REPRESENTATIONS OF SEMISIMPLE LIE GROUPS. III. CHARACTERS

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We shall adhere strictly to the notation of the preceding note.[1] Making use of an unpublished result of Chevalley one can prove the following theorem.

THEOREM 1. *Let π be a quasisimple irreducible representation of G on a Hilbert space $\mathfrak{H}$. Then there exists an integer N such that*

$$dim\ \mathfrak{H}_{\mathfrak{D}} \leq N(d(\mathfrak{D}))^2$$

for any $\mathfrak{D} \epsilon \Omega$.

Moreover if $\mathfrak{H}_{\mathfrak{D}} \neq \{0\}$ for some $\mathfrak{D} \epsilon \Omega_F$ then it can be shown that we may take N equal to the order of the Weyl group W.

Let π be as above and let $C_c^{\infty}(G)$ denote the class of all complex-valued functions on G which are indefinitely differentiable everywhere and which vanish outside a compact set. Let A be a bounded operator on $\mathfrak{H}$. We say that A has a trace if for every complete orthonormal set[2] $(\psi_1, \psi_2, \ldots, \psi_n, \ldots)$ in $\mathfrak{H}$ the series $\sum_{i \geq 1} (\psi_i, A\psi_i)$ converges to a finite number independent of the choice of this orthonormal set. We denote this number by spA. Let A^* be the adjoint of A. We say that A is of the Hilbert-Schmidt class if AA^* has a trace.

THEOREM 2. *Let π be a quasisimple irreducible representation of G on a Hilbert space $\mathfrak{H}$. Then for any $f \epsilon C_c^{\infty}(G)$ the operator[3] $\int_G f(x)\pi(x)\,dx$ has a trace. Put*

$$T_{\pi}(f) = sp(\textstyle\int_G f(x)\pi(x)\,dx)$$

and for any $a \epsilon G$ let ${}_af_a$ denote the function ${}_af_a(x) = f(a^{-1}xa)$ $(x \epsilon G)$. Then T_{π} is a distribution in the sense of L. Schwartz[4] and

$$T_{\pi}({}_af_a) = T_{\pi}(f) \qquad (f \epsilon C_c^{\infty}(G),\ a \epsilon G).$$

We shall call the distribution T_{π} the character of the representation π.

THEOREM 3. *Let π_1 and π_2 be quasisimple irreducible representations of G on two Hilbert spaces. If $T_{\pi_1} = cT_{\pi_2} (c \epsilon C)$ then π_1 and π_2 are infinitesimally equivalent. Conversely if π_1 and π_2 are infinitesimally equivalent $T_{\pi_1} = T_{\pi_2}$.*

Since for irreducible unitary representations infinitesimal equivalence is the same as ordinary equivalence, such a representation is complete'y determined within equivalence by its character.

THEOREM 4. *Let π be a quasisimple irreducible representation of G and let f be any measurable function on G such that f vanishes outside a compact set and* $\int_G |f(x)|^2 \, dx < \infty$. *Then the operator* $\int_G f(x)\pi(x) \, dx$ *is of the Hilbert-Schmidt class.*

For any $X \in \mathfrak{g}_0$ and $f \in C_c^\infty(G)$ put

$$({}^*Xf)(x) = \left\{\frac{d}{dt} f(\exp(-tX)x)\right\}_{t=0}.$$

Then the mapping $X \to {}^*X$ is a representation of $\mathfrak{g}_0$ on $C_c^\infty(G)$ which can be extended to a representation $b \to {}^*b$ $(b \in \mathfrak{B})$ of $\mathfrak{B}$. Let φ be the anti-automorphism of $\mathfrak{B}$ such that $\varphi(X) = -X$ $(X \in \mathfrak{g})$. If T is any distribution on G we define bT $(b \in \mathfrak{B})$ as follows:

$$bT(f) = T({}^*(\varphi(b))f) \qquad (f \in C_c^\infty(G)).$$

Moreover for any function f on G we denote by ${}_yf, f_z$ and ${}_yf_z$ $(y, z, \in G)$ the functions

$${}_yf(z) = f(y^{-1}x),\ f_z(x) = f(xz),\ {}_yf_z(x) = f(y^{-1}xz)\ (z \in G).$$

