THE PROPAGATION OF LIGHT IN ABSORBING BIAXIAL CRYSTALS—I. THEORETICAL

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1. INTRODUCTION

ABSORBING biaxial crystals in general display a variety of remarkable optical phenomena in the vicinity of both the optic axes. For example, if an extended source of unpolarised light is viewed through a plate of highly pleochroic material cut normal to an optic axis, two dark brushes—the Brewster's brushes—are generally seen in the field of view; while if a polariser be inserted in front of the plate, the so-called idiophanic rings are observed—similar to the interference rings that can appear in the case of a transparent crystal *if an analyser be also present.*

The theoretical investigations of Waldemar Voigt focussed attention on the fact that certain of the features relating to the propagation of light in the vicinity of an optic axis differ radically from those obtaining in transparent media. Thus, whereas along any general direction in a transparent crystal there are two particular linearly polarised vibrations that can be propagated without change of form, this is no longer the case in absorbing crystals. As a matter of fact, close to an optic axis and on either side of it, there even exist two directions—the singular axes—with the following remarkable properties: *only* a right-circular vibration can be propagated without change of form along one of these axes, and *only* a left-circular vibration along the other.* In this paper it will be shown that the various features of the propagation of light in absorbing media may also be conveniently regarded as due to the superposed effects of birefringence and dichroism. Because of the simplicity of the method, it has also been possible to make a more detailed investigation of the following interesting question: what will happen when, for example, a right-circular vibration is incident in the direction of a singular axis where only a left-circular vibration can be propagated without change of form? The results obtained in this connection are at variance with those expected by Voigt (Section 6).

* A non-mathematical summary, in English, of the main results of Voigt's investigations may be found in Reference 1.

The analysis of the propagation of light in absorbing biaxial media from the standpoint of the electromagnetic theory has been discussed by several authors. The comparatively simple case of orthorhombic crystals (where the principal axes of the dielectric tensor and the complex conductivity tensor necessarily coincide) is discussed in Drude's treatise.[2] But the somewhat oversimplified presentation there given omits entirely those theoretical and experimental features with which we shall be particularly concerned. These may be found described in more detailed treatments,[3-5] particularly those of Voigt and Pockels. In view of the complexity of the phenomena involved, it would appear that a consideration of the problem from a simpler though less rigorous standpoint would certainly be useful. Such an approach is provided in the present paper, and, as we shall show, the method adopted leads to results that are practically identical with those of the electromagnetic theory.

2. General Features of Light Propagation in Absorbing Media

(*a*) *The Index and Absorption Ellipsoids*

In an absorbing biaxial medium not possessing optical activity, the two waves propagated along any direction appreciably inclined to both the optic axes may be regarded as practically plane polarised (though not rigorously so, as in the transparent crystal). And as in a transparent crystal their 'vibration-directions' may then be considered to lie on the principal planes, their velocities being determined by their vibration-directions thus: the reciprocal of any radius of a so-called *index ellipsoid* gives the velocity for vibrations parallel to that radius. In addition, the two waves have different coefficients of extinction κ_1 and κ_2, these being determined again by their vibration-directions thus: the reciprocal of any radius of a so-called *absorption ellipsoid* gives the value of $\sqrt{(2\kappa v^3/c)}$ for vibrations parallel to that radius, v being the velocity for that vibration-direction.

By assuming that the above statements hold good even for directions in the vicinity of an optic axis, it is indeed possible to explain *some* of the phenomena observed there—and such a procedure is in fact followed in Drude's treatise. For example, the occurrence of Brewster's brushes can be explained along the following lines. In the neighbourhood of an optic axis, a comparatively small change in the direction of propagation will in general cause an appreciable change in the inclinations of the two principal planes to the axial plane; this in turn will lead to a large variation in the total absorption, since the absorption coefficients of the two waves will be determined by the orientation of their vibration-directions.

(b) *The Elliptical Polarisation of the Waves*

The appearance of idiophanic interference rings with a polarizer *alone* cannot however be explained on the assumption that the light incident along any direction is split up into two linearly polarized beams with their vibrations at right angles to one another; their states of polarization being *orthogonal,* two such beams will be incapable of interference with one another (unless brought to the same plane of vibration by an analyser).

The fact is that when we turn to directions of propagation in the vicinity of an optic axis, we are no longer justified in neglecting a remarkable and important consequence of the phenomenological theory: namely, that the two waves propagated in any general direction in an absorbing biaxial medium are in reality, *elliptically polarized.* Though the two elliptic vibrations have their axes *majores* at right angles, and their ellipticities equal, they are rendered *non-orthogonal* by the fact that they are of the *same* handedness; and this last mentioned feature (together with the fact that the major axes do not in general coincide with the principal planes) distinguishes the situation sharply from that obtaining in optically active (transparent) crystals.

In the context of the elliptical polarisation of the waves, the index and absorption ellipsoids—strictly speaking—retain significance only in terms of the dielectric and conductivity-like tensors by means of which they are *defined.* Nevertheless, as we shall show, the existence of the two non-orthogonal elliptically polarised waves may be conveniently treated as due to the superposed effects of birefringence and dichroism—just as the propagation of two orthogonal elliptically polarised waves near to an optic axis in an optically active transparent medium, may (by Gouy's hypothesis[4,6]) be conveniently treated as due to the superposed effects of birefringence and rotation.

3. The Superposition of Birefringence and Dichroism

Consider a plate cut perpendicular to an arbitrary direction z which is also taken as being normal to the plane of the paper. Let OX_r and OY_r (Fig. 1) be the trace of the principal planes of refraction—defined as usual, either in terms of the index ellipsoid or the optic binormals. Similarly let OX_k and OY_k be the trace of the *principal planes of absorption*—which we shall define analogously, *either* as containing the major and minor diameters of the elliptical section of the absorption ellipsoid made by the plane of the paper, *or* as the internal and external bisectors of the angle subtended on the z-direction by the two absorption-binormals (normals to the circular sections of the absorption ellipsoid).

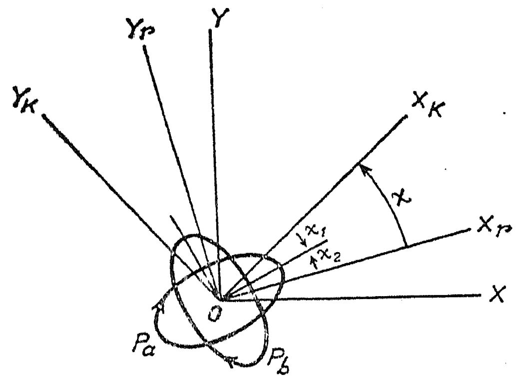

FIG. 1

P_a and P_b—States of polarisation propagated unchanged.
OX_k and OY_k—Trace of the principal planes of absorption.
OX_r and OY_r—Trace of the principal planes of refraction.

Consider an arbitrary elliptic vibration P (which, for the sake of concreteness, may be temporarily identified with the one marked P_a in the figure). In the absence of absorption the arbitrary elliptic vibration P will be resolved into two vibrations (along the principal planes of refraction, OX_r and OY_r) between which an infinitesimal phase difference δdz will be introduced corresponding to a passage dz. Since, however, anisotropic absorption is also present, we perform, in addition, the infinitesimal operation of linear dichroism; the elliptic vibration—as modified by the infinitesimal operation of birefringence—is resolved into two linear vibrations (this time, along the principal planes of absorption, OX_k and OY_k), the amplitudes of which are then reduced by the multiplying factors $(1 - k_1dz)$ and $(1 - k_2dz)$ respectively. The differential absorption of the two components will cause the state of the elliptic vibration to 'move towards' the state of polarisation of the less absorbed component OX_k (a phrase which acquires a more vivid meaning in the Poincaré sphere representation). Those states of polarisation alone can be propagated without change of form, which under the successive infinitesimal operations of birefringence and dichroism (applied in either order) remain unaltered in form and orientation—and to these states of polarisation alone can definite velocities of propagation and coefficients of absorption be assigned.

