

Projective structures on a Riemann surface*

Indranil Biswas and A. K. Raina

1 Introduction

A *projective structure* (also called a *projective connection*) on a Riemann surface is an equivalence class of coverings by holomorphic coordinate charts such that the transition functions are all Möbius transformations. There are several equivalent notions of a projective structure [D], [G].

For a compact Riemann surface X of any genus g , let L denote the line bundle $K_{X \times X} \otimes \mathcal{O}_{X \times X}(2\Delta)$ on $X \times X$, where $K_{X \times X}$ is the canonical bundle of $X \times X$ and Δ is the diagonal divisor. This line bundle L is trivialisable over a Zariski open neighborhood of Δ and has a *canonical trivialisatation* over the nonreduced divisor 2Δ .

Our main result [Theorem 3.2] is that the space of projective structures on X is canonically identified with the space of all trivialisations of L over 3Δ which restrict to the canonical trivialisatation of L over 2Δ mentioned above.

In ([D], page 31, Definition 5.6 bis) Deligne gave another definition of a projective structure (what he calls “forme infinitésimale”). We give a direct identification of this definition with our definition of a projective structure [Theorem 4.2].

In Section 5, which is independent of the rest of the paper, we describe briefly the origin of this work in the study of the so-called “Sugawara form” of the energy-momentum tensor in a conformal quantum field theory.

2 Trivialisability of the line bundle L

Let X be a compact connected Riemann surface, equivalently, a smooth connected projective curve over \mathbb{C} , of genus g . We denote by S the complex surface $X \times X$, by Δ the diagonal divisor of S , and by K_S the canonical bundle of S . Thus $K_S =$

*To appear in International Mathematics Research Notices

$p_1^*K_X \otimes p_2^*K_X$, where p_i ($i = 1, 2$) is the projection of $X \times X$ onto the i -th factor and K_X is the canonical bundle of X .

Let σ be the involution of S defined by $(x, y) \mapsto (y, x)$, of which Δ is the fixed point set. We note that σ has a canonical lift $\tilde{\sigma}$ to $L = K_S \otimes \mathcal{O}_S(2\Delta)$; in other words, $\tilde{\sigma}$ is an isomorphism between L and $\sigma^*(L)$ with $\tilde{\sigma} \circ \tilde{\sigma}$ being the identity isomorphism.

The aim of this section is to establish the following theorem:

Theorem 2.1. *The line bundle $L := K_S \otimes \mathcal{O}_S(2\Delta)$ on S is (a), trivialisable on every infinitesimal neighborhood $n\Delta$ of Δ in S and (b), has a canonical trivialisation on the first infinitesimal neighborhood 2Δ , which is the unique trivialisation of L on 2Δ invariant under the action of $\tilde{\sigma}$ and coinciding with the canonical trivialisation of L on Δ .*

We denote by J^d the component of the Picard group of X consisting of all line bundles of degree d and by $\mathcal{O}(\Theta)$ the line bundle on J^{g-1} corresponding to the theta divisor, viz. the reduced theta divisor on J^{g-1} defined by the subset $\{\xi \in J^{g-1} | H^0(X, \xi) \neq 0\}$ (when $g = 0$, it is the zero divisor). Let θ denote the natural section of the line bundle $\mathcal{O}(\Theta)$ on J^{g-1} given by the constant function 1; it vanishes precisely on the theta divisor. For $\xi \in J^{g-1}$ let $\xi^*\Theta$ denote the divisor $\{\zeta \otimes \xi^{-1} | \zeta \in \Theta\} \subset J^0$. We now recall [NR] that the linear equivalence class of $\xi^*\Theta + (K \otimes \xi^{-1})^*\Theta$ on J^0 is independent of $\xi \in J^{g-1}$ and defines *canonically* a line bundle on J^0 , which we denote by $\mathcal{O}(2\Theta_0)$.

We require the following property of the line bundle L in the proof of Theorem 2.1:

Lemma 2.2. *Let $\phi : S \rightarrow J^0$ be the morphism defined by $(x, y) \mapsto \mathcal{O}_X(x - y)$. Then*

$$(2.3) \quad L = \phi^*\mathcal{O}(2\Theta_0)$$

Proof. Clearly we can write

$$(2.4) \quad L = \mathcal{M}_\alpha \otimes \sigma^*\mathcal{M}_\alpha$$

where, for $\alpha \in J^{g-1}$, the line bundle \mathcal{M}_α on S is defined as follows:

$$(2.5) \quad \mathcal{M}_\alpha := p_1^*(K_X \otimes \alpha^{-1}) \otimes p_2^*(\alpha) \otimes \mathcal{O}_S(\Delta)$$

As shown in [R1], \mathcal{M}_α is isomorphic to $\phi_\alpha^*\mathcal{O}(\Theta)$, where

$$(2.6) \quad \phi_\alpha : S \rightarrow J^{g-1}$$

is the morphism defined by $(x, y) \mapsto \alpha \otimes \mathcal{O}_X(x - y)$. Theorem 2.2 of [BR] is, however, preferable, since it gives, in this special case, a *natural* isomorphism between $\phi_\alpha^*\mathcal{O}(\Theta)$

and $\mathcal{M}_\alpha \otimes \zeta_\alpha$, where ζ_α denotes the trivial line bundle on S with fiber Θ_α , the fiber of $\mathcal{O}(\Theta)$ at the point α . (For any α outside the theta divisor, the nonzero vector $\theta(\alpha) \in \Theta_\alpha$ identifies ζ_α with the trivial line bundle.) Using this in (2.4), we see that Lemma 2.2 follows immediately from the definition of $\mathcal{O}(2\Theta_0)$. \square

