*AUTOMORPHIC FORMS ON A SEMISIMPLE LIE GROUP**

By Harish-Chandra

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

Communicated by Paul A. Smith, February 19, 1959

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. We denote the corresponding Lie algebras by $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Put $B(X, Y) = tr(ad\,X\,ad\,Y)$ $(X, Y \in \mathfrak{g})$ and let $\mathfrak{p}$ denote the space of all $Y \in \mathfrak{g}$ which are orthogonal to $\mathfrak{k}$ under the bilinear form B. Then $\mathfrak{g}$ is the direct sum of $\mathfrak{k}$ and $\mathfrak{p}$. Put $\theta(X + Y) = X - Y$ $(X \in \mathfrak{k}, Y \in \mathfrak{p})$. Then θ is an automorphism of $\mathfrak{g}$ and the quadratic form $||Z||^2 = -B(Z, \theta(Z))$ $(Z \in \mathfrak{g})$ is positive definite. We regard $\mathfrak{g}$ as a (real) Hilbert space under this quadratic form and put $||x||^2 = tr(Ad\ (x)\ Ad(x)^*)$ $(x \in G)$ where $Ad(x)^*$ is the adjoint of the linear transformation $Ad(x)$.

Let $\mathfrak{Z}$ be the algebra of all differential operators on G which are invariant under both left- and right-translations of G. Fix a homomorphism χ of $\mathfrak{Z}$ into the field of complex numbers. Let Γ be a discrete subgroup of G and σ and μ unitary representations of K and Γ respectively on a finite-dimensional complex Hilbert space U. We assume that σ is continuous and U is a left K-module (under σ) and a right Γ-module (under μ). We shall denote the norm of u by $|u|$ $(u \in U)$. An automorphic form on G of type (σ, μ, χ) is a C^∞ function f on G with values in U such that (1) $f(kx\gamma) = \sigma(k)f(x)\mu(\gamma)$ $(k \in K, x \in G, \gamma \in \Gamma)$ and (2) $zf = \chi(z)f$ $(z \in \mathfrak{Z})$. It is not difficult to show that such a function is always analytic. Let $\mathfrak{F} = \mathfrak{F}(\sigma, \mu, \chi)$ denote the space of all such automorphic forms. If G/Γ is compact one proves easily that $\dim \mathfrak{F} < \infty$. The main problem here is to establish a similar result under some weaker hypothesis on G/Γ.

A form $f \in \mathfrak{F}$ is called normal at infinity if there exists an integer $m \geq 0$ and a real number $c > 0$ such that $|f(x)| \leq c||x||^m$ for all $x \in G$. Let $\mathfrak{F}_0 = \mathfrak{F}_0(\sigma, \mu, \chi)$ be the space of all $f \in \mathfrak{F}$ which are normal at infinity. Also let dx denote the Haar measure on G.

LEMMA 1. *Let f be a function in $\mathfrak{F}$ such that $\int_{G/\Gamma} |f(x)|dx < \infty$. Then $f \in \mathfrak{F}_0$.*

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. We introduce an arbitrary but fixed lexicographic order in the space $\mathfrak{a}'$ of (real) linear functions on $\mathfrak{a}$. For any $\alpha \in \mathfrak{a}'$, let $\mathfrak{n}_\alpha$ denote the space of all $X \in \mathfrak{g}$ such that $[H, X] = \alpha(H)X$ for every $H \in \mathfrak{a}$. Consider the set Σ of those $\alpha > 0$ for which $\mathfrak{n}_\alpha \neq \{0\}$. Then Σ is a finite set. A subset Σ' of Σ is called closed if $\alpha + \beta \in \Sigma'$ whenever $\alpha, \beta \in \Sigma'$ and $\alpha + \beta \in \Sigma$. Let $\Sigma' \supset \Sigma''$ be two subsets of Σ such that Σ' is closed. We say that Σ'' is an ideal in Σ' if $\alpha + \beta \in \Sigma''$ whenever $\alpha \in \Sigma''$ and $\beta, \alpha + \beta \in \Sigma'$. For any closed Σ' put $\mathfrak{n}(\Sigma') = \sum_{\alpha \in \Sigma'} \mathfrak{n}_\alpha$. Then $\mathfrak{n}(\Sigma')$ is a subalgebra of the nilpotent Lie algebra $\mathfrak{n} = \mathfrak{n}(\Sigma)$. Moreover if Σ' is an ideal in Σ, $\mathfrak{n}(\Sigma')$ is an ideal in $\mathfrak{n}$. Let $N(\Sigma')$, N and A denote the analytic subgroups of G corresponding to $\mathfrak{n}(\Sigma')$, $\mathfrak{n}$ and $\mathfrak{a}$ respectively.