Several particular cases may first be noted. Along the optic axial directions where the birefringence vanishes, the two waves (propagated with different coefficients of absorption) are linearly polarised along the principal planes of absorption. Similarly, the waves propagated along directions appreciably inclined to the optic axes will be practically plane polarised along the principal planes of refraction, since the absolute values of the dichroism (k_2-k_1) is usually such that it is very small compared with the birefringence $(\delta_1-\delta_2)$ along such directions. Also, where the principal planes of absorption and refraction coincide (as for example along the axial plane in orthorhombic crystals) the waves will be rigorously linearly polarised along the common principal planes.

The more general case, where the solution is not so apparent, is discussed analytically in Section 7, but the main results will first be proved more briefly and elegantly by the use of the Poincaré sphere. For this purpose, the form of the arbitrary elliptic vibration P—as distinct from its intensity and absolute phase—must first be specified by means of certain parameters. The principal planes of absorption and refraction form the two natural co-ordinate systems to which the vibration P may be referred. The ratio tan ϕ of the amplitudes of the components of the vibration P along OY_r and OX_r does not by itself completely specify the form of the elliptic vibration (since the phase difference θ_r between these components has also to be given). Similarly the ratio tan ψ of the amplitudes of the components of P along OY_k and OX_k does not by itself completely specify the form of P (since the phase difference θ_k between these components has also to be given). But ϕ and ψ together form two convenient symmetrical parameters completely specifying the form of the elliptic vibration P (ϕ, ψ)—provided we separately give the sense of description of the ellipse. It may also be noted that apart from an intensity factor, cos 2ϕ and cos 2ψ are the two values of the second Stokes parameter M of the vibration P (ϕ, ψ) when it is referred successively to co-ordinate systems along the principal planes of refraction and absorption respectively.

As indicated in the figure, it turns out that there are two particular elliptic vibrations described in the same sense, P_a (ϕ_a, ψ_a) and P_b (ϕ_b, ψ_b), that can be propagated without change of form under the superposed effects of birefringence and dichroism. The form of the vibration P_b can be obtained from that of P_a merely by rotating the latter by 90° in its own plane—which means that ϕ_b and ψ_b are complementary to ϕ_a and ψ_a respectively.

4. Use of the Poincaré Sphere for Superposition

(*a*) *The General Method*

The Poincaré sphere,[4,7] which has proved very useful for the analysis of the propagation of polarised light in *transparent* media, turns out also to be of great use in our present discussion on absorbing crystals.

As is well known, a one-to-one correspondence can be set up between all the points on the surface of a sphere (the Poincaré sphere) and all the possible forms of elliptic vibrations that can be conceived (circular and linear vibrations being regarded as particular cases of elliptic vibrations). In particular the arbitrary elliptic vibration P referred to in the previous section is represented by a corresponding point P on the Poincaré sphere, while a linear vibration along OX_k will be represented by some other point X_k. The infinitesimal operation of anisotropic absorption described in the last section will obviously alter the form of the elliptic vibration P in such a manner that one may say it gets more polarised in the direction of OX_k, since this is the less absorbed component. This infinitesimal alteration in the state of the elliptic vibration P corresponds (on the Poincaré sphere) to an infinitesimal movement of the point P directly towards X_k, *i.e.*, along the direction of the shortest arc joining them.

Similarly the infinitesimal alteration in the state of the (initial) elliptic vibration P due to the operation of birefringence alone, corresponds to an infinitesimal movement ds_r of the representative point P. If the elliptic vibration is to be propagated without change of form, this movement ds_r should be equal and opposite to the displacement of P due to dichroism alone; and the problem of finding the states of polarisation that can be propagated without change of form is therefore reduced to the simple geometrical problem of finding the points P on the Poincaré sphere which satisfy the above requirement.

(b) The Operations of Dichroism and Birefringence

Referring to Fig. 2, let X_k and Y_k give the orientations of the principal planes of absorption. (The arbitrary elliptic vibration P has not been indicated on the sphere, but for the sake of concreteness, may be temporarily identified with the particular state P_a in the figure.) If the elliptic vibration P is resolved into two orthogonal linear vibrations in the states X_k and Y_k, then the amplitudes F_k and G_k of these components will be proportional to $\cos\psi$ and $\sin\psi$, where 2ψ is the angular distance of the point P from X_k on the Poincaré sphere. (For a proof of this statement, see reference 7.) Hence $G_k/F_k = \tan\psi$. If the amplitudes of these components are reduced by the multiplying factors $e^{-k_1 z}$ and $e^{-k_2 z}$, the angular distance of P from X_k will change from 2ψ to $2\psi'$ where:

$$\tan\psi' = \frac{G_k\, e^{-k_2 z}}{F_k\, e^{-k_1 z}} = \tan\psi\, e^{-(k_2-k_1)\, z} \tag{1}$$

Since the phases of the X_k and Y_k components of P are to be left unaltered in this operation, the movement of P will be entirely on the meridional arc

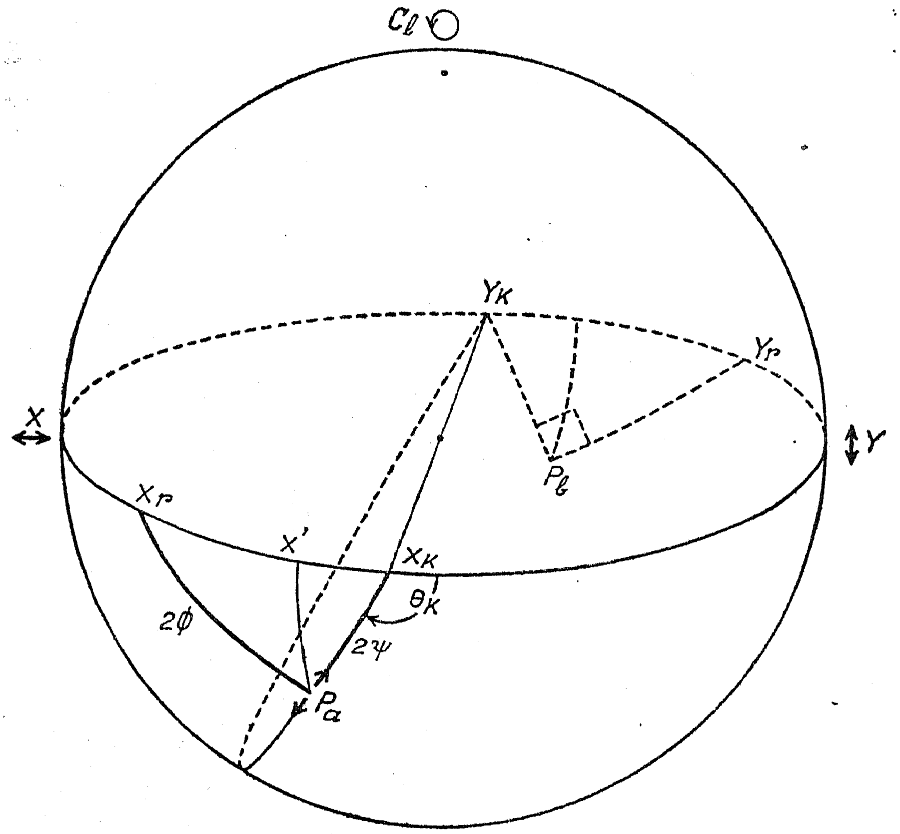

FIG. 2

(X_k, Y_k) and (X_r, Y_r)—Principal planes of absorption and refraction, respectively.

