PLANCHEREL FORMULA FOR COMPLEX SEMISIMPLE LIE GROUPS

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Communicated by P. A. Smith, October 5, 1951

Let R and C be the fields of real and complex numbers respectively and let G be a connected (but not necessarily simply connected) complex semisimple Lie group and $\mathfrak{g}_0$ its Lie algebra over R. Let K be a maximal compact subgroup of G and let $\mathfrak{K}_0$ be the corresponding subalgebra of $\mathfrak{g}_0$. Define $\mathfrak{P}_0$, $\mathfrak{h}_{\mathfrak{K}_0}$, $\mathfrak{h}_{\mathfrak{P}_0}$ and $\mathfrak{h}_0$ as in a previous note.[1] Since G is a complex group there exists a 1-1 linear mapping Γ of $\mathfrak{K}_0$ on $\mathfrak{P}_0$ such that $[X, \Gamma(Y)] = \Gamma([X, Y])$ and $[\Gamma(X), \Gamma(Y)] = -[X, Y]$ $(X, Y \epsilon \mathfrak{K})$. We extend Γ to a linear mapping of $\mathfrak{g}_0$ on itself by defining $\Gamma(\Gamma(X)) = -X(X \epsilon \mathfrak{K}_0)$. Let $\sqrt{-1}$ be a fixed square root of -1 in C. For any $c \epsilon C$ and $X \epsilon \mathfrak{g}_0$ put $c * X = aX + b\Gamma(X)$ where $c = a + \sqrt{-1}\, b$ $(a, b \epsilon R)$. Under this multiplication $\mathfrak{g}_0$ becomes a Lie algebra over C. We shall denote this complex algebra by $\mathfrak{g}^*$. Similarly the algebra $\mathfrak{h}_0$ regarded as a (complex) subalgebra of $\mathfrak{g}^*$ will be denoted by $\mathfrak{h}^*$. Then $\mathfrak{h}^*$ is a Cartan subalgebra of $\mathfrak{g}^*$. Let $X \rightarrow ad\, X(X \epsilon \mathfrak{g}^*)$ be the adjoint representation of $\mathfrak{g}^*$ and let

$B(X, Y) = sp(ad\ X\ ad\ Y)\ (X, Y \in \mathfrak{g}^*)$. Given any linear function λ on $\mathfrak{h}^*$ we denote by H_λ the unique element in $\mathfrak{h}^*$ such that $\lambda(H) = B(H, H_\lambda)$ for all $H \in \mathfrak{h}^*$. Let $H_1, \ldots, H_l$ be a base for $\mathfrak{h}_{\mathfrak{P}_0}$ over R. Then it is also a base for $\mathfrak{h}^*$ over C. We shall say that λ is real if $H_\lambda = \sum_{1 \leq i \leq l} c_i H_i (c_i \in R)$ and furthermore that $\lambda > 0$ if $\lambda \neq 0$ and $c_j > 0$ where j is the least index $(1 \leq j \leq l)$ such that $c_j \neq 0$. For every root α of $\mathfrak{g}^*$ (with respect to $\mathfrak{h}^*$) we choose an element $X_\alpha \neq 0$ in $\mathfrak{g}^*$ such that $[H, X_\alpha] = \alpha(H) * X_\alpha\ (H \in \mathfrak{h}^*)$. We can do this in such a way that $B(X_\alpha, X_{-\alpha}) = 1$ and $X_\alpha + X_{-\alpha}$, $\sqrt{-1} * (X_\alpha - X_{-\alpha})$ are both in $\mathfrak{K}_0$. Put $H_\alpha = \sum_{1 \leq i \leq l} \alpha^i H_i (\alpha^i \in R)$ and let $\mathfrak{N}^* = \sum_{\alpha \in P} C * X_\alpha$ where P is the set of all positive roots. Then $\mathfrak{N}^*$ is a nilpotent subalgebra of $\mathfrak{g}^*$ to which there corresponds an analytic subgroup N of G.

