operator $D(\gamma, n)$ in the context of a Riemann surface equipped with a projective structure.

Appendix B: A Differential operator on a Riemann surface with a projective structure

Let X be a Riemann surface equipped with a projective structure, which we denote by $\mathfrak{P}$. Fix a pair $(\mathcal{L}, \Psi)$, where

$$\Psi : \mathcal{L}^{\otimes 2} \longrightarrow T_X \tag{B.1}$$

is an isomorphism of line bundles.

Take a coordinate chart (U_α, ϕ_α) compatible with $\mathfrak{P}$. Recall the pair (L_W, Ψ_W) constructed in Section 2 for any open set W of $\mathbb{C}$. Substituting $\phi_\alpha(U_\alpha)$ for W, fix an isomorphism

$$\theta_\alpha : \mathcal{L}|_{U_\alpha} \longrightarrow \phi_\alpha^* L_{\phi_\alpha(U_\alpha)} \tag{B.2}$$

such that $\theta_\alpha^{\otimes 2} \circ \Psi^{-1} = d\phi_\alpha$. The isomorphism from $\mathcal{L}^{\otimes i}|_{U_\alpha}$ to $(\phi_\alpha^* L_{\phi_\alpha(U_\alpha)})^{\otimes i}$, induced by θ_α, is denoted by $\theta_\alpha^{\otimes i}$. Note that there is a natural isomorphism between $L_W^{\otimes 2}$ and T_W. Let $\mathcal{D}_\alpha(n)$ be the differential operator on U_α obtained from the operator $\mathcal{D}_W(n)$ defined in (A.9), using the isomorphism θ_α, namely,

$$\mathcal{D}_\alpha(n) := (\theta_\alpha^{\otimes(-n-1)})^{-1} \circ \mathcal{D}_{\phi_\alpha(U_\alpha)}(n) \circ \theta_\alpha^{\otimes(n-1)} \in \Gamma\left(U_\alpha, \mathrm{Diff}^n_{U_\alpha}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right), \tag{B.3}$$

which maps sections of $\mathcal{L}^{\otimes(n-1)}$ to sections of $\mathcal{L}^{\otimes(-n-1)}$. Note that although there are two choices of the homomorphism θ_α, differing by multiplication with -1, the operator $\mathcal{D}_\alpha(n)$ does not depend upon the choice of θ_α.

Let (U_β, ϕ_β) be another coordinate chart compatible with the projective structure $\mathfrak{P}$. Since $\phi_\beta \circ \phi_\alpha^{-1}$ is a Möbius transformation, Lemma A.10 and the identity (A.4) combine together to imply that the restrictions of the two differential operators $\mathcal{D}_\alpha(n)$ and $\mathcal{D}_\beta(n)$ to the open set $U_\alpha \cap U_\beta$ actually coincide. Thus patching these operators, we have a global differential operator

$$\mathcal{D}_{\mathfrak{P}}(n) \in H^0\left(X, \mathrm{Diff}^n_X\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right) \tag{B.4}$$

on X, which depends on the projective structure $\mathfrak{P}$. This differential operator is called the *Bol's operator* (see [Bo]). (An alternative construction of this operator can be found in [B1].)

Take another projective structure $\mathfrak{P}_1$ on X. Let ϕ and ϕ_1 be two coordinate functions on $U_0 \subseteq X$ compatible with $\mathfrak{P}$ and $\mathfrak{P}_1$, respectively. As in (B.2), let θ and θ_1 be two isomorphisms from $\mathcal{L}|_{U_0}$ to $\phi^* L_{\phi(U_0)}$ and $\phi_1^* L_{\phi_1(U_0)}$, respectively. With a slight abuse of notation, the isomorphisms of tensor powers induced by θ (and θ_1) are also denoted by θ (and θ_1).

Consider the differential operator $D(\gamma, n)$ defined in (A.2), after substituting $U = \phi(U_0)$ and $\gamma = \phi_1 \circ \phi^{-1}$. Finally, let

$$\mathcal{D}^n_{U_0}(\mathfrak{P}, \mathfrak{P}_1) \in \Gamma\left(U_0, \mathrm{Diff}^{n-2}_{U_0}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right) \tag{B.5}$$

be the differential operator of order $n-2$ defined by

$$\mathcal{D}^n_{U_0}(\mathfrak{P}, \mathfrak{P}_1)\left(\theta_1^{-1}\left(f\Psi^{\otimes(n-1)}_{\phi_1(U_1)}\right)\right) = \theta^{-1}(\Psi^{\otimes(-n-1)}_{\phi(U)} D(\gamma, n) f),$$

where $f \in \Gamma(\phi_1(U_0), \mathcal{O})$ and Ψ_U is as in Appendix A. The identities (A.4), (A.5), and (A.6) together imply that for any two open sets U_0 and U_1 of X, the two corresponding differential operators, defined in (B.5), actually coincide over $U_0 \cap U_1$. Thus, patching up these locally defined operators, we get a global differential operator

$$\mathcal{D}_X^n(\mathfrak{P}, \mathfrak{P}_1) \in H^0\left(X, \mathrm{Diff}_X^{n-2}\left(\mathcal{L}^{\otimes(n-1)}, \mathcal{L}^{\otimes(-n-1)}\right)\right) \tag{B.6}$$

of order $n-2$ on X.

The following lemma is immediate from Lemma A.10.

Lemma B.7. Let $\mathcal{D}_{\mathfrak{P}}(n)$ and $\mathcal{D}_{\mathfrak{P}_1}(n)$ be the two differential operators for $\mathfrak{P}$ and $\mathfrak{P}_1$, respectively, as constructed in (B.4). Then the identity

$$\mathcal{D}_{\mathfrak{P}_1}(n) - \mathcal{D}_{\mathfrak{P}}(n) = \mathcal{D}_X^n(\mathfrak{P}, \mathfrak{P}_1)$$

is valid. □

The space of all projective structures on the Riemann surface X is an affine space for $H^0\left(X, K_X^{\otimes 2}\right)$ (see [Gu]). Let $\omega \in H^0\left(X, K_X^{\otimes 2}\right)$ be such that

$$\mathfrak{P}_1 = \mathfrak{P} + \omega. \tag{B.8}$$

Note that $\mathcal{D}_{\mathfrak{P}_1}(2) - \mathcal{D}_{\mathfrak{P}}(2) \in H^0\left(X, K_X^{\otimes 2}\right)$. The equality

$$\mathcal{D}_{\mathfrak{P}_1}(2) = \mathcal{D}_{\mathfrak{P}}(2) + \frac{\omega}{2}$$

is valid (see [HS], [CMZ]).

Lemma B.7 and the equality (A.8) together immediately imply the following lemma.

Lemma B.9. The following equality is valid:

$$\mathrm{symbol}\left(\mathcal{D}_{\mathfrak{P}_1}(n) - \mathcal{D}_{\mathfrak{P}}(n)\right) = \mathrm{symbol}\left(\mathcal{D}_X^n(\mathfrak{P}, \mathfrak{P}_1)\right) = \frac{n(n^2-1)\omega}{12}.$$ □

Acknowledgments

Indranil Biswas wishes to thank the International Centre for Theoretical Physics, Trieste, where a part of the work was carried out. A. K. Raina gratefully acknowledges the support and hospitality of the Max-Planck-Institut für Mathematik, Bonn, during this work.

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