P_a and P_b —States of polarisation propagated unchanged.

$X_rX_k = 2\chi$; $X_kX' = 2\chi_1$; $X_rX' = 2\chi_2$; $P_aX' = 2\theta$.

Y_kPX_k. This follows from the fact that *the X_k and Y_k components of all elliptic vibrations on this arc have the same phase difference* θ_k (where θ_k is the angle indicated in Fig. 2); for, if an additional phase difference $-\theta_k$ be introduced between these components, any such elliptic vibration will be reduced to a linear vibration on the equatorial arc $X_kY_rY_k$ (by a well known property of the Poincaré sphere).

The infinitesimal operation of linear dichroism (corresponding to a passage dz) will, apart from reducing the intensity, cause the initial state of polarisation P to move along the arc PX_k towards the state X_k (the less absorbed component), through an arc $ds_k = -2d\psi$. From (1) we have

$$\tan\psi + d(\tan\psi) = [1-(k_2-k_1)\,dz]\tan\psi$$

or

$$2\sec^2\psi d\psi = -2(k_2-k_1)\,dz.\tan\psi$$

which leads to the simple relation

$$ds_k = (k \sin 2\psi)\, dz \tag{2}$$

where

$$k \text{ denotes } (k_2 - k_1)$$

Let X_r and Y_r represent the orientations of the principal planes of refraction, the former corresponding to the slower wave in the absence of absorption; let δ denote the phase difference $(\delta_1 - \delta_2)$ introduced per unit distance (in the absence of dichroism). The infinitesimal operation of birefringence (corresponding to a passage dz) consists in rotating the sphere clockwise about the equatorial diameter $X_r\, Y_r$ through the infinitesimal angle δdz. This operation will cause the initial state of polarization P to move along the arc of a small circle with X_r as pole through an arc ds_r, where

$$ds_r = (\delta \sin 2\phi)\, dz \tag{3}$$

(c) *The States of Polarisation Propagated Unchanged*

In order that the simultaneous superposition of linear dichroism and birefringence should cause no change in the state of P, the movements ds_k and ds_r must be equal in magnitude and opposite in direction. Since arc ds_k is along PX_k while arc ds_r is perpendicular to PX_r we must have firstly, $X_r PX_k = \pi/2$, or

$$\cos 2\chi = \cos 2\phi \cos 2\psi \tag{4}$$

and secondly

$$\delta \sin 2\phi = k \sin 2\psi \tag{5}$$

together with the condition that P will be a right- or left-elliptic vibration according as χ (the angle between the X_k and X_r axes) is positive (0 to $\pi/2$) or negative (0 to $-\pi/2$).

In general there are two positions P which simultaneously satisfy these conditions—the relations (4) and (5) being unchanged when we alter 2ϕ and 2ψ to $\pi - 2\phi$ and $\pi - 2\psi$ respectively. Thus in the figure the state P_b whose distances from the points Y_k and Y_r are 2ψ and 2ϕ, is also propagated unchanged. The states P_a and P_b have the same latitudes, their longitudes differing by π.

Hence we arrive at the result, also obtained from the electromagnetic theory, that the states of polarisation propagated unchanged along any general direction are two similarly rotating elliptic vibrations which have their major axes crossed and their ellipticities equal.

In order to construct these two elliptic vibrations (see Fig. 1) we must first determine the orientations of one of the principal diameters—for example the orientation of OX′ which is the major axis of P_a and also the minor axis of P_b. Let the inclination of OX′ be χ_2 (anti-clockwise) with respect to OX_r and χ_1 (clockwise) with respect to OX_k. The direction of OX′ may be determined by the relation

$$\frac{\sin 4\chi_1}{\sin 4\chi_2} = \frac{\delta^2}{k^2} \tag{6}$$

Next the ratio $\tan\theta$ of the minor to the major axis of the elliptic vibrations may be obtained from

$$\sin^2 2\theta = \tan 2\chi_1 \tan 2\chi_2 \tag{7}$$

—a relation which gives the ellipticity ($\tan\theta = \operatorname{cosec}^2 2\theta + \sqrt{\operatorname{cosec}^2 2\theta - 1}$) in terms of the orientation of the axes of the elliptic vibrations. It may be noted that equations (6) and (7) remain unaltered when we change χ_1 and χ_2 to $(\pi/2 + \chi_1)$ and $(\pi/2 + \chi_2)$ respectively.

The relations stated in the previous paragraph will now be proved with the aid of the Poincaré sphere (Fig. 2). The direction OX′ is given by the point X′ on the equator having the same longitude as P. Then since the triangles X_rPX' and X_kPX' are both right-angled, we have

$$\left.\begin{aligned} \cos 2\phi &= \cos 2\chi_2 \cos 2\theta \\ \cos 2\psi &= \cos 2\chi_1 \cos 2\theta \end{aligned}\right\} \tag{8}$$

Multiplying these equations and comparing with (4),

$$\cos(2\chi_1 + 2\chi_2) = \cos 2\chi_1 \cos 2\chi_2 \cos^2 2\theta$$

which on simplification gives the relation (7).

To prove (6) we consider the right-angled triangles X_kPX_r and X_kPX':

$$\cos \hat{X}_k = \frac{\tan 2\psi}{\tan 2\chi} = \frac{\tan 2\chi_1}{\tan 2\psi}$$

Hence

$$\tan^2 2\psi = \tan 2\chi_1 \tan 2\chi$$

$$\tan^2 2\phi = \tan 2\chi_2 \tan 2\chi$$

or,

$$\frac{\tan^2 2\psi}{\tan^2 2\phi} = \frac{\tan 2\chi_1}{\tan 2\chi_2} \tag{9}$$

From relations (8) we also have

$$\frac{\cos^2 2\psi}{\cos^2 2\phi} = \frac{\cos^2 2\chi_1}{\cos^2 2\chi_2} \tag{10}$$

Multiplying (9) and (10) and comparing with (5) we get the required relation (6).