Let $C_c^\infty(G)$ be the class of complex-valued functions on G which are everywhere defined and indefinitely differentiable and which vanish outside a compact set. For any complex number c we denote by $\bar{c}$ its complex conjugate. Moreover if $z = x + \sqrt{-1}\, y\ (x, y \in R)$ is a complex variable and f a complex-valued differentiable function of x and y, we write

$$\frac{\partial}{\partial z} f = \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right) f,\quad \frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right) f.$$

We shall now first prove a formula which has been obtained by Gelfand and Naimark[2] in the case when G is the $n \times n$ complex unimodular group. Let $X \to \exp X (X \in \mathfrak{g}_0)$ denote the exponential mapping of $\mathfrak{g}_0$ into G. Put $\rho = \frac{1}{2} \sum_{\alpha \in P} \alpha$ and let du and dn denote the elements of the invariant Haar measures on K and N respectively. We assume that $\int_K du = 1$.

THEOREM 1. *Put* $H_a = \sum_{1 \leq i \leq l} a_i * H_i (a_i \in C)$ *and*

$$D_\alpha = \sum_{1 \leq i \leq l} \alpha^i \frac{\partial}{\partial a_i},\quad \bar{D}_\alpha = \sum_{1 \leq i \leq} \alpha \frac{\partial}{\partial \bar{a}_i} \qquad (\alpha \in P).$$

Then with a suitable normalization of dn we have

$$f(1) = \lim_{H_a \to 0} \prod_{\alpha \in P} D_\alpha \bar{D}_\alpha \left\{ e^{\rho(H_a) + \overline{\rho(H_a)}} \int_{K \times N} f[u(\exp H_a) n u^{-1}]\, du\, dn \right\}$$

for any[3] $f \in C_c^\infty(G)$.

We give below a rapid sketch of the various steps leading to the proof of this theorem. Let θ denote the automorphism of $\mathfrak{g}_0$ over R given by $\theta(X + Y) = X - Y\ (X \in \mathfrak{K}_0, Y \in \mathfrak{P}_0)$. Let $\mathfrak{N}_0$ denote the set $\mathfrak{N}^*$ regarded as a vector space over R and let $dX (X \in \mathfrak{N}_0)$ be the element of the usual Euclidean measure on $\mathfrak{N}_0$.

LEMMA 1. *There exists a real constant $c > 0$ such that*

$$\lim_{\to 0} \frac{d}{dt}\left\{e^{t\rho(H - \theta H)} \int_N f[(\exp tH)n]\, dn\right\} = c \int_{\mathfrak{N}_0} \left\{\frac{d}{dt} f[\exp(X + tH)]\right\}_{t=0} dX \qquad (t \in R)$$

for any $f \in C_c^\infty(G)$ and $H \in \mathfrak{h}_0$.

Since $\mathfrak{g}_0$ is a real Euclidean space it may be regarded as an analytic manifold. Let $C_c^\infty(\mathfrak{g}_0)$ be the class of all complex-valued functions on $\mathfrak{g}_0$ which are everywhere indefinitely differentiable and which vanish outside a compact set. Put

$$X = \sum_{1 \leq i \leq l} a_i{*}H_i + \sum_{\alpha \in P} z_\alpha{*}X_\alpha + \sum_{\alpha \in P} z_{-\alpha}{*}X_{-\alpha}$$

where a_i, z_α, $z_{-\alpha}(1 \leqq i \leqq l,\ \alpha \in P)$ are independent complex variables. For any complex variable $z = x + \sqrt{-1}\, y$ $(x, y \in R)$ let $d\mu(z)$ denote the element $dx\, dy$ of Euclidean measure on the corresponding complex plane. Let $x \to Ad(x)$ $(x \in G)$ be the adjoint representation of G. Consider a function $F \in C_c^\infty(\mathfrak{g}_0)$ such that $F(Ad(u)X) = F(X)$ $(u \in K)$. Put

$$g(Y) = \frac{1}{(2\pi)^n} \int_{\mathfrak{g}_0} \exp\left(\frac{1}{2}\sqrt{-1}\,[B(X, Y) + \overline{B(X, Y)}]\right) F(X)\, dX \qquad (Y \in \mathfrak{g}_0). \quad (1)$$

Here $n = \frac{1}{2}\dim_R \mathfrak{g}_0$ and $dX = \prod_{1 \leqq i \leqq l} d\mu(a_i) \prod_{\alpha \in P} d\mu(z_\alpha) d\mu(z_{-\alpha})$. Then if we assume, as we may, that $B(H_i, H_j) = \delta_{ij}$ $(1 \leqq i, j \leqq l)$ it follows that

$$F(0) = \frac{1}{(2\pi)^n} \int_{\mathfrak{g}_0} g(X)\, dX \qquad (2)$$

and $g(Ad(u)X) = g(X)$ $(u \in K, X \in \mathfrak{g}_0)$. Now it is known that $\bigcup_{u \in K} Ad(u)\,(\mathfrak{h}_0 + \mathfrak{N}_0) = \mathfrak{g}_0$ and from this we can deduce the following lemma.