Put $l = \dim \mathfrak{a}$. Then one can select l linearly independent elements $\alpha_1, \ldots, \alpha_l$ in Σ such that every $\alpha \in \Sigma$ is of the form $\alpha = m_1\alpha_1 + \ldots + m_l\alpha_l$ where $m_1, \ldots, m_l$ are nonnegative integers. Let $\mathfrak{a}^+$ denote the set of all points $H \in \mathfrak{a}$ where $\alpha(H) > 0$ for every $\alpha \in \Sigma$ and put $A^+ = \exp \mathfrak{a}^+$.

Γ being a discrete subgroup of G, we shall say that Γ is of the type I if $\int_{G/\Gamma} dx < \infty$. Moreover we say that Γ is of type II if the following conditions hold:

(1) For every ideal Σ' in Σ, $N(\Sigma')/N(\Sigma') \cap \Gamma$ is compact.

(2) There exists an element $a_0 \in A$ such that $G = Ka_0(A^+)^{-1}N\Gamma$.

It is easy to show that if Γ is of type II then it is also of type I.

Let S be the class of all subsets of the set $(\alpha_1, \ldots, \alpha_l)$. For every $Q \in S$ let Σ_Q denote the smallest closed set in Σ containing Q and let Σ_Q' be the complement of Σ_Q in Σ. Then Σ_Q' is an ideal in Σ. Put $N_Q = N(\Sigma_Q)$ and $V_Q = N(\Sigma_Q')$. Select an element $H_0 \in \mathfrak{a}$ such that $\alpha(H_0) = 0$ $(\alpha \in \Sigma)$ if and only if $\alpha \in \Sigma_Q$. This is always possible. Let $\mathfrak{c}_Q$ denote the centralizer of H_0 in $\mathfrak{g}$. Then $\mathfrak{c}_Q$ (which depends only on Q and not on the choice of H_0) is reductive in $\mathfrak{g}$. Put $\mathfrak{g}_Q = [\mathfrak{c}_Q, \mathfrak{c}_Q]$ and $\mathfrak{a}_Q = \mathfrak{a} \cap \mathfrak{g}_Q$. Then $\mathfrak{g}_Q$ is semisimple. Let G_Q and A_Q denote the analytic subgroups of G corresponding to $\mathfrak{g}_Q$ and $\mathfrak{a}_Q$ respectively. Put $A_Q^+ = \exp \mathfrak{a}_Q^+$ where $\mathfrak{a}_Q^+$ is the set of those $H \in \mathfrak{a}_Q$ where $\alpha(H) > 0$ for every $\alpha \in Q$. Also let $\Gamma_Q = G_Q \cap ((G_Q V_Q \cap \Gamma) V_Q)$. Then Γ_Q is a discrete subgroup of G_Q. We shall say that Γ is of type III, if Γ_Q is of type II in G_Q for every $Q \in S$.

THEOREM 1. *Let Γ_μ denote the kernel of μ in Γ. Suppose $N \cap \Gamma / N \cap \Gamma_\mu$ is finite and Γ is of type III. Then* $\dim \mathfrak{F}_0(\sigma, \mu, \chi) < \infty$.

The proof, which proceeds by induction on l, depends on a lemma of Godement.[1]

Put $<H, H'> = B(H, H')$ $(H, H' \in \mathfrak{a})$ and, for any $\lambda \in \mathfrak{a}'$, define the dual element $H_\lambda \in \mathfrak{a}$ by the condition that $<H, H_\lambda> = \lambda(H)$ for every $H \in \mathfrak{a}$. For any $\alpha \in \Sigma$ we define the Weyl reflexion s_α in $\mathfrak{a}$ by $s_\alpha H = H - 2\{\alpha(H)/\alpha(H_\alpha)\} H_\alpha (H \in \mathfrak{a})$. Then s_α can be "extended" to an automorphism $a \to a^{s_\alpha}$ of A. Also s_α operates on $\mathfrak{a}'$ by duality. Select a base $H_1, \ldots, H_l$ for $\mathfrak{a}$ such that $\alpha_i(H_j) = \delta_{ij}$ $1 \leq i, j \leq l$. By a theorem of Iwasawa,[2] corresponding to any $x \in G$, there exists a unique element $H(x) \in \mathfrak{a}$ such that $x \in K \exp H(x) N$.

