SPHERICAL FUNCTIONS ON A SEMISIMPLE LIE GROUP

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Let R be the field of real numbers and G a connected semisimple Lie group with the Lie algebra $\mathfrak{g}_0$ over R. We assume that the center of G is finite. Let K be a maximal compact subgroup of G. By a spherical function f we mean a complex-valued function on G such that $f(k_1xk_2) = f(x)$ $(k_1, k_2 \in K;\ x \in G)$. Let $\mathfrak{k}_0$ be the Lie algebra of K. Define $\mathfrak{p}_0$, $\mathfrak{h}_{\mathfrak{p}_0}$, and $\mathfrak{n}_0$ as in an earlier paper,[1] and let A and N be the analytic subgroups of G corresponding to $\mathfrak{h}_{\mathfrak{p}_0}$ and $\mathfrak{n}_0$, respectively. Then, for any $x \in G$, we denote by $H(x)$ the unique element in $\mathfrak{h}_{\mathfrak{p}_0}$ such that $x = k(\exp H(x))n$ for some $k \in K$ and $n \in N$. Introduce a linear function ρ and a polynomial function p on $\mathfrak{h}_{\mathfrak{p}_0}$ by means of the equations $e^{2\rho(H)} = \det\,(\mathrm{Ad}\,(\exp H))_{\mathfrak{n}_0}$ and $p(H) = \det\,(\mathrm{ad}\, H)_{\mathfrak{n}_0}$ $(H \in \mathfrak{h}_{\mathfrak{p}_0})$, where the subscript indicates the restriction on $\mathfrak{n}_0$. Then p is a product of real linear factors. Put $\pi = \alpha_1\alpha_2\ldots\alpha_r$, where $\alpha_1, \ldots, \alpha_r$ are all the distinct prime factors of p. Let $\mathfrak{F}$ denote the space of linear functions on $\mathfrak{h}_{\mathfrak{p}_0}$. Sometimes it would be convenient to identify $\mathfrak{F}$ with $\mathfrak{h}_{\mathfrak{p}_0}$ by means of the fundamental bilinear form on $\mathfrak{g}_0$. Then π becomes a polynomial function also on $\mathfrak{F}$. Let M be the centralizer and M' the normalizer of $\mathfrak{h}_{\mathfrak{p}_0}$ in G. Then $W = M'/M$ is a finite group whose elements operate as linear transformations on $\mathfrak{h}_{\mathfrak{p}_0}$. Hence W operates also on the ring of polynomial functions on $\mathfrak{h}_{\mathfrak{p}_0}$. It can be shown that π^2 is invariant under W, and therefore $\pi^s = \varepsilon(s)\pi$ $(s \in W)$, where $\varepsilon(s) = \pm 1$. Let $\mathfrak{h}_{\mathfrak{p}_0}'$ denote the set of those points $H \in \mathfrak{h}_{\mathfrak{p}_0}$, where $\pi(H) \neq 0$. There exists a unique connected component $\mathfrak{h}_{\mathfrak{p}_0}^+$ of $\mathfrak{h}_{\mathfrak{p}_0}'$ such that $\rho(H) > \rho(sH)$ for $s \in W$ and $H \in \mathfrak{h}_{\mathfrak{p}_0}^+$. Let A_+ denote the closure of $\exp(\mathfrak{h}_{\mathfrak{p}_0}^+)$. Then $G = KA_+K$, and therefore a spherical function is completely determined by its restriction on A_+. We shall say that $H \to \infty$ $(H \in \mathfrak{h}_{\mathfrak{p}_0}^+)$ if $|\alpha_j(H)| \to \infty$ $(1 \leqq j \leqq r)$.

Put $\phi_\lambda(x) = \int_K \exp\{i\lambda(H(xk)) - \rho(H(xk))\}dk$ $(\lambda \in \mathfrak{F},\ x \in G)$, where dk is the normalized Haar measure on K. Then ϕ_λ is spherical, and it is also an elementary function of the positive-definite type.[2]

THEOREM 1. *There exists a unique analytic function β on the real Euclidean space $\mathfrak{F}$ such that*

$$\lim_{H\to\infty} \left|\pi(\lambda)e^{\rho(H)}\phi_\lambda(\exp H) - \sum_{s \in W} \epsilon(s)\beta(s\lambda)\exp\,(i\lambda(s^{-1}H))\right| = 0 \quad (H \in \mathfrak{h}_{\mathfrak{p}_0}^+)$$

for any $\lambda \in \mathfrak{F}$. Moreover, $|\beta(s\lambda)| = |\beta(\lambda)|$ for $s \in W$ and $\lambda \in \mathfrak{F}$.