(*d*) *Comparison with the Electromagnetic Theory*

The relation (7) giving the ellipticity in terms of the orientation of the axes of the elliptic vibration, is identical with that deduced from the electromagnetic theory (Pockels, *loc. cit.*, p. 399, eq. 54); while the relation (6) giving the orientation of the axes of the elliptic vibrations has to be compared with the following similar relation (Pockels, *loc. cit.*, p. 397, eq. 53):

$$\frac{\sin 4\chi_1}{\sin 4\chi_2} = \frac{\rho^2}{\sigma^2} \tag{6'}$$

where

$$\rho = \tfrac{1}{2}(a_2 - a_1) \text{ and } \sigma = \tfrac{1}{2}(b_2 - b_1)$$

The tensor components a_1, a_2, etc., may be easily shown to have the following geometrical meanings. The major and minor semi-axes of the elliptical section of the index ellipsoid made by the plane of the paper have lengths $1/\sqrt{a_1}$ and $1/\sqrt{a_2}$ respectively, while the major and minor semi-axes of the elliptical section of the absorption ellipsoid have lengths $1/\sqrt{b_1}$ and $1/\sqrt{b_2}$ respectively. Relations (6) and (6′) will be identical if

$$\delta^2/k^2 = \rho^2/\sigma^2 \tag{11}$$

As pointed out in Section 2, the waves propagated along directions appreciably inclined to the optic axes may be considered as linearly polarised; and for such directions of propagation, if $1/\sqrt{b}$ be the length of any radius of the absorption ellipsoid, then $b = 2\kappa v^3/c$, where κ is the extinction coefficient and v the velocity for that vibration-direction. In consonance with this it would be natural to use the following relation for the hypothetical extinction coefficients κ_1 and κ_2 in the absence of birefringence:

$$\frac{2\kappa_1 v_m{}^3}{c} = b_1\,; \quad \frac{2\kappa_2 v_m{}^3}{c} = b_2 \tag{12}$$

where v_m is a mean velocity. We then have

$$k = \frac{2\pi}{\lambda_0}(\kappa_2 - \kappa_1) = \frac{2\pi c}{\lambda_0}\,\frac{\sigma}{v_m{}^3}$$

Relation (11) will be obviously satisfied if we analogously set

$$\delta = \frac{2\pi c}{\lambda_0} \cdot \frac{\rho}{v_m{}^3} = \frac{2\pi c}{\lambda_0} \cdot \frac{a_2 - a_1}{2v_m{}^3} \tag{13}$$

Since the velocoties v_1 and v_2 in the absence of absorption are equal to $\sqrt{a_1}$ and $\sqrt{a_2}$ respectively, relation (13) will be *exactly* satisfied if we define the mean velocity v_m by

$$v_m{}^3 = \tfrac{1}{2}(v_1 + v_2) \cdot v_1 v_2 \tag{14}$$

5. The Absorption Coefficients and Refractive Indices of the Waves

It is well known that in the case of a transparent crystal, it is simpler to specify the velocities of the waves as functions of the vibration directions than as functions of the directions of propagation: the former leads to the simple index-ellipsoid representation, the latter to the comparatively more complex wave surface of two sheets. We shall show that for absorbing crystals too, if we choose to express the velocities and absorption coefficients as functions of the states of polarisation (ϕ, ψ) of the waves, the resulting expressions (as deduced both by the method of superposition and by the electromagnetic theory) may be put in a very simple form.

When an elliptic vibration of unit intensity in any state of polarisation travels a distance dz, the diminution in its intensity may be calculated directly from the reduction of intensity involved in the infinitesimal operation of dichroism corresponding to the passage dz—since the operation of birefringence produces no reduction in intensity. If, in addition, the elliptic vibration be in a state of polarisation P_a that can be propagated without change of form, this reduction in intensity may be equated to $2k_a\,dz$ where k_a is the coefficient of absorption for that wave. The amplitudes of the X_k and Y_k components of the elliptic vibration P_a will be $\cos\psi_a$ and $\sin\psi_a$ respectively; hence the reduction in intensity of these components will obviously be $2k_1 \cos^2\psi_a\,dz$ and $2k_2 \sin^2\psi_a\,dz$ respectively. Therefore,

$$\left.\begin{aligned} & k_a = k_1 \cos^2\psi_a + k_2 \sin^2\psi_a \\ \text{Similarly} \quad & \\ & k_b = k_1 \sin^2\psi_a + k_2 \cos^2\psi_a \\ \text{So that} \quad & \\ & (k_a - k_b) = (k_1 - k_2)\cos 2\psi_a \end{aligned}\right\} \tag{15}$$

Here 2ψ being the arc PX_k on the Poincare sphere may be evaluated by the relations (4) and (5) of Section 4*c*, which determine the states of polarisation of the waves.

Expressions for the refractive indices of the waves (in terms of the state of polarisation of one of them) are equally simple, being given by:

$$\left.\begin{aligned} n_a &= n_1 \cos^2 \phi_a + n_2 \sin^2 \phi_a \\ n_b &= n_1 \sin^2 \phi_a + n_2 \cos^2 \phi_a \\ (n_a - n_b) &= (n_1 - n_2) \cos 2\phi_a \end{aligned}\right\} \tag{16}$$

The proofs of these relations do not have the same simplicity as those of (15), and will be given only at the end of Section 7, since recourse must be taken to the analytically derived equations obtained there.

(6) The Propagation of Light along the Axes of Circular Polarisation

(*a*) *The Singular Axes*

The electromagnetic theory predicts that close to an optic axis and on either side of it, there exist two directions along each of which only *one* state of polarisation (and not two) can be propagated unchanged: only a right-circularly polarised wave can be propagated along one of these axes, and only a left-circularly polarised wave along the other. These directions have been termed the *Windungsachsen*; Voigt has also referred to them as *singular axes* and we shall follow this simpler nomenclature. At these two axes of circular polarisation the inclinations of the principal planes of absorption with respect to the *corresponding* principal planes of refraction are $+45°$ and $-45°$ respectively. Further, along these two directions the pure bi-refringence term δ is equal to the dichroic term k.

The remarkable property of these axes follows very simply from the standpoint of the method of superposition by the use of the Poincaré sphere. Let us suppose for example that the principal plane of absorption OX_k makes an angle of $-45°$ with respect to the corresponding principal plane of refraction OX_r. In Fig. 3, the diameter X_kY_k will then be at right angles to X_rY_r as shown. If we consider a state of polarisation initially coincident with the pole C_l, it can be seen that its movement ds_r (due to an infinitesimal clockwise rotation δdz, about X_rY_r) will be oppositely directed to the movement ds_k towards the less absorbed component X_k; and the movements will be equal in magnitude if $\delta = k$. Thus a left-circular vibration can be propagated unchanged along such a direction. Further, there can be no other state which can also be propagated unchanged, since the 2 elliptic vibrations propagated unchanged along any direction must have the same sense of description and the same ellipticity. Similarly, where OX_k makes an angle