Let Z denote the center of G and let π be a quasisimple irreducible representation of G on a Hilbert space. Let χ be the infinitesimal character of π and let η be the homomorphism of Z into C such that $\pi(a) = \eta(a)\pi(1)$ $(a \in Z)$. Then if T_π is the character of π it is easily seen that

$$zT_\pi = \chi(z)T_\pi \qquad (z \in \mathfrak{Z})$$

$$T_\pi({}_yf_y) = T_\pi(f),\ T_\pi(f_a) = \eta(a)T_\pi(f)\ (f \in C_c^\infty(G),\ y \in G,\ a \in Z).$$

Now put[5] $M_1 = MZ$ and let $x \to x^*$ denote the adjoint representation of G. Put $(x)^{y^*} = yxy^{-1}(x, y \in G)$ and let $C_c(G)$ denote the class of all continuous functions on G which vanish outside a compact set. Let α and g be continuous functions[5] on A_+ and M_1, respectively. Put[6]

$$T(f) = \int f((n^{-1}h^{-1}m^{-1})^{u^*})\alpha(h)g(m)\, dm\, dh\, dn\, du^* \qquad (f \in C_c(G))$$

Here dm, dh, dn, du^* are the left invariant Haar measures on M_1, A_+, N, K^*, respectively, and the integral extends over $K^* \times M_1 \times A_+ \times N$. It can be shown that $T(f) = T({}_yf_y)(y \in G)$. Let σ be an irreducible representation of M_1 on a finite-dimensional Hilbert space U. Let δ be the equivalence class of the representation σ_0 of M defined by σ. Let $\psi_0 \neq 0$ be an element in U belonging to the highest weight λ_δ of σ_0 and let η be the homomorphism of Z into C such that $\sigma(a) = \eta(a)\sigma(1)$ $(a \in Z)$. Put $g(m) = (\psi_0, \sigma(m^{-1})\psi_0)$ $(m \in M_1)$ and $\alpha(h) = e^{-(\nu+2\rho)(\log h)}$ where ν is a linear function on $\mathfrak{h}_{\mathfrak{B}}$ and $\log h$ is the unique element in $\mathfrak{h}_{\mathfrak{p}_0}$ such that $\exp(\log h) = h$. If we regard T as a distribution it is easily seen that

$$zT = \chi_\Lambda(z)T \qquad (z \in \mathfrak{Z})$$

where $\Lambda(H_1 + H_2) = \nu(H_1) + \lambda_\delta(H_2)$ $(H_1 \in \mathfrak{h}_{\mathfrak{r}}, H_2 \in \mathfrak{h}_{\mathfrak{k}})$. Moreover

$$T(f_a) = \eta(a)T(f) \qquad (a \in Z, f \in C_c(G))$$

and it is easy to check that

$$T(f) = \int f((n^{-1}h^{-1}m^{-1})^{u^*})e^{-(\nu+2\rho)(\log h)}\,\xi(m^{-1})\,dm\,dh\,dn\,du^* \quad (f \in C_c(G))$$

where

$$\xi(m) = \frac{1}{d(\delta)}|\psi_0|^2\, sp\sigma(m)$$

and $d(\delta)$ is the degree of σ. Let A_- be the analytic subgroup of M corresponding to $\mathfrak{h}_{\mathfrak{k}_0}$. Put $A = A_+A_-$ and $A_1 = AZ$. Let V be the set of all elements in G which can be written in the form xyx^{-1} with $x \in G$ and $y \in A_1N$. We shall say that an element $y \in V$ is regular if $y = xhx^{-1}$ for some $x \in G$ and $h \in A_1$ and y^* has exactly[5] l eigen-values equal to 1. Let V_0 be the set of all elements in V which are regular. Then V_0 is open in G and V is the closure of V_0. Let W_0 be the subgroup of the Weyl group W consisting of those elements $s \in W$ for which there exists an $x \in G$ such that $sH = x^*H$ for all $H \in \mathfrak{h}$. It is easily seen that every $s \in W_0$ leaves both $\mathfrak{h}_{\mathfrak{k}}$ and $\mathfrak{h}_{\mathfrak{p}}$ invariant. Put

$$\Delta^-(H) = \prod_{\alpha \in P_-} (e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)}),\ \Delta^+(H) = \prod_{\alpha \in P_+} (e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)}) \qquad (H \in \mathfrak{h})$$