Proof of part (a) of Theorem 2.1. Simply observe that the image $\phi(\Delta) = 0 \in J^0$. In fact, since $\mathcal{O}(2\Theta_0)$ has no base points, L has a global section which is nowhere zero on Δ . \square

Corollary 2.7. *Let $\alpha \in J^{g-1} \setminus \Theta$. Then the section*

$$\omega_\alpha = \phi_\alpha^* \theta \otimes (\phi_\alpha \circ \sigma)^* \theta \in H^0(S, L)$$

is 1 at any point of the diagonal Δ . In particular, this section gives a trivialisation of L over some Zariski open neighborhood of Δ . The existence of ω_α implies that

$$(2.8) \quad \dim H^0(S, L) \geq \dim H^0(S, K_S) + 1 = \dim H^0(X, K_X)^{\otimes 2} + 1 = g^2 + 1$$

Proof. Using the natural trivialisation of $\mathcal{O}(\Theta)$ outside the theta divisor given by the section θ and the above identification of \mathcal{M}_α with the pullback of $\mathcal{O}(\Theta)$, we have a trivialisation of \mathcal{M}_α over some Zariski open neighborhood of Δ . This gives a trivialisation of $\sigma^* \mathcal{M}_\alpha$ over some Zariski open neighborhood of Δ . Now the equality (2.4) completes the proof. \square

Notation: For $n \geq 1$, we shall denote the restriction of L to the divisor $n\Delta$ (the $(n-1)$ -th order infinitesimal neighborhood of Δ) by $L|n\Delta$.

Proof of part (b) of Theorem 2.1. Now $L| \Delta = \mathcal{O}_\Delta$ and \mathcal{O}_Δ has a one-dimensional space of sections invariant under the action induced by the involution σ on S . Hence L has a canonical trivialisation on Δ defined by the section “1”.

The situation on 2Δ is more complicated. We know that ω_α in Corollary 2.7, which defines a trivialisation of L on 2Δ , is symmetric under σ . The claim that L has a *canonical* trivialisation on 2Δ will then follow from the following lemma:

Lemma 2.9. *The restriction of L to 2Δ has a one-dimensional space of sections invariant under the action induced by the involution $\sigma : (x, y) \mapsto (y, x)$ on S .*

Proof. Consider the exact sequence

$$0 \longrightarrow K_\Delta \longrightarrow L|2\Delta \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

where we have made use of the canonical trivialisation of L on Δ . Now note that the global sections form a short exact sequence. Observe that the natural invariant

section “1” of \mathcal{O}_Δ lifts (by averaging over $\tilde{\sigma}$) to an invariant section of $L|_{2\Delta}$, so that the dimension of the space of invariant sections of the latter is at least one. On the other hand, $\tilde{\sigma}$ operates on $H^0(K_\Delta)$ as $-Id$. Indeed, the tangent space at $(x, x) \in \Delta$ is $T_x X \oplus T_x X$, and it is the direct sum of the subspace spanned by (v_x, v_x) with the subspace spanned by $(v_x, -v_x)$, where v_x is a nonzero vector in $T_x X$. The former are invariant under $\tilde{\sigma}$ and belong to the tangent bundle of Δ , while the latter are anti-invariant under $\tilde{\sigma}$ and belong to the normal bundle of Δ . Now, since K_Δ is the conormal bundle to Δ , the involution $\tilde{\sigma}$ operates as $-Id$ on $H^0(K_\Delta)$. Thus we conclude that K_Δ has no nonzero section which is invariant under $\tilde{\sigma}$. This proves the lemma and also completes the proof of Theorem 2.1.

What is happening is that, under the quotient map $q : S \rightarrow S/\sigma$, the line bundle L descends to \tilde{L} on S/σ , since $\tilde{\sigma}$ acts trivially on the fibers of L at each point of the fixed point set Δ of σ . The trivialisation of L over Δ induces a trivialisation of \tilde{L} over Δ/σ . Since the scheme-theoretic inverse image $q^{-1}(\Delta/\sigma)$ is 2Δ and $q^*\tilde{L} = L$, the trivialisation of \tilde{L} over Δ/σ induces a trivialisation of L over 2Δ . \square

It is useful to have an alternative view of the canonical trivialisation of L on 2Δ :

Proposition 2.10. *The canonical trivialisation of L on 2Δ is given by the unique section of $L|_{2\Delta}$ which restricts to the canonical trivialisation on Δ and lifts to a global section of L .*

The proof rests on the following lemma, which shows that the inequality (2.8) is actually an equality.