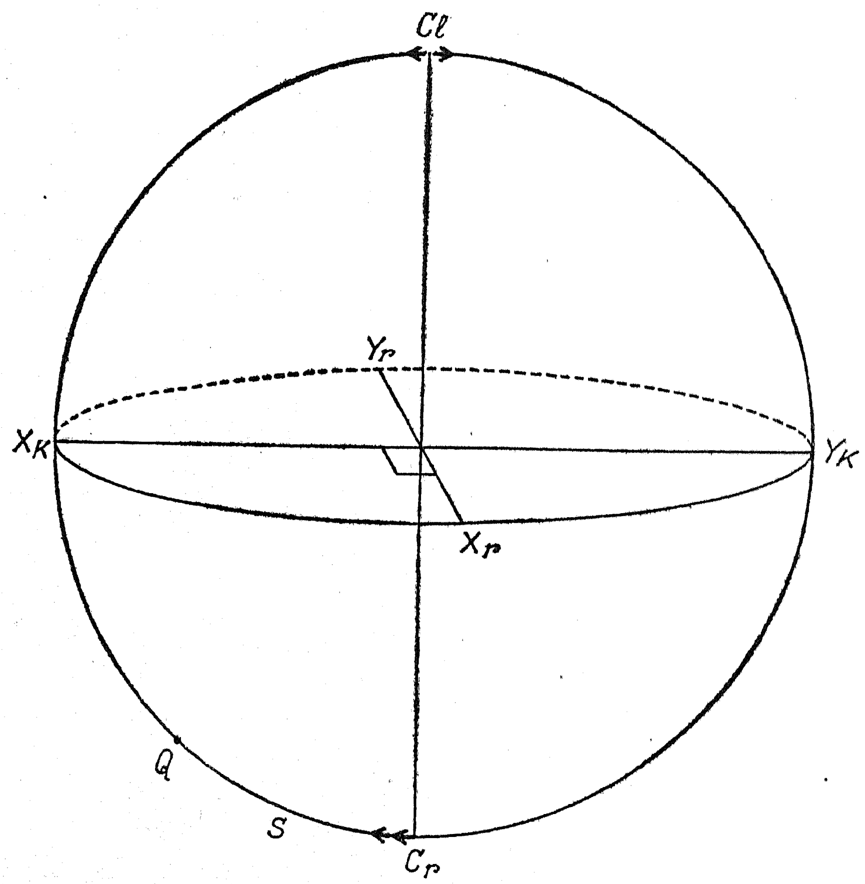

FIG. 3

C_r—Right circular vibration incident in the direction of a singular axis (where only the left-circular vibration C_l can be propagated unchanged).

Q—The state of vibration at a depth z (specified by the arc s).

of $+45°$ with respect to OX_r, and where in addition $\delta = k$, only a right-circular vibration can be propagated unchanged. The refractive index of the circularly polarised wave that can be propagated unchanged along a singular axis is $\frac{1}{2}(n_1 + n_2)$ and its absorption coefficient $\frac{1}{2}(k_1 + k_2)$ as may be seen by setting $\phi = \psi = \pi/4$ in relation (18) and (19).

Before proceeding to discuss in more detail the propagation of light along the singular axes, we consider it relevant to point out that the functions with which we are concerned show no discontinuity at the singular axes. Thus both the elliptic vibrations propagated without change of form along any general direction, gradually degenerate into two (identical) circular vibrations as we approach a singular axis from any side whatsoever. [This can be seen by making $\chi \to 45$ and $\delta \to k$ in relations (4) and (5) of Section 4 *c*.] The refractive indices and absorption coefficients of these two waves, being determined by their states of polarisation (by relations 15 and

16 of Section 5) tend towards the common values $\frac{1}{2}(n_1 + n_2)$ and $\frac{1}{2}(k_1 + k_2)$ respectively, as we approach a singular axis. (See also reference 8.)

(*b*) *Effects with Incident Circularly Polarised Light*

In this section we shall inquire as to what will happen when, for example right circularly polarised light C_r is incident in the direction of a singular axis where only a left circular vibration C_l can be propagated unchanged (Fig. 3). Our results in this connection are at variance with those expected by Voigt. It was supposed by Voigt[3, 4] that if a plate cut normal to an optic axis is viewed in convergent circularly polarised light, then along the singular axis where the incident vibration can be propagated unchanged, more light would get through than in the neighbourhood of the other singular axis where only the oppositely directed circular vibration can be propagated unchanged; and that the latter direction should in consequence appear darker than the former.* On performing an actual experiment, he observed a dark and a bright spot in the field of view, one on either side of the optic axis and this was considered by him as confirming his view. According to our analysis, however, it is the singular axis where the incident vibration can be propagated unchanged that should appear *darker* than the other singular axis (where only the oppositely directed circular vibration can be propagated *unchanged*).

We shall apply *directly* the method of superposition, according to which, given the state of vibration at a particular plane in the medium, the state of the vibration at a further distance dz is obtained by superposing the effects of pure birefringence and pure dichroism corresponding to that passage. The state of vibration should then get progressively altered as we proceed into the medium.

Referring to Fig. 3, if the state of polarisation be initially coincident with the pole C_r, its movement ds_r (due to a clockwise rotation δdz about X_rY_r) is in the *same* direction as its movement ds_k towards the less absorbed component X_k; and the sum of these movements will give the alteration in the state of vibration corresponding to a passage dz. Continuing this procedure, it can be seen that as we proceed into the medium, the state of polarisation progressively moves along the arc $C_rX_kC_l$. At a particular depth the vibration would be linearly polarised along the principal plane

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* In a later paper [8], Voigt has suggested that if we could get a plate exactly normal to a singular axis, and have circularly polarised light of the proper sense incident precisely along this normal, the light would be *totally* reflected—the reflection being partial in practical cases. This idea receives no support from the results of the present investigation.

of absorption OX_k. After this stage the movement ds_k due to dichroism opposes the movement ds_r due to birefringence; but the latter being greater in magnitude, the state of polarisation continues to alter as we proceed further into the medium, tending towards the state C_l that can be propagated without change of form.

Thus if we consider the state of polarisation at successive depths within the medium, we see that the incident right-circular vibration will first get modified to an elliptic vibration (with major axis always at 45° to the principal planes of refraction), which in turn gets reduced to a linear vibration; as we proceed further the linear vibration opens out into a *left-handed* elliptic vibration, which gradually tends towards the state of a left-circular vibration that can be propagated unchanged. Nevertheless, as may be seen physically, this last state is never attained at any finite depth; for as the state of polarisation comes close to that of a left-circular vibration, the modification of the state corresponding to an additional passage dz becomes correspondingly reduced.

It is easy to deduce an explicit expression for the state of polarisation P that should be expected (according to the above line of argument) at any depth z inside the medium. The state P may be specified by giving the length s of the arc C_rP. Then the state $s + ds$ at the depth $z + dz$ will be given by

$$ds = (\delta + k \cos s)\, dz \tag{17}$$

according to relations (2) and (3) of Section 4 *b*. Since $\delta = k$, we have on integration,

$$\tan \tfrac{1}{2}s = kz \tag{18}$$

This relation shows that the transformation from a right-circular vibration ($s = 0$), to a linear vibration at 45° to the principal planes of refraction ($s = \pi/2$), occurs within a smaller depth than if the crystal had been transparent; whereas the corresponding alteration from the linear vibration to a left circular vibration (which for a transparent crystal would have occurred at a finite depth) requires here an infinite passage, due to the 'retarding' effect of the dichroism.

We shall next calculate the intensity I_z of the vibration P at a distance z inside the medium. The diminution of intensity $-dI_z$ corresponding to an additional passage dz is given by

$$-dI_z/I_z = 2k_z dz \tag{19}$$

where, it must be noted k_z is not a constant but a function of the state of polarisation and hence also of the depth z. We will have for k_z an expression analogous to (15), Section 5:

$$k_z = k_1 \cos^2 \psi + k_2 \sin^2 \psi$$
$$= \tfrac{1}{2}(k_1 + k_2) - \tfrac{1}{2}k \cos 2\psi \tag{20}$$

Thus k_z is always less than the coefficient of absorption $\frac{1}{2}(k_1 + k_2)$ of the left-circularly polarised wave that can be propagated without change of form along the same direction. (This is more directly seen by the fact that the state of polarisation is always nearer to the less absorbed component X_k than is a left-circular vibration.) Hence *when the sense of description of a circular vibration incident in the direction of a singular axis is* opposed *to that which can be propagated unchanged along that direction, the emergent intensity should in fact be* greater *than when the sense of description of the incident vibration is reversed.* An expression for the ratio of the emergent intensities in the two cases will now be deduced. Since, from our point of view, the incident disturbance can propagate into the medium in both cases (though in one case with a progressive change in the state of polarisation) we have no particular reason to assume that the reflection losses would be different in the two cases.