and define $\epsilon(s) = \pm 1$ $(s \in W_0)$ in such a way that

$$\Delta^-(sH) = \epsilon(s)\Delta^-(H)$$

for all $H \in \mathfrak{h}_{\mathfrak{k}}$. In particular if P_- is empty $\epsilon(s) = 1$ for all $s \in W_0$. Consider the function $\Theta_{\Lambda,\eta}$ on V_0 defined as follows:

$$\Theta_{\Lambda,\eta}(y) = \eta(\gamma)\frac{\sum_{s \in W_0} \epsilon(s)e^{s(\Lambda+\rho)(H_1+H_2)}}{\Delta^-(H_1)|\Delta^+(H_1 + H_2)|} \qquad (y \in V_0)$$

where $y = x(\gamma \exp(H_1 + H_2))x^{-1}$ for some $x \in G$, $\gamma \in Z$, $H_1 \in \mathfrak{h}_{\mathfrak{k}_0}$ and $H_2 \in \mathfrak{h}_{\mathfrak{p}_0}$. It can be shown that in spite of the ambiguity in the choice of γ, H_1 and H_2, $\Theta_{\Lambda,\eta}$ is well defined on V_0. We extend $\Theta_{\Lambda,\eta}$ on G by defining it to be zero outside V_0. Then it can be proved that[6]

$$T(f) = \int_G f(x)\,\Theta_{\Lambda,\eta}(x)\,dx \qquad (f \in C_c(G))$$

provided the Haar measure dx on G is suitably normalized. Let $s_1 = 1$ $s_2, \ldots, s_r$ be a maximal set of distinct elements in the Weyl group W with the following properties:

(I) Let $\Lambda_i = s_i(\Lambda + \rho) - \rho$ $1 \le i \le r$. Then $\Lambda_i + \rho \neq s(\Lambda_j + \rho)$ if $i \neq j$ $(1 \le i, j \le r)$ and $s \in W_0$.

(II) For each i $(1 \le i \le r)$ there exists a $\delta_i \in \omega$ such that Λ_i coincides on

$\mathfrak{h}_{\mathfrak{k}}$ with the highest weight[5] λ_{δ_i} of δ_i. Moreover if $\sigma \in \delta_i$ and $\gamma \in M \cap Z$

$$\sigma(\gamma) = \eta(\gamma)\sigma(1).$$

Put $\Theta_i = \Theta_{\Lambda_i, \eta}$ $1 \leq i \leq r$ and $T_i(f) = \int_G f(x)\Theta_i(x)\, dx \qquad (f \in C_c(G))$. Then we see that the distributions T_i $1 \leq i \leq r$ are solutions of the equations,

$$zT = \chi_\Lambda(z)T \qquad (z \in \mathfrak{Z})$$

$$T({}_yf_y) = T(f),\ T(f_a) = \eta(a)T(f)\ (f \in C_c^\infty(G), y \in G, a \in Z).$$

We have seen above that if π is any quasisimple irreducible representation of G such that χ_Λ is the infinitesimal character of π and $\pi(a) = \eta(a)\pi(1)$ $(a \in Z)$, then its character T_π is also a solution of the above equations. Hence one might hope that in most cases $T\pi$ would be a linear combination of T_i $1 \leq i \leq r$.

In conclusion I should like to thank Professor C. Chevalley for his help and advice on several questions connected with the results of this note.

[1] "Representations of semisimple Lie groups. II," PROC. NATL. ACAD. SCI., **37**, 362 (1951), quoted hereafter as *RII*.

[2] Since π is irreducible it follows easily that $\mathfrak{H}$ is separable.

[3] Here dx denotes the left invariant Haar measure on G.

[4] Schwartz, L., *Theorie des distributions*, Hermann, Paris, 1950.

[5] See *RII* for the meaning of the various symbols.

[6] Compare with Gelfand I. M., and Naimark M. A., *Izvestiya Akad. Nauk SSR*, Ser. Mat., 1947, vol. 11, pp. 411–504; *Mat. Sbornik N. S.*, 1947, vol. 21 (63), pp. 405–434; *Doklady Akad. Nauk SSSR* (*N. S.*), 1948, vol. 61, pp. 9–11.