Lemma 2.11. $\dim H^0(S, L) = g^2 + 1.$

Proof. In view of (2.8), we merely have to establish the upper bound. Indeed, tensoring the following exact sequence of sheaves on S

$$(2.12) \quad 0 \longrightarrow \mathcal{O}_S(-\Delta) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

by L and passing to cohomology, this follows from the observation that $K_S(\Delta)$ has g^2 sections. To establish the latter, tensor (2.12) by $K_S(\Delta)$ and pass to cohomology; it then suffices to show that the injection $H^0(S, K_S) \rightarrow H^0(S, K_S(\Delta))$ is an isomorphism. Taking the direct image of this short exact sequence by the projection, p_1 , to the first factor of S , gives the long exact sequence

$$0 \longrightarrow K_X \otimes \mathbb{C}^g \xrightarrow{\iota} p_{1*}(K_S(\Delta)) \longrightarrow K_X \longrightarrow \dots$$

of which the first three terms are locally free sheaves. The first two terms are rank g vector bundles and hence ι must be an isomorphism, which completes the proof. \square

Proof of Proposition 2.10. From the exact sequence

$$0 \longrightarrow K_S \longrightarrow L \longrightarrow L|_{2\Delta} \longrightarrow 0$$

and the fact that L has only $g^2 + 1$ sections, we conclude that the space of sections of L has a one-dimensional image in $L|_{2\Delta}$. \square

3 Projective structures and the line bundle L

We will recall the definition of a *projective structure on a Riemann surface subordinate to the complex structure*. This is defined (see [G] page 167) to be a holomorphic coordinate covering, $\{U_i, z_i\}_{i \in I}$, of X such that for any pair $i, j \in I$, the holomorphic transition function $f_{i,j}$ (defined by $z_i = f_{i,j}(z_j)$) is a Möbius transformation, i.e., a function of the form

$$(3.1) \quad z \longmapsto \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$. The space \mathcal{Q} of all projective structures on X subordinate to the complex structure is an affine space for the complex vector space $H^0(X, K_X^2)$ ([G] page 172).

The main result of this section is the following theorem:

Theorem 3.2. *Let \mathcal{Q} denote the space of all trivialisations of $L|_{3\Delta}$, which, on restriction to 2Δ , give the canonical trivialisation of $L|_{2\Delta}$. Then \mathcal{Q} is an affine space for the vector space $H^0(X, K_X^2)$, which is canonically isomorphic to the affine space \mathcal{Q} of projective structures on X .*

Proof. The obvious exact sequence

$$(3.3) \quad 0 \longrightarrow K_X^2 \longrightarrow L|_{3\Delta} \longrightarrow L|_{2\Delta} \longrightarrow 0$$

shows that \mathcal{Q} is an affine space for the vector space $H^0(X, K_X^2)$.

We shall now construct a map from \mathcal{Q} to \mathcal{Q} .

Let $M = CP^1 \times CP^1$, and consider the *trivial* line bundle $L_M := K_M \otimes \mathcal{O}_M(2\Delta_M)$ on M , where Δ_M is the diagonal on M . Let

$$(3.4) \quad s \in H^0(M, L_M)$$

be the trivialisation of L_M whose restriction to Δ_M coincides with the canonical trivialisation given by Theorem 2.1(b). The group of all automorphisms of CP^1 , namely $\text{Aut}(CP^1)$, acts naturally on M by the diagonal action; this action lifts to L_M . The

section s in (3.4) is evidently invariant under the induced action of $\text{Aut}(CP^1)$ on $H^0(M, L_M)$. This invariance property of the section s immediately implies that if we have a projective structure on X , the section s induces a trivialisation of L on some analytic open neighborhood of the diagonal Δ . Now, restricting this trivialisation of L to 3Δ we get an element in \mathcal{Q} . This gives the required map

$$(3.5) \quad F : \mathcal{O} \longrightarrow \mathcal{Q}$$

The above construction of the map F has been motivated by [Bi].

The proof of Theorem 3.2 is now completed by the following lemma which describes how the map F relates the affine structures on \mathcal{O} and \mathcal{Q} .

Lemma 3.6. *For any $\mathcal{I} \in \mathcal{O}$ and $\gamma \in H^0(X, K_X^2)$, the following equality holds.*

$$F(\mathcal{I} + \gamma) = F(\mathcal{I}) + \frac{\gamma}{6}$$

Proof. Let us first recall how the affine $H^0(X, K_X^2)$ structure of \mathcal{O} is defined ([G] page 170, Theorem 19). We start by recalling the definition of the *Schwarzian derivative*, denoted by \mathcal{S} , which is the differential operator:

$$\mathcal{S}(f)(z) := \frac{2f'(z)f'''(z) - 3(f''(z))^2}{2(f'(z))^2}$$

defined over \mathbb{C} .

Take any $\mathcal{I} = \{U_i, z_i\}_{i \in I} \in \mathcal{O}$ and $\gamma \in H^0(X, K_X^2)$. On each U_i there is a holomorphic function h_i such that $\gamma = h_i dz_i \otimes dz_i$. For $i \in I$, let w_i be a holomorphic function on $z_i(U_i)$ satisfying the equation

$$(3.7) \quad h_i = \mathcal{S}(w_i)(z_i)$$

Another function w'_i satisfies the equation (3.7) if and only if $w'_i(z_i) = \rho \circ w_i(z_i)$, where ρ is a Möbius transformation. The element $\mathcal{I} + \gamma \in \mathcal{O}$ is given by $\{U_i, w_i \circ z_i\}_{i \in I}$. (Actually we may have to shrink each U_i a bit so that $w_i \circ z_i$ is a coordinate function.)