Substituting the value of k_z given by (20) in (19) we get

$$-dI_z/I_z = (k_1 + k_2)\,dz - k \sin s\,dz$$

Expressing $\sin s$ in terms of z by using relation (18), and integrating, we have —if I_1 be the emergent intensity and I_0 the intensity entering the medium,

$$\log (I_0/I_1) = (k_1 + k_2)\,z - \log (1 + k^2z^2) \tag{21}$$

On the other hand if I_2 be the emergent intensity when the incident circular vibration is of the sense which can be propagated unchanged,

$$\log (I_0/I_2) = (k_1 + k_2)\,z \tag{22}$$

From (21) and (22) we have the following simple relation for the ratio of the intensities emerging in the two cases:

$$I_1/I_2 = 1 + k^2z^2$$

a ratio which is always greater than unity.

Though our results regarding the properties of the singular axes are at variance with those expected by Voigt, it must not therefore be concluded that the method of superposition leads to results differing from the electromagnetic theory—since it is possible to regard the former merely as a mathematical artifice: for (as has been pointed out in Section 8), by reversing the entire scheme of arguments, the propagation of two elliptically polarised waves with different absorption coefficients and velocities (*as given by the electromagnetic theory*) may for mathematical convenience be rigorously treated by the method of superposition; the convenience of this artifice being particularly manifest when we wish to find the limiting effects as the two elliptically polarised waves gradually tend towards the state of two (identical) circularly polarised waves with equal velocities and absorption coefficients.

7. Analytical Discussion of Superposition

Referring to Fig. 1, let us suppose that we are given the equation of the elliptic vibration described at any particular plane z in the medium. If the initial state of polarisation is to be propagated without change of form, then the equation of the vibration at the plane $z + dz$ can be obtained not only by the method of superposition but also from the usual equation for the propagation of a damped wave. By equating these two expressions we can determine not only the states of polarisation that can be propagated without change of form but also their velocities and extinction coefficients.

Let OX and OY be two arbitrary rectangular axes taken in the plane of the figure, the inclinations of OX_k and OX_r with respect to the positive x-axis being α_1 and α_2 respectively. Let the components of the arbitrary elliptic vibration P along the axes OX, OY, have the following equations (using complex notation and indicating the complex quantities by bars):

$$\left.\begin{aligned} x &= \mathrm{F}e^{i(\omega t-\theta_1)} = \bar{f}e^{i\omega t} \\ y &= \mathrm{G}e^{i(\omega t-\theta_2)} = \bar{g}e^{i\omega t} \end{aligned}\right\} \tag{23}$$

so that

$$\frac{\bar{g}}{\bar{f}} = \frac{\mathrm{G}}{\mathrm{F}}e^{i\theta}$$

where $\bar{g}/\bar{f}$ is the ratio of the complex amplitudes, G/F the ratio of the real amplitudes and θ the difference of phase $(\theta_1 - \theta_2)$ between the x and y components. (It may be noted that G/F and θ will have—on the Poincaré sphere—geometrical interpretations essentially similar to those that have been described in Section 4 c for $\mathrm{G}_k/\mathrm{F}_k$ and θ_k.)

Let the complex amplitudes become $\bar{f}'$, $\bar{g}'$, after the infinitesimal operation of dichroism alone, and $\bar{f}''$, $\bar{g}''$, after both the infinitesimal operations of dichroism and birefringence corresponding to a passage dz. But if the initial state of polarisation $(\bar{f}, \bar{g})$ is to be propagated without change of form, and with a specific extinction coefficient κ and refractive index n, then its state $(\bar{f}'', \bar{g}'')$ after propagating a distance dz should also be given by:

$$\bar{f}'' e^{i\omega t} = \bar{f} e^{i\left(\omega t - \frac{2\pi}{\lambda_0} \bar{n}\, dz\right)}$$

or

$$\left.\begin{aligned} \bar{f}'' &= \bar{f}\left(1 - i\frac{2\pi}{\lambda_0}\bar{n}\, dz\right) \\ \text{and} \quad \bar{g}'' &= \bar{g}\left(1 - i\frac{2\pi}{\lambda_0}\bar{n}\, dz\right) \end{aligned}\right\} \tag{24}$$

where $\bar{n}$ is the complex refractive index $(n - i\kappa)$ of the elliptically polarised wave. We shall now, by the method of superposition, proceed to determine expressions for the final state $(\bar{f}'', \bar{g}'')$ in terms of the initial state $(\bar{f}, \bar{g})$, and then substitute these expressions in (24).

The Operation of Dichroism.—The elliptic vibration P, given by (23), is first referred to the axes OX_k, OY_k along the principal planes of absorption; the complex amplitudes $(\bar{f}_k, \bar{g}_k)$ of the components along these directions can be obtained from the amplitudes $(\bar{f}, \bar{g})$ by the usual transformation scheme for the rotation of co-ordinate axes through an angle α_1. These amplitudes $\bar{f}_k$, $\bar{g}_k$ are then multiplied by $(1 - 2\pi/\lambda_0 \cdot \kappa_1 dz)$ and $(1 - 2\pi/\lambda_0 \cdot \kappa_2 dz)$ respectively, to give the amplitudes $\bar{f}_k'$, $\bar{g}_k'$ of the x_k, y_k components after the operation of dichroism. Finally the elliptic vibration $\bar{f}_k'$, $\bar{g}_k'$ thus obtained is referred back to the axes OX, OY; the complex amplitudes $\bar{f}'$, $\bar{g}'$ of the components along these directions are obtained from the amplitudes $\bar{f}_k'$, $\bar{g}_k'$ by the usual transformation scheme for the rotation of the co-ordinate axes through an angle $-\alpha_1$.

If we go through the straightforward but lengthy algebraic substitutions involved in the procedure described in the last paragraph, it can be shown that the amplitudes $\bar{f}'$, $\bar{g}'$ of the x, y components after the operation of dichroism are related to the corresponding initial amplitudes $\bar{f}$, $\bar{g}$ by relations which may be put in a form analogous to (24):

$$\left.\begin{aligned} \bar{f}' &= \bar{f}\left[1 - \frac{2\pi}{\lambda_0}\left(\kappa_{11} + \frac{\bar{g}}{\bar{f}} \cdot \kappa_{12}\right) dz\right] \\ \bar{g}' &= \bar{g}\left[1 - \frac{2\pi}{\lambda_0}\left(\kappa_{22} + \frac{\bar{f}}{\bar{g}} \cdot \kappa_{12}\right) dz\right] \end{aligned}\right\} \tag{25}$$

where, if α_1 be the orientation of the positive OX_k axis,

$$\left.\begin{aligned} \kappa_{11} &= \kappa_1 \cos^2\alpha_1 + \kappa_2 \sin^2\alpha_1 \\ \kappa_{22} &= \kappa_1 \sin^2\alpha_1 + \kappa_2 \cos^2\alpha_1 \\ \kappa_{12} &= \tfrac{1}{2}(\kappa_1 - \kappa_2)\sin 2\alpha_1 \end{aligned}\right\} \tag{26}$$