Extend the automorphism[1] θ of $\mathfrak{g}_0$ to G, and put $n' = \theta(n^{-1})$ $(n \in N)$.

THEOREM 2. *It is possible to normalize the Haar measure dn on N in such a way that*[3]

$$\beta(\lambda) = \lim_{\epsilon\to 0} \pi(\lambda_\epsilon)\int_N \exp\{-i\lambda_\epsilon(H(n')) - \rho(H(n'))\}\,dn \qquad (\varepsilon > 0,\ \lambda \in \mathfrak{F}),$$

where $\lambda_\epsilon = \lambda - i\varepsilon\rho$. This normalization is characterized by the condition that $\int_N e^{-2\rho(H(n'))}dn = 1$.

Define the space $\mathfrak{C}(\mathfrak{F})$ as in a previous note,[4] and for any $a \in \mathfrak{C}(\mathfrak{F})$, put $\phi_a(x) = \int_{\mathfrak{F}} \pi(\lambda)a(\lambda)\phi_\lambda(x)d\lambda$, where $d\lambda$ stands for the (suitably normalized) Euclidean measure on $\mathfrak{F}$. Then it can be shown that, for any $\mu \in \mathfrak{F}$, $\int_G |\phi_\mu(x)\phi_a(x)|\,dx < \infty$, where dx denotes the Haar measure of G. Moreover, $\beta a \in \mathfrak{C}(\mathfrak{F})$ for a in $\mathfrak{C}(\mathfrak{F})$.

THEOREM 3. *$d\lambda$ can be so normalized that*

$$\pi(\mu) \int_G (\text{conj } \phi_\mu(x)) \phi_a(x)\, dx = |\beta(\mu)|^2 \sum_{s \in W} \epsilon(s)\, a(s\mu)$$

for $\mu \in \mathfrak{F}$ and $a \in \mathcal{C}(\mathfrak{F})$.

COROLLARY 1. $\int_G |\phi_a(x)|^2 dx = w^{-1} \int_{\mathfrak{F}} |\beta(\lambda)|^2 |\sum_{s \in W} \epsilon(s)a(s\lambda)|^2 d\lambda \quad (a \in \mathcal{C}(\mathfrak{F}))$, *where w is the order of W.*

It is possible to show that $\int_N |\phi_a(hn)|\, dn < \infty$ for $h \in A$ and $a \in \mathcal{C}(\mathfrak{F})$.

COROLLARY 2. $e^{\rho(H)} \int_N \phi_a((\exp H)n)\, dn =$ $\int_{\mathfrak{F}} |\beta(\lambda)|^2 \{\pi(\lambda)^{-1} \sum_{s \in W} \epsilon(s)\, a(s\lambda)\}\, e^{i\lambda(H)} d\lambda$ *for $\lambda \in \mathfrak{F}$, $a \in \mathcal{C}(\mathfrak{F})$, and $H \in \mathfrak{h}_{\mathfrak{p}_0}$.*

In certain special cases it is possible to compute β explicitly. For example, β is a constant if G is complex.[6]

We shall now indicate very briefly the central idea of our method of proof. Since ϕ_λ is spherical, it can be regarded as a function on the factor space G/K. Let $\mathfrak{D}$ be the algebra of all differential operators on G/K which are invariant under operations of G. It is known that ϕ_λ is an eigenfunction of every D in $\mathfrak{D}$. Let $\chi_\lambda(D)$ denote the corresponding eigenvalue. Then the above results can be obtained by a detailed study of the system of differential equations $D\phi_\lambda = \chi_\lambda(D)\phi_\lambda$ $(D \in \mathfrak{D})$.

Let $L_2(G)$ denote the Hilbert space of all square-integrable functions on G, and $I_2(G)$ the closure of the subspace consisting of those continuous spherical functions which vanish outside a compact set. Then, if it could be shown that the functions ϕ_a $(a \in \mathcal{C}(\mathfrak{F}))$ are dense in $I_2(G)$, the Plancherel formula[7] for functions on G/K would follow in an explicit form from Corollary 1 of Theorem 3.

[1] *Trans. Am. Math. Soc.*, **75**, 187–188, 1953.

[2] See *Trans. Am. Math. Soc.*, **76**, 64, 1954, Theorem 5.

[3] Here we extend π to a polynomial function on the complexification of $\mathfrak{F}$ in the obvious way.

[4] These PROCEEDINGS, **42**, 252–253, 1956.

[5] "conj c" denotes the conjugate of a complex number c.

[6] See *Trans. Am. Math. Soc.*, **76**, 253, 1954, Theorem 7.

[7] See these PROCEEDINGS, **40**, 203–204, 1954.