We require an explicit description of the section s defined in (3.4) in terms of local coordinates. Identify CP^1 with $\mathbb{C} \cup \{\infty\}$ and let (z_1, z_2) be the natural coordinates on M . In these coordinates the section s can be written as:

$$(3.8) \quad s_z := \frac{dz_1 \wedge dz_2}{(z_1 - z_2)^2}$$

Let $\mathcal{I} := \{U_i, z_i\}_{i \in I}$ be a projective structure on X , as before. Take a coordinate chart (U, z) in \mathcal{I} . On the open set $U \times U \subset S$ there is a natural coordinate function (z_1, z_2) obtained from z . Now s_z in (3.8) gives a trivialisation of L over $U \times U$.

Let (V, y) be another coordinate chart in \mathcal{I} with $y = (az + b)/(cz + d)$ as in (3.1). This implies that the following identity holds:

$$s_z := \frac{dz_1 \wedge dz_2}{(z_1 - z_2)^2} = \frac{dy_1 \wedge dy_2}{(y_1 - y_2)^2} =: s_y$$

where (y_1, y_2) is the coordinate function on $V \times V$. This equality implies that the two local sections of L , viz. s_z and s_y , coincide on the intersection $(U \cap V) \times (U \cap V) \subset S$.

Thus various local trivialisations of L of the form s_z patch together to give a trivialisaton of L on some analytic open neighborhood of Δ . In particular, we get a trivialisaton of $L|n\Delta$ for any n . Since the section s_z takes the value 1 on $U \times U$ and is invariant under the involution $\tilde{\sigma}$, the trivialisaton of $L|2\Delta$ obtained this way is the canonical trivialisaton. Thus the trivialisaton of $L|3\Delta$ is actually an element of \mathcal{Q} . Evidently, the element of \mathcal{Q} obtained in this way coincides with $F(\mathcal{I})$, where F is the map defined in (3.5).

Let (U_i, z_i) , $i \in I$, be a coordinate chart around $x \in X$, with $z_i(x) = 0$. Assume that

$$(3.9) \quad w_i = z_i + \sum_{j=2}^{\infty} a_j z_i^j$$

is a solution of (3.7) (we may assume that w_i is of this form since we may compose w_i with any Möbius transformation). Then equation (3.7) gives

$$(3.10) \quad h(0) = 6a_3 - 6a_2^2$$

Set $y = w_i \circ z_i$, and define s_y as in (3.8). For $\tilde{x} := (x, x) \in S$, using (3.9) we find that

$$s_y(\tilde{x}) = s_{z_i}(\tilde{x}) + (a_3 - a_2^2) dz_i \otimes dz_i$$

Comparing this with (3.10) we get that $s_y(\tilde{x}) = s_{z_i}(\tilde{x}) + \gamma(x)/6$. This completes the proof of the lemma and also of Theorem 3.2. \square

Remark 3.11. A consequence of Theorem 3.2 is the following alternative definition of the Schwarzian derivative.

Let f be a holomorphic function around $z_0 \in \mathbb{C}$ such that $f'(z_0) \neq 0$. Then the function $\bar{f} := (f, f)$ is a biholomorphism defined on some neighborhood, U , of $(z_0, z_0) \in \mathbb{C} \times \mathbb{C}$. Consider the section s defined in (3.4). The restriction of

$$\hat{s} := \bar{f}^* s - s$$

to the (nonreduced) divisor $3\Delta_U$ is actually a local section of $K_{\mathbb{C}}^2$ around z_0 . From the computation in the proof of Lemma 3.6 it follows that \hat{s} is actually $\mathcal{S}(f)(dz)^{\otimes 2}/6$.

Remark 3.12. An interesting question, arising naturally from Theorem 3.2, is whether an element of \mathcal{Q} comes necessarily from a global section of L . Thus let $\Lambda \subset H^0(S, L)$ denote the affine subspace consisting of those sections of L which restrict to the canonical trivialisation on 2Δ . Then Λ is an affine space for the subspace $H^0(S, K_S) = H^0(X, K_X)^{\otimes 2}$. Associating to any $s \in \Lambda$, the corresponding trivialisation of L over 3Δ , we get a map from Λ to \mathcal{Q} . Then from Theorem 3.2 we have a (holomorphic) map λ from Λ to \mathcal{Q} , the space of all projective structures on X . Our question is now whether λ is surjective. Let

$$(3.13) \quad R : H^0(X, K_X)^{\otimes 2} = H^0(S, K_S) \longrightarrow H^0(\Delta, K_S|_{\Delta}) = H^0(X, K_X^2)$$

denote the obvious restriction map. From Lemma 3.6 it follows that for any $s \in \Lambda$ and $\beta \in H^0(X, K_X)^{\otimes 2}$, the equality

$$(3.14) \quad \lambda(s + \beta) = \lambda(s) + R(6\beta) \in \mathcal{Q}$$

holds. This equality implies that λ is surjective if and only if the homomorphism R in (3.13) is surjective. From M. Noether's theorem ([ACGH], page 117) we know that if X is non-hyperelliptic then R is surjective. Moreover, for elements $s, t \in \Lambda$ to have the same image under λ , we must have $s - t \in H^0(S, K_S(-\Delta))$. A similar argument shows that in the non-hyperelliptic case λ remains surjective when restricted to the subspace of Λ consisting of sections symmetric under the map $\tilde{\sigma}$ induced from $\sigma : S \rightarrow S ((x, y) \mapsto (y, x))$. Related observations have been made by Tyurin [T].