If an ellipse be drawn with its principal semi-axes of lengths $1/\sqrt{\kappa_1}$ and $1/\sqrt{\kappa_2}$ lying along OX_k and OY_k, then $1/\sqrt{\kappa_{11}}$ and $1/\sqrt{\kappa_{22}}$ are the lengths of the radii vectors intercepted by the OX and OY directions, the equation of the ellipse being

$$\kappa_{11}\, x^2 + \kappa_{22}\, y^2 + 2\kappa_{12}\, xy = 1$$

The Operation of Birefringence.—The mathematical procedure involved in the infinitesimal operation of birefringence is essentially the same as in the operation of dichroism. The elliptic vibration $\bar{f}', \bar{g}'$ obtained after the operation of dichroism is first referred to the axes OX_r, OY_r along the principal planes of refraction; the complex amplitudes $\bar{f}_r'\ \bar{g}_r'$ of the components along these directions are then multiplied by $\exp(-i\, 2\pi/\lambda_0 . n_1 dz)$ and $\exp(-i\, 2\pi/\lambda_0 . n_2 dz)$ respectively—where n_1 and n_2 are the refractive indices in the absence of absorption. On referring the final vibration $(\bar{f}_r'', \bar{g}_r'')$ back to the axes OX, OY we will have the complex amplitudes $(\bar{f}'', \bar{g}'')$ of the x, y components, related to the corresponding amplitudes $\bar{f}', \bar{g}'$ (before the birefringence operation) by equations essentially similar to (25), though put in the form of (24):

$$\left.\begin{aligned} \bar{f}'' &= \bar{f}'\left[1 - i\frac{2\pi}{\lambda_0}\left(n_{11} + \frac{\bar{g}'}{\bar{f}'}\cdot n_{12}\right) dz\ \right] \\ \bar{g}'' &= \bar{g}'\left[1 - i\frac{2\pi}{\lambda_0}\left(n_{22} + \frac{\bar{f}'}{\bar{g}'}\cdot n_{12}\right) dz\ \right] \end{aligned}\right\} \tag{27}$$

Here n_{11}, n_{22} and n_{12} are to be regarded as *defined* by the relations (analogous to (26):

$$\left.\begin{aligned} n_{11} &= n_1 \cos^2 \alpha_2 + n_2 \sin^2 \alpha_2 \\ n_{22} &= n_1 \sin^2 \alpha_2 + n_2 \cos^2 \alpha_2 \\ n_{12} &= \tfrac{1}{2}(n_1 - n_2) \sin 2\alpha_2 \end{aligned}\right\} \tag{28}$$

where α_2 gives the angle made by the positive OX_r axis with the x-axis.

Since we shall omit terms involving dz^2, the value of $(\bar{g}'/\bar{f}')$ to be substituted in (27) need not include even the terms of the first order in dz, *i.e.*, we may write $(\bar{g}/\bar{f})$ for $(\bar{g}'/\bar{f}')$ in (27). We then obtain as the equation connecting the final state of polarisation $\bar{f}'', \bar{g}''$ (after both the infinitesimal operation of birefringence and dichroism corresponding to a passage dz), with the initial state $\bar{f}, \bar{g}$:

$$\left.\begin{aligned} \bar{f}'' &= \bar{f}\left\{1 - i\frac{2\pi}{\lambda_0}\left[\overline{n_{11} - i\kappa_{11}} + \frac{\bar{g}}{\bar{f}}\cdot \overline{n_{12} - i\kappa_{12}}\,\right] dz\ \right\} \\ \bar{g}'' &= \bar{g}\left\{1 - i\frac{2\pi}{\lambda_0}\left[\overline{n_{22} - i\kappa_{22}} + \frac{\bar{f}}{\bar{g}}\cdot \overline{n_{12} - i\kappa_{12}}\right] dz\right\} \end{aligned}\right\} \tag{29}$$

The Final Equations for Wave Propagation.—We now introduce the values of $\bar{f}''$, $\bar{g}''$ given by (29), into equation (7). Conciseness will obviously be attained if we first introduce the complex quantities:

$$\bar{n}_{11} = n_{11} - i\kappa_{11}$$
$$\bar{n}_{22} = n_{22} - i\kappa_{22}$$
$$\bar{n}_{12} = n_{12} - i\kappa_{12}$$

where the n_{hk} and κ_{hk} have already been defined in relations (26) and (28). We then obtain as our final equations:

$$\left.\begin{aligned} \bar{n} - \bar{n}_{11} &= \frac{\bar{g}}{\bar{f}} \cdot \bar{n}_{12} \\ \bar{n} - \bar{n}_{22} &= \frac{\bar{f}}{\bar{g}} \cdot \bar{n}_{12} \end{aligned}\right\} \tag{30}$$

There will be two pairs of values $(n_a, \bar{g}_a/\bar{f}_a)$ and $(\bar{n}_b, \bar{g}_b/\bar{f}_b)$ which simultaneously satisfy (30); and—since $\bar{g}/\bar{f}$ and $\bar{n}$ are both in general complex—this means that there should be two elliptically polarized waves that can be propagated, each with a specific velocity and coefficient of extinction. Eliminating $\bar{n}$ between the two equations of (30) by subtracting, we get the following quadratic in $\bar{g}/\bar{f}$ determining the states of polarisation propagated without change of form:

$$\frac{\bar{f}}{\bar{g}} - \frac{\bar{g}}{\bar{f}} = \frac{\bar{n}_{11} - \bar{n}_{22}}{\bar{n}_{12}} \tag{31}$$

The two roots of this equation are obviously connected by the relation $(\bar{g}_a/\bar{f}_a) = -(\bar{f}_b/\bar{g}_b)$, from which it follows that the two elliptically polarised vibrations have their major axes at right angles and their ellipticities equal, but are described in the same sense (see, *e.g.*, McLaurin[9]).

Eliminating $\bar{g}/\bar{f}$ between the two equations (30) by multiplying the two, we get the following quadratic in $\bar{n}$ determining the complex refractive indices of the waves:

$$(\bar{n} - \bar{n}_{11})(\bar{n} - \bar{n}_{22}) = \bar{n}_{12}{}^2 \tag{32}$$

From this we get the expressions for the sum and difference of the complex refractive indices:

$$(\bar{n}_a - \bar{n}_b)^2 = (\bar{n}_{11} - \bar{n}_{22})^2 + 4\bar{n}_{12}{}^2 \tag{33}$$

and

$$(\bar{n}_a + \bar{n}_b) = \bar{n}_{11} + \bar{n}_{22} \tag{34}$$

Velocities and Absorption Coefficients of the Waves.—The expressions for the velocities and absorption coefficients given in Section 4 may be derived from the equations (30) which give the complex refractive $\bar{n}$ in terms of the corresponding state of polarisation $\bar{g}/\bar{f}$. If we choose axes of co-ordinates along OX_r, OY_r then $n_{12} = 0$, $\alpha_2 = 0$ and $\alpha_1 = 2X$. The first of the two relations in (30) gives:

$$(n - i\kappa) = (n_1 - i\kappa_{11}) + \frac{G_r}{F_r} e^{i\theta_r} \cdot (-i\kappa_{12})$$

Equating real parts

$$n = n_1 + (G_r/F_r)\,\kappa_{12} \sin\theta_r$$
$$= n_1 + (G_r/F_r) \cdot \tfrac{1}{2}(\kappa_1 - \kappa_2) \sin 2X \sin\theta_r$$