LEMMA 2. *Let $g(X)$ be a measurable function on $\mathfrak{g}_0$ such that*

$$g[Ad(u)X] = g(X)\ (u \in K, X \in \mathfrak{g}_0) \text{ and } \int_{\mathfrak{g}_0} |g(X)|\, dX < \infty.$$

Then

$$\int_{\mathfrak{g}_0} g(X)\, dX = \int_{\substack{Z \in \mathfrak{N}_0 \\ H \in \mathfrak{h}_0}} \prod_{\alpha \in P} |\alpha(H)|^2 g(Z + H)\, dZ\, dH$$

where dZ and dH are the elements of the (suitably normalized) Euclidean measures on $\mathfrak{N}_0$ and $\mathfrak{h}_0$ respectively.

Applying this lemma to equation (2) we get

$$F(0) = \lim_{H_a \to 0} \left\{ c \int_{Z \in \mathfrak{N}_0} \prod_{\alpha \in P} D_\alpha \bar{D}_\alpha F(Z + H_a)\, dZ \right\}$$

where c is a positive real constant depending only on the normalization of dZ. The assertion of the theorem now follows without much difficulty if we take into account lemma 1.

Now we assume that the base $H_1, \ldots, H_l$ is so chosen that $\exp H_a = 1$ if and only if $\sqrt{-1}\,\frac{a_i}{2\pi}$ $1 \leqq i \leqq l$ are all rational integers. Let A, $A+$ and $A-$ be the analytic subgroups of G corresponding to $\mathfrak{h}_0$, $\mathfrak{h}_{\mathfrak{P}_0}$ and $\mathfrak{h}_{\mathfrak{L}_0}$ respectively. Then $A-$ is compact while $A+$ is simply connected. For any $h \in A$ we denote by $h+$ and $h-$ the unique elements in $A+$ and $A-$ respectively such that $h = h+h-$. Also let $\log h+$ denote the unique element $H \in \mathfrak{h}_{\mathfrak{P}_0}$ such that $h+ = \exp H$. Let $\mathfrak{F}+$ be the set of all linear functions ν on $\mathfrak{h}^*$ such that $\nu(H)$ is real for all $H \in \mathfrak{h}_{\mathfrak{P}_0}$. Moreover let $\mathfrak{F}-$ denote the set of all linear functions Λ on $\mathfrak{h}^*$ such that $\Lambda(H_i) 1 \leqq i \leqq l$ are all integers. Given any $\nu \in \mathfrak{F}+$ and $\Lambda \in \mathfrak{F}-$ put

$$\xi_{\nu,\Lambda}(h) = e^{\sqrt{-1}\,\nu(\log h_+)} e^{\Lambda(\log h_-)} \qquad (h \in A)$$

$\log h-$ being any element in $\mathfrak{h}_{\mathfrak{L}_0}$ such that $\exp(\log h-) = h-$. It is known that the mapping $(u, h, n) \to uhn (u \in K, h \in A+, n \in N)$ is a topological mapping of $K \times A+ \times N$ on G. We may normalize the Haar measure on G in such a way that

$$dx = e^{4\rho(\log h)}\, du\, dh\, dn \qquad (x = uhn,\ u \in K,\ h \in A+,\ n \in N)$$

dh being the Haar measure on $A+$. Moreover we assume that the normalization of the various Haar measures is such that Theorem 1 holds and

$$\int_K du = 1,\ \int_{A_-} dh_- = 1,\ dh = dh_+ dh_- \qquad (h \in A).$$

For any $x \in G$ and $u \in K$ define $u_x \in K$ and $H(x, u) \in \mathfrak{h}_{\mathfrak{P}_0}$ by the relation

$$xu = u_x[\exp H(x, u)]n \qquad (n \in N).$$

Given any $\Lambda \in \mathfrak{F}-$ let $\mathfrak{H}_\Lambda'$ denote the set of all continuous functions ψ on K such that

$$\psi(u \exp H) = e^{-\Lambda(H)}\psi(u) \ (u \in K,\ H \in \mathfrak{h}_{\mathfrak{L}_0}).$$