Remark 3.15. Let $(X_T, \Gamma_T) \longrightarrow T$ be a family of Riemann surfaces with theta characteristic. This means that Γ_T is a holomorphic line bundle on X_T , the total space of the family of Riemann surfaces, and for any $t \in T$, the restriction Γ_t to the Riemann surface X_t satisfies the condition that $\Gamma_t^{\otimes 2} = K_{X_t}$. Consider the line bundle

$$\mathcal{M}_T := p_1^*(K_{\text{rel}} \otimes \Gamma_T^*) \otimes p_2^*(\Gamma_T) \otimes \mathcal{O}(\Delta_T)$$

on the fiber product $X_T \times_T X_T$, where p_i denote the projection to the i -th factor, Δ_T is the diagonal divisor in the fiber product, and K_{rel} is the relative canonical bundle on X_T . Let \mathcal{M}_t be the line bundle on $X_t \times X_t$ obtained by setting $\alpha = \Gamma_t$ in the proof of Lemma 2.2. Clearly the restriction of \mathcal{M}_T to $X_t \times X_t$ is \mathcal{M}_t . The natural isomorphism between $\phi_\alpha^* \mathcal{O}(\Theta)$ and $\mathcal{M}_\alpha \otimes \zeta_\alpha$ mentioned in the proof of Lemma 2.2 shows that the restriction of \mathcal{M}_T to the diagonal Δ_T is the trivial line bundle. We may extend this trivialisation to some analytic neighborhood of Δ_T . Now using the equality (2.4) for the given family of Riemann surfaces we get a holomorphic family of trivialisations of the restriction of L to some neighborhood of the diagonal. Using Theorem 3.2 this family of trivialisations equips the family X_T with a holomorphic family of projective structures.

Given a family of Riemann surfaces, $X_{T'} \longrightarrow T'$, consider the corresponding family of Riemann surfaces with theta characteristic

$$(X_T, \Gamma_T) \longrightarrow T$$

where $p : T \longrightarrow T'$ is the finite étale Galois cover with the fiber of p over $t \in T'$ being the set of all theta characteristics on the corresponding Riemann surface X_t . We earlier saw that there is a holomorphic family of projective structures for the family $X_T \longrightarrow T$. For $x \in T$ let \wp_x denote the projective structure on the Riemann surface over x . For any $t \in T'$ consider the projective structure on X_t given by the average

$$\frac{1}{\#p^{-1}(t)} \sum_{x \in p^{-1}(t)} \wp_x$$

which is defined using the affine space structure on the space of all projective structures on X_t . Using this construction we conclude that the family of Riemann surfaces, $X_{T'}$, admits a holomorphic family of projective structures.

4 Relation with Deligne's definition

We shall now recall another definition of a projective structure given in [D] (Definition 5.6 bis).

The fibers of the natural projection, ν , of the second order infinitesimal neighborhood of the diagonal Δ (in S) onto Δ are isomorphic to $\text{Spec}(R)$, where R is the algebra $\mathbb{C}[\epsilon]/\epsilon^3$. Let P denote the principal $\text{Aut}(\text{Spec}(R))$ bundle on X whose fiber over $x \in X$ is the space of all isomorphisms between $\text{Spec}(R)$ and the fiber of ν over x . On the other hand, $\text{Aut}(\text{Spec}(R))$ is same as the group of all automorphisms of CP^1 that fix the point $0 \in \mathbb{C} \cup \{\infty\} = CP^1$. Let P_{tg} denote the projective bundle on X associated to P . Since $\text{Aut}(\text{Spec}(R))$ fixes a point in CP^1 , the bundle P_{tg} has a natural section which we shall denote by τ . There is a natural isomorphism between the second order infinitesimal neighborhood of Δ and the second order neighborhood of the image of τ (in P_{tg}).

Definition 4.1 “Definition 5.6 bis of [D]”. A projective structure on X is an isomorphism between the third order infinitesimal neighborhood of the diagonal Δ (in S) with the third order infinitesimal neighborhood of τ (in P_{tg}) such that the restriction of this isomorphism to the second order infinitesimal neighborhood of Δ is the canonical isomorphism with the second order infinitesimal neighborhood of the image of τ mentioned above.

If $X = CP^1$, the projective line, then $P_{tg} = CP^1 \times CP^1$. Thus there is a canonical projective structure on CP^1 in the sense of ([D], Definition 5.6 bis) given by the identity map of the third order neighborhood of the diagonal in $CP^1 \times CP^1$.

Let \mathcal{H} denote the sheaf on X which to any open set, $U \subset X$, associates the space of all embeddings of the third order infinitesimal neighborhood of the diagonal of $U \times U$ into the restriction of P_{tg} to U which lift the canonical embedding of the second order neighborhood of the diagonal of $U \times U$.

Let us recall [D] that a K_X^2 -torsor is a holomorphic fiber bundle over X such that its fiber over any $x \in X$ is equipped with a free, transitive holomorphic action of the fiber K_x^2 . In other words, the result of the action of a local holomorphic section of K_X^2 on a local holomorphic section of the torsor is again a local holomorphic section.

Proposition 5.8 (page 32) of [D] says that the sheaf \mathcal{H} defined above is a K_X^2 -torsor.

We shall denote the restriction of L to $n\Delta$ by $L(n)$. For an analytic open set U of X let Δ_U be the diagonal divisor on $U \times U$. Let \mathcal{G} denote the sheaf on X which to any open set $U \subset X$ associates the space of all trivialisations of the restriction of $L(3)$ to $3\Delta_U$ giving the canonical trivialisations on $2\Delta_U$. From (3.3) it follows that the restriction $\mathcal{G}(U)$ is an affine space for $H^0(U, K_U^2)$, where K_U is the canonical bundle of U . In other words, \mathcal{G} is a torsor for the sheaf K_X^2 .