Referring to Fig. 2, since $\theta_r = PX_r X_k$ we have

$$\sin 2X \sin\theta_r = \sin 2\psi$$

Hence on using eq. (5), Section 4 *c*, we have

$$n = n_1 - (G_r/F_r) \cdot \tfrac{1}{2}(n_1 - n_2) \sin 2\phi$$

Since $(G_r/F_r) = \tan\phi$, we get as our final expression for the refractive index,

$$n = n_1 \cos^2\phi + n_2 \sin^2\phi$$

Similarly we will have for the extinction coefficient,

$$\kappa = \kappa_1 \cos^2\psi + \kappa_2 \sin^2\psi$$

8. Comparison with the Electromagnetic Theory

Let the sections of the index and absorption ellipsoids made by the *xy* plane be given by the respective equations:

$$\left.\begin{array}{l} a_{11}x^2 + a_{22}y^2 + 2a_{12}xy = 1 \\ b_{11}x^2 + b_{22}y^2 + 2b_{12}xy = 1 \end{array}\right\} \quad (35)$$

Let us introduce the quantities

$$\bar{c}_{hk} = a_{hk} + ib_{hk} \quad (36)$$

Then the equations (30) giving the states of polarisation and the complex refractive indices of the two waves propagated along any direction have to be compared with the following similar relations obtained from the electromagnetic theory:

$$\left.\begin{aligned}(\bar{v}^2 - \bar{c}_{11}) &= \frac{\bar{g}}{\bar{f}} \cdot \bar{c}_{12}\\(\bar{v}^2 - \bar{c}_{22}) &= \frac{\bar{f}}{\bar{g}} \cdot \bar{c}_{12}\end{aligned}\right\} \tag{37}$$

where the complex velocity $\bar{v} = v\,(1 + ix')$

We may first remark that it would indeed be possible to *define* the quantities $\bar{n}_{11}$, $\bar{n}_{22}$ and $\bar{n}_{12}$ (in terms of $\bar{c}_{11}$, $\bar{c}_{22}$ and $\bar{c}_{12}$) in such a manner that the results obtained by the method of superposition would be identical with the results of the electromagnetic theory. But in order to retain the physical content of the method of superposition it is necessary to regard the velocities v_1 and v_2 in the absence of absorption as being equal to $\sqrt{a_1}$ and $\sqrt{a_2}$ respectively (where $1/\sqrt{a_1}$ and $1/\sqrt{a_2}$ are the lengths of the principal radii of the elliptical section of the index ellipsoid). And once this is done, at least some of the results obtained by the method of superposition have necessarily to be regarded as approximations. We shall however show that for directions near an optic axis where the birefringence is necessarily very small and where alone the ellipticity of the waves play an important role, the error involved is negligible. To this end we shall start by assuming the relations (12) and (14) which give a connection between the extinction coefficients κ_1 and κ_2 on the one hand, and the lengths $1/\sqrt{b_1}$ and $1/\sqrt{b_2}$ of the principal radii of the elliptical section of the absorption ellipsoid on the other. The quantities a_{hk} occurring in equation (35) can obviously be expressed in terms of a_1 and a_2 thus:

$$\left.\begin{aligned}a_{11} &= a_1 \cos^2 \alpha_2 + a_2 \sin^2 \alpha_2\\a_{22} &= a_1 \sin^2 \alpha_2 + a_2 \cos^2 \alpha_2\\a_{12} &= \tfrac{1}{2}\,(a_1 - a_2) \sin 2\alpha_2\end{aligned}\right\} \tag{38}$$

Similar relations analogous to (26) hold for the b_{hk}.

On examining the equations [(30) and 37)] obtained by the method of superposition and by the electromagnetic theory, we notice that the two are entirely similar in form, the only difference being the occurrence of the quantities $\bar{c}_{hk}$ instead of $\bar{n}_{hk}$ and $\bar{v}^2$ instead of $\bar{n}$. Hence it follows that given any equation obtained by the method of superposition, a corresponding exact equation obtainable from the electromagnetic theory can be written down, merely by changing the symbols occurring in the equation according to the following scheme:

n	n_1	n_2	κ	κ_1	κ_2
v^2	a_1	a_2	$-\dfrac{2\kappa v^3}{c}$	$-b_1$	$-b_2$

Thus to obtain the states of polarisation (ϕ, ψ) that can be propagated without change of form, we have to replace the ratio δ/k in eqn. (5) by the ratio ρ/σ where $\rho = \frac{1}{2}(a_2 - a_1)$ and $\sigma = \frac{1}{2}(b_2 - b_1)$. This replacement will however leave the equation unaltered, as we have already shown in Section 4 (*d*) that $\delta/k = \rho/\sigma$. It is also possible to show that the *sense* of description of the two vibrations as obtained by the method of superposition is the same as that obtained by the electromagnetic theory.

Expressions for the velocities and absorption coefficients according to the electromagnetic theory may be similarly written down from the relations (15) and (16):

$$v^2 = a_1 \cos^2 \phi + a_2 \sin^2 \phi \tag{39}$$

$$\frac{2\kappa v^3}{c} = b_1 \cos^2 \psi + b_2 \sin^2 \psi \tag{40}$$

We hence obtain

$$(v_a{}^2 - v_b{}^2) = (a_1 - a_2) \cos 2\phi_a$$

Or, if v_m be the mean velocity introduced in Section 4 (*d*)

$$\frac{v_b{}^2 - v_a{}^2}{2v_m{}^3} \cdot c = (n_1 - n_2) \cos 2\phi_a$$

On comparing this with the last equation in (16) we see that the approximation involved in using the method of superposition is to regard the expression on the left-hand side of the above equation as being practically equal to the difference in refractive indices $(n_a - n_b)$. This is justifiable along directions where the birefringence is low, and in fact this same approximation is also made when the propagation in transparent optically active crystals is regarded from the standpoint of superposition (Pockels, *loc. cit.*, p. 312).

From Eqn. (40) we get

$$2\kappa_a v_a{}^3 - 2\kappa_b v_b{}^3 = (b_1 - b_2) \cos 2\psi_a$$

Or, using (12)

$$\frac{\kappa_a v_a{}^3 - \kappa_b v_b{}^3}{v_m{}^3} = (\kappa_1 - \kappa_2) \cos 2\psi_a$$

On comparing this with the last equation in (15), we see that the approximation we have to make is to regard the expression on the left as being practically equal to the difference in extinction coefficients ($\kappa_a - \kappa_b$)—an approximation that is again justifiable where the birefringence is low.

It may be similarly shown that the mean refractive index and the mean extinction coefficient are also only negligibly in error. We shall not however give the proof here since the absolute values of the velocities and extinction coefficients are not as important as their *differences*.

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9. Summary

The optical behaviour of pleochroic biaxial crystals in the vicinity of an optic axis can be elegantly interpreted as due to the effects of linear birefringence and linear dichroism superposed continuously along the depth of the material. This idea—followed up geometrically with the Poincaré sphere, and analytically by direct algebraic methods—explains the *elliptical* polarisation of the two waves propagated in any general direction, as due to the non-coincidence of the principal planes for the usual operation of birefringence with those for the operation of dichroism. In particular, it also explains the existence, on either side of an optic axis, of two *singular axes*—along any one of which *only one* circular vibration with a definite sense of description can be propagated unchanged: for these directions the principal planes of absorption and refraction make angles of 45° with each other, the linear birefringence and dichroism being also equal in magnitude.

A discussion is also given of the effects to be expected when a circular vibration incident in the direction of a singular axis has a sense of description *opposite* to that which can be propagated unchanged; and it is shown that—contrary to what Voigt expected—the emergent intensity will not be either zero or negligible, but will in fact be *greater* than when the sense of description of the incident vibration is reversed.

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