Let $L_2(K)$ be the Hilbert space consisting of all measurable and square-integrable functions on K, taken with the usual norm. Then the closure $\mathfrak{H}_\Lambda$ of $\mathfrak{H}_\Lambda'$ in $L_2(K)$ is a Hilbert space. For any $\nu \in \mathfrak{F}+$ we define a unitary representation $\pi_{\nu, \Lambda}$ of G on $\mathfrak{H}_\Lambda$ as follows. If $\psi \in \mathfrak{H}_\Lambda$ its transform $\pi_{\nu, \Lambda}(x) \psi = \varphi$ is given by

$$\varphi(u) = e^{-\sqrt{-1}\nu(H(x^{-1}, u))} e^{-2\rho(H(x^{-1}, u))} \psi(u_{x^{-1}}) \qquad (u \in K).$$

It is easily proved that if $f \in C_c^\infty(G)$, the operator

$$\int_G f(x)\pi_{\nu, \Lambda}(x)\, dx$$

has a trace $T_{\nu, \Lambda}(f)$ which is given by

$$T_{\nu, \Lambda}(f) = \int f(uhnu^{-1})\xi_{\nu, \Lambda}(h)e^{2\rho(\log h_+)}\, du\, dh\, dn$$

where the integral extends over all $u \in K$, $h \in A$, $n \in N$. Now $\mathfrak{F}_+$ is clearly a vector space over R of finite dimension. Let $d\nu$ denote the element of Euclidean measure in $\mathfrak{F}_+$. Then the following result is easily obtained from Theorem 1.

THEOREM 2. *Put*

$$m(\nu, \Lambda) = \prod_{\alpha \in P} \left| \sqrt{-1}\, \nu(H_\alpha) + \Lambda(H_\alpha) \right|^2 \qquad (\nu \in \mathfrak{F}_+,\ \Lambda \in \mathfrak{F}_-).$$

Then if $d\nu$ is suitably normalized we have the formula

$$f(1) = \sum_{\Lambda \in \mathfrak{F}_-} \int_{\mathfrak{F}_+} m(\nu, \Lambda) T_{\nu, \Lambda}(f)\, d\nu \qquad [f \in C_c^\infty(G)]$$

the series being absolutely convergent.

Now suppose $f \in C_c^\infty(G)$ and

$$F(x) = \int_G \overline{f(y)} f(yx)\, dy.$$

Then $F \in C_c^\infty(G)$ and the operator $\int_G F(x)\pi_{\nu, \Lambda}(x)\, dx$ is self-adjoint and positive semidefinite. Hence $T_{\nu, \Lambda}(F)$ is real and non-negative. In fact

$$T_{\nu, \Lambda}(F) = \int_{v,\, u \in K} |f_{\nu, \Lambda}(v, u)|^2\, dv\, du$$

where

$$f_{\nu, \Lambda}(v, u) = \int_{AN} f(vhnu^{-1})\xi_{\nu, \Lambda}(h)e^{2\rho(\log h_+)}\, dh\, dn.$$

Therefore

$$\int_G |f(x)|^2\, dx = F(1) = \sum_{\Lambda \in \mathfrak{F}_-} \int_{\mathfrak{F}_+} m(\nu, \Lambda)\, d\nu \int_{K \times K} |f_{\nu, \Lambda}(v, u)|^2\, dv\, du$$

from Theorem 2. Since $m(\nu, \Lambda)$ is real and non-negative the following analogue of the Plancherel theorem is now easily obtained.

THEOREM 3. *Let f be a measurable function on G such that*

$$\int_G |f(x)|^2\, dx < \infty \text{ and } \int_G |f(x)|\, dx < \infty.$$

Then

$$\int_G |f(x)|^2\, dx =$$

$$\sum_{\Lambda \in \mathfrak{F}_-} \int_{\mathfrak{F}_+} m(\nu, \Lambda)\, d\nu \int_{K \times K} dv\, du \left| \int_{AN} f(vhnu^{-1})\, \xi_{\nu, \Lambda}(h) e^{2\rho(\log h_+)}\, dhdn \right|^2$$

[1] Harish-Chandra, PROC. NATL. ACAD. SCI., **37**, 362–365 (1951).

[2] Gelfand and Naimark, *Trudi Mat. Inst. Steklova*, **36**, 198 (1950).

[3] We denote the unit element of G by 1.