Our aim in this section is to prove the following theorem:

Theorem 4.2. *The two K_X^2 -torsors on X , namely \mathcal{G} and \mathcal{H} , are canonically isomorphic.*

Theorem 4.2 gives a natural identification of the space \mathcal{Q} , the space of global sections of \mathcal{G} , with the space of global sections of \mathcal{H} , which is the space of all projective structures on X in the sense of ([D], Definition 5.6 bis).

Proof of Theorem 4.2. We shall prove the theorem by constructing a third K_X^2 -torsor, \mathcal{T} , on X and identifying both \mathcal{G} and \mathcal{H} with \mathcal{T} . A reason for introducing \mathcal{T} as the intermediate step is that its construction might be of some interest.

For $n \geq 0$, let $J^n(X)$ denote the sheaf of jets of order n on X , which is a vector bundle on X of rank $n + 1$. Define $J_0^n(X)$ to be the kernel of the obvious projection of $J^n(X)$ onto $J^0(X)$. Note that there is a canonical splitting of the inclusion of $J_0^n(X)$ into $J^n(X)$ given by the constant functions. Let $\mathcal{P}(X)$ denote the subset of the total space of $J_0^3(X)$ given by the inverse image of $\{K_X - 0\}$ (the set of all nonzero vectors in the total space of K_X) under the projection of $J_0^3(X)$ onto $J_0^1(X) = K_X$. The space $\mathcal{P}(X)$ admits a natural action (by composition of functions) of the group $M(0)$, the isotropy group of $0 \in \mathbb{C}$ for the Möbius group action on CP^1 . The action

of $M(0)$ on $\mathcal{P}(X)$ is free, since the only Möbius transformation of CP^1 , which acts as the identity map on the second order neighborhood of a point, is actually the identity transformation ([D], page 29).

Let \mathcal{T} denote the quotient of $\mathcal{P}(X)$ by $M(0)$. A projective structure on X gives maps from neighborhoods of points of X into CP^1 which differ only by a Möbius transformation, and hence gives a section of the obvious projection of \mathcal{T} onto X . An identification between the space of all sections of \mathcal{T} and \mathcal{G} , the space of projective structures on X , is obtained in this way.

We shall now give a K_X^2 -torsor structure on \mathcal{T} . Let $f \in \mathcal{P}(X)$ be an element over $x \in X$, and let $v \in (K_X^2)_x$ be an element of the fiber of K_X^2 over x . Let \bar{f} be a function defined around x which represents f . Since $d\bar{f}(x) \neq 0$ (by the definition of $\mathcal{P}(X)$), there is a number $\lambda \in \mathbb{C}$ such that $v = \lambda d\bar{f}(x) \otimes d\bar{f}(x)$. Consider the function

$$(4.3) \quad \bar{f}_\lambda := \bar{f} + \lambda \bar{f}^3$$

defined around x . The element in $\mathcal{P}(X)$ over x represented by the function \bar{f}_λ clearly does not depend upon the choice of the representative \bar{f} of f . An action of K_X^2 on $\mathcal{P}(X)$ is obtained by mapping the pair (v, f) to the element of $\mathcal{P}(X)_x$ represented by \bar{f}_λ . This is a free (but not transitive) action of the abelian group scheme K_X^2 over X . This action of K_X^2 on $\mathcal{P}(X)$ induces a K_X^2 -torsor structure on the quotient space \mathcal{T} of $\mathcal{P}(X)$. Indeed, this is a consequence of the following fact: let $J_0^n(0)$ be the jets of order n of functions vanishing at $0 \in \mathbb{C}$; in this notation, the group $M(0)$ acts freely and transitively on the subset of $J_0^2(0)$ consisting of all elements whose image in $J_0^1(0)$ is nonzero. (If i denotes the isomorphism from the space of all sections of \mathcal{T} to \mathcal{G} , then $i(A + \gamma) = i(A) + 6\gamma$ for any $\gamma \in H^0(X, K_X^2)$.)

Theorem 4.2 is a consequence of the assertion that both \mathcal{G} and \mathcal{H} coincide with this K_X^2 -torsor \mathcal{T} . We shall first show that \mathcal{G} coincides with \mathcal{T} .

Take any $x \in X$, and let $f \in \mathcal{T}_x$ be an element of the fiber over x . Let z be a function defined in a neighborhood, U , of x , that represents f . Since $dz(x) \neq 0$, we may assume that z is a biholomorphism onto its image. Let $\bar{z} = (z, z)$ be the biholomorphism defined on $U \times U$. Pull back the section s (defined in (3.4)) to $U \times U$ using this map \bar{z} . Let \hat{f} denote the local section of \mathcal{G} obtained by restricting this section to the second order infinitesimal neighborhood of the diagonal. The evaluation at x , namely $\hat{f}(x)$, depends only on f and not on the representing function z . Thus we have a map from \mathcal{T} to \mathcal{G} which is evidently an isomorphism. We want to check that this isomorphism preserves the K_X^2 -torsor structures of \mathcal{T} and \mathcal{G} .