Let π be a representation of G on a finite-dimensional complex vector space $U \neq \{0\}$. We denote the corresponding representation of $\mathfrak{g}$ also by π. One can always introduce a Hilbert-space structure in U in such a way that the adjoint of $\pi(X)$ is $-\pi(\theta(X))$ $(X \in \mathfrak{g})$. We shall always tacitly assume that such a structure on U is given. For any $\lambda \in \mathfrak{a}'$, let U_λ denote the set of all $u \in U$ such that $\pi(H)u = \lambda(H)u$ for every $H \in \mathfrak{a}$. λ is called a weight of π if $U_\lambda \neq \{0\}$. Λ being the highest weight of π, we denote the orthogonal projection of U on U_Λ by E_Λ. A vector $u \in U$ is said to belong to a weight λ if $u \in U_\lambda$.

Put $G_i = G_Q$ where $Q = \{\alpha_i\}$ and let $s_i = s_{\alpha_i}$ $(1 \leq i \leq l)$. We shall say that Γ is of type IV if the following conditions hold:

(1) $N(\Sigma')/N(\Sigma') \cap \Gamma$ is compact for every ideal Σ' in Σ.

(2) $\inf_{\gamma \in \Gamma} \langle H_i, H(\gamma) \rangle > -\infty \qquad (1 \leq i \leq l)$.[3]

(3) We can choose a compact set N_0 in N, elements $\gamma_i \in \Gamma \cap G_i$ $(1 \leq i \leq l)$, a representation π of G on a finite-dimensional complex vector space $U \neq \{0\}$ and a unit vector ψ belonging to the highest weight Λ of π such that:

(a) $N_0(N \cap \Gamma) = N$,

(b) $\gamma_i a \gamma_i^{-1} = a^{s_i}$ $(a \in A, 1 \leq i \leq l)$,

(c) $s_i \Lambda < \Lambda$ $(1 \leq i \leq l)$,

(d) $|E_\Lambda \pi(n\gamma_i)\psi| < 1 \qquad$ for $n \in N_0$ $(1 \leq i \leq l)$.

One can prove that if Γ is of type IV then it is also of type III.

Let R and C be the fields of real and complex numbers respectively, Z the ring of rational integers and $Z' = Z[(-1)^{1/2}]$ the ring of Gaussian integers. Then it is easy to check that Γ is of type IV in G in the following cases: (1) $G = SL(n, R)$, $\Gamma = SL(n, Z)$, (2) $G = Sp(n, R)$, $\Gamma = Sp(n, Z)$, (3) $G = SL(n, C)$, $\Gamma = SL(n, Z')$, (4) $G = Sp(n, C)$, $\Gamma = Sp(n, Z')$. (Here n is any positive integer.) Hence our results are applicable to these four cases.

Now Γ' being any subgroup of Γ, the Haar measure dx on G defines an invariant measure on G/Γ'. Consider the Hilbert space $\mathfrak{H} = L_2(G/\Gamma')$ corresponding to this measure. Since G operates on G/Γ' on the left in the obvious way, we get a unitary representation λ of G on $\mathfrak{H}$. Let π be an irreducible unitary representation of G on some Hilbert space. Given an integer $m \geq 0$, we say that π occurs as a discrete component of λ at least m times, if we can find m mutually orthogonal closed invariant subspaces $\mathfrak{H}_i$ $(1 \leq i \leq m)$ of $\mathfrak{H}$ such that the representation of G defined on $\mathfrak{H}_i$

under λ is equivalent to π. If π occurs at least once we say π is a discrete component of λ. Moreover we say that π occurs only a finite number of times if there exists an integer $m \geq 1$ such that it is impossible to choose $\mathfrak{H}_i$ $(1 \leq i \leq m)$ with the above properties.

The following result is an immediate consequence of Lemma 1 and Theorem 1.

THEOREM 2. *Let* Γ *be a discrete subgroup of* G *of type* III *and* Γ' *a subgroup of finite index in* Γ. *Put* $\Gamma_0 = \bigcap_{\gamma \in \Gamma} \gamma\Gamma'\gamma^{-1}$ *and suppose that* $N \cap \Gamma/N \cap \Gamma_0$ *is finite. Then every discrete irreducible component of the representation* λ *of* G *on* $L_2(G/\Gamma')$ *occurs only a finite number of times in* λ.

* This work was supported in part by a grant from the National Science Foundation.

[1] See Séminaire H. Cartan, 1957/58, *Exposé 8*, pp. 8–10.

[2] *Ann. of Math.*, **50**, 525 (1949).

[3] This condition was suggested by Godement.