Take an element $v = \lambda(dz)^{\otimes 2} \in K_x^2$, where $\lambda \in \mathbb{C}$. From the definition of the K_X^2 -torsor structure on \mathcal{T} in (4.3) it follows that the local function $z + \lambda z^3$ represents the

result of the action of v on f . Remark 3.11 says that the two sections of \mathcal{G} , represented by z and $z + \lambda z^3$ respectively, differ by $\mathcal{S}(z + \lambda z^3)(0)/6$. Since

$$\mathcal{S}(z + \lambda z^3)(0) = 6\lambda$$

the preservation of the K_X^2 -torsor structures of \mathcal{T} and \mathcal{G} is established.

Next we want to show that \mathcal{H} coincides with \mathcal{T} . Take x , f and z as above. We noted earlier that CP^1 has a canonical projective structure (in the sense of [D], Definition 5.6 bis) given by the identity map of the third order neighborhood of the diagonal. This projective structure induces a projective structure on U by the biholomorphism z . The evaluation of the section (over U) of \mathcal{H} , thus obtained, at the point x , does not depend upon the choice of the representative z of f . This gives the required K_X^2 -torsor structure preserving isomorphism between \mathcal{T} and \mathcal{H} .

As an alternative proof of Theorem 4.2 we shall give a direct identification between \mathcal{G} and \mathcal{H} using coordinate charts.

Let (U, z) be a coordinate chart around $x \in X$ with $z(x) = 0$. Using (3.8) we get a section of \mathcal{G} over U . We shall denote this section as f_z .

Since the only Möbius transformation of CP^1 , which acts as the identity map on the second order neighborhood of a point, is actually the identity map, and the group of Möbius transformations acts transitively on CP^1 , there is a natural projective structure on CP^1 in the sense of ([D], Definition 5.6 bis).

Since the function z identifies U with an open set in CP^1 , we get a local section of \mathcal{H} , which we shall denote by g_z .

By mapping the section f_z to g_z we get an identification of the restriction $\mathcal{G}(U)$ with $\mathcal{H}(U)$ which preserves the torsor structures. We shall show that this identification does not depend upon the choice of the coordinate function z .

Let (V, w) be another coordinate chart around x . Thus

$$(4.4) \quad w = \sum_{i=0}^{\infty} a_i z^i$$

with $a_1 \neq 0$. Let f_w (resp. g_w) denote the local section of \mathcal{G} (resp. \mathcal{H}) for (V, w) . It is a simple calculation using (4.4) to check that

$$f_w(x) - f_z(x) = \frac{a_1 a_3 - a_2^2}{a_1^2} dz \otimes dz = g_w(x) - g_z(x)$$

This completes the proof of the theorem. □

5 Genesis in conformal field theory

In this section we explain how the above definition of projective connection in terms of trivialisations of L on 3Δ came out of some investigations on a model quantum field theory on a curve (see [R1]-[R3]), which give it the intuitive picture of a *generalized cross ratio* on a compact Riemann surface, in the limit when all of its arguments are made to coalesce.

The application of algebraic geometry to quantum field theory in [R1]-[R3] rests on replacing the study of “quantum fields”, which are not geometric objects, by their so-called “ n -point functions” which are hypothesised to be so. Thus in [R1] and [R2] we identified the “ n -point functions” of the defining “quantum fields” of the model with meromorphic sections of certain line bundles on the n -fold Cartesian product of the curve X . In [R3] we showed how the n -point functions of the “current” j , which is a “regularised product” of the defining fields, could be computed by the use of *schemes* having *nilpotent elements* to give a precise meaning to the coalescing of arguments involved in the definition of the current.

A similar regularised product of currents gives the “energy-momentum tensor” T of the system, a fact usually expressed by saying that T is in “Sugawara form”. This is a feature of many conformal quantum field theories and plays an important role in the theory of the Virasoro algebra [KR]. The heuristic expectation in (conformal) quantum field theory [BPZ] is that its “one point function” $\langle T(z) \rangle$ is a *projective connection*.

Our study of $\langle T(z) \rangle$ proceeds from the calculation of the two point function of currents $\langle j(z)j(w) \rangle$ in [R3]. The salient point is the introduction of the remarkable line bundle $\mathcal{A} := \mathcal{O}(D_{12} + D_{34} - D_{14} - D_{23})$ on $X^4 := X_1 \times X_2 \times X_3 \times X_4$, the product of 4 copies of X , where D_{ij} denotes the divisor of X^4 defined by the diagonal of X_i and X_j . It was pointed out in [R3] that the canonical meromorphic section $1_{\mathcal{A}}$, associated with the divisor defining \mathcal{A} , is a natural generalisation to an arbitrary compact, connected Riemann surface of the *cross ratio* of 4 points in the complex plane. It was shown in [R3] that the calculation of $\langle j(z)j(w) \rangle$ requires the trivialisability of \mathcal{A} on the product scheme $Z := 2\Delta_{13} \times 2\Delta_{24}$, where Δ_{ij} is the diagonal of $X_i \times X_j$ and $2\Delta_{ij}$ denotes its first infinitesimal neighborhood.

Proposition 5.1. *The line bundle $\mathcal{A} := \mathcal{O}(D_{12} + D_{34} - D_{14} - D_{23})$ is trivialisable on $Z := 2\Delta_{13} \times 2\Delta_{24}$ and, moreover, if $\rho \in H^0(Z, \mathcal{A} | Z)$ denotes such a trivialisation, then*

$$(5.2) \quad 1_{\mathcal{A}} | Z - \rho = \omega_B$$

where ω_B denotes a symmetric meromorphic section of $K_{X \times X}$ with double pole on the diagonal, defined by a holomorphic section of $K_{X \times X}(2\Delta)$ which restricts to 1 on the

diagonal.

As pointed out in [R3], equation (5.2) can be regarded as the precise algebro-geometric formulation of the following well known formula expressing the meromorphic bidifferential ω_B in terms of the “prime form” $E(x, y)$ (see Fay[F], eqn.(28) p.20):

$$(5.3) \quad \omega_B(x, y) = \frac{\partial^2 \ln E(x, y)}{\partial x \partial y}$$

In this way it was shown in [R3] that the two point function of currents $\langle j(z)j(w) \rangle$ is a *symmetric meromorphic bidifferential with a double pole on the diagonal*. The computation of the “one point function” $\langle T(z) \rangle$ from $\langle j(z)j(w) \rangle$ now requires that the line bundle $K_{X \times X}(2\Delta)$ should be trivialisable on the *second* infinitesimal neighborhood 3Δ of Δ in $X \times X$ (see [R4] for further details), the validity of which follows from results in [R1]. In this way we arrive at our proposed definition of projective connection, having started with the generalised cross ratio and ended with the coalescing of all of its arguments.

The fact that a symmetric meromorphic bidifferential gives rise to a projective connection appears to have been first observed in [HS] (see also [F] p.20, following eqn.(28) cited above). The techniques used in these references, however, do not give a *characterisation* of a projective structure, as is provided by Theorem 3.2, nor an understanding of when all projective structures arise in this way, as is provided by Remark 3.12. Moreover, their approach cannot be adapted to the study of $\langle T(z) \rangle$, for which we require an algebro-geometric approach, which will make possible the study of the *higher* point functions as well as other related problems. It also appears to be a fact that [HS] and [F] are inaccessible to most geometers and so we hope that the present treatment clarifies some of these results.

The present formulation of the concept of projective connection was announced in several conferences and also in [R4], where the interested reader will find, in addition, a survey for mathematicians of the papers [R1], [R2] and [R3].

Acknowledgments: The authors are very grateful to Prof. M. S. Narasimhan for his useful comments. The first named author is thankful to the Institut Fourier and the Académie des Sciences, Paris, for their hospitality and support. The second named author thanks the International Centre for Theoretical Physics, Trieste, for its hospitality.

References

- [ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris : Geometry of Algebraic Curves, Volume I. Grundlehren der Math. Wiss. **267**. Springer-Verlag, Berlin-Heidelberg- New York, 1985.
- [BPZ] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov : Infinite conformal symmetry in two-dimensional quantum field theory. Nuclear Phys. **B241** (1984) 333-380.
- [BR] I. Biswas, A.K. Raina : Relative curves, Theta divisor, and Deligne pairing. Int. Math. Res. Not. **No. 9** (1995) 425-435.
- [Bi] I. Biswas : A remark on the jet bundles over the projective line, to appear in Math. Res. Lett.
- [D] P. Deligne : Equations Différentielles à Points Singuliers Réguliers. Lecture Notes in Math. **163**. Springer-Verlag, Berlin-Heidelberg-New York, 1970.
- [F] J. Fay : Theta functions on Riemann surfaces. Lecture Notes in Math. **352**. Springer-Verlag, Berlin- Heidelberg-New York, 1973.
- [G] R.C. Gunning : Lectures on Riemann surfaces. Mathematical notes 2. Princeton University Press, Princeton, New Jersey, 1966.
- [HS] N.S. Hawley, M. Schiffer : Half-order differentials on Riemann surfaces. Acta. Math. **115** (1966) 199-236.
- [KR] V.G. Kac, A.K. Raina : Highest weight representations of infinite dimensional Lie algebras. World Scientific, Singapore, 1987.
- [NR] M.S. Narasimhan, S. Ramanan : 2θ -linear systems in Abelian varieties, in: Vector Bundles on Algebraic Varieties, T.I.F.R. and Oxford Univ. Press, 1984.
- [R1] A. K. Raina : Fay's trisecant identity and conformal field theory. Commun. Math. Phys. **122** (1989), 625-641; Fay's trisecant identity and Wick's theorem: an algebraic geometry viewpoint. Expositiones Math. **8** (1990) 227-245.
- [R2] A. K. Raina : An algebraic geometry study of the $b - c$ system with arbitrary twist fields and arbitrary statistics. Commun. Math. Phys. **140** (1991) 373-397.
- [R3] A. K. Raina : An algebraic geometry view of currents in a model quantum theory on a curve. C. R. Acad. Sci. Paris, **318** (1994) 851-856.

- [R4] A. K. Raina : An algebraic geometry view of a model quantum field theory on a curve. Preprint series of Max-Planck-Institute, Bonn, no. 95-59 (1995). To appear in : Proc. of Pacific Rim Geometry Conference, Singapore 1994, editors: J. Berrick, B. Loo, H. Wang (W. de Gruyter, in press).
- [T] A. N. Tyurin : On periods of quadratic differentials. Russian Math. Surveys **33**:6 (1978) 169-221

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, INDIA

E-mail : indranil@math.tifr.res.in

Theoretical Physics Group, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, INDIA

E-mail : raina@theory.tifr.res.in