

Moduli of regular holonomic \mathcal{D} -modules with normal crossing singularities

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Abstract

This paper solves the global moduli problem for regular holonomic \mathcal{D} -modules with normal crossing singularities on a nonsingular complex projective variety. This is done by introducing a level structure (which gives rise to “pre- \mathcal{D} -modules”), and then introducing a notion of (semi-)stability and applying Geometric Invariant Theory to construct a coarse moduli scheme for semistable pre- \mathcal{D} -modules. A moduli is constructed also for the corresponding perverse sheaves, and the Riemann-Hilbert correspondence is represented by an analytic morphism between these moduli spaces.

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1 Introduction

The moduli problem for regular holonomic \mathcal{D} modules on a non-singular complex projective variety X has the following history. Around 1989, Carlos Simpson solved the problem in the case when the \mathcal{D} -modules are \mathcal{O} -coherent, which first appeared in a preliminary version of his famous paper [S]. In this case, a \mathcal{D} -module M on X is the same as a vector bundle together with an integrable connection. The next case, that of meromorphic connections with regular singularity along a fixed normal crossing divisor $Y \subset X$ was solved in [N]. As explained there, one has to consider a level structure in the form of logarithmic lattices for the meromorphic connections in order to have a good moduli problem (or an Artin algebraic stack), and secondly, a notion of semi-stability has to be introduced in order to be able to apply Geometric Invariant Theory. In collaboration with Claude Sabbah, a more general case was treated in [N-S], where the divisor $Y \subset X$ is required to be smooth, but the only restriction on the regular holonomic module M is that its characteristic variety should be contained in $X \cup N_{S,X}^*$ (this is more general than being a non-singular or meromorphic connection). Here, we introduced the notion of pre- \mathcal{D} -modules, which play the same role for these regular holonomic modules that logarithmic connections play for regular meromorphic connections. A notion of semistability was introduced for the pre- \mathcal{D} -modules, and a moduli was constructed. We also constructed a moduli for the corresponding perverse sheaves, and showed that the Riemann-Hilbert correspondence defines an analytic morphism from the first moduli to the second, and has various good properties. This is already the most general case if X is 1 dimensional.

The present paper solves the moduli problem in the general case where we have a divisor with *normal crossings*, and the characteristic variety of the regular holonomic \mathcal{D} -modules is allowed to be any subset of the union of the conormal bundles of the nonsingular strata of the divisor. This is done by extending the notion of pre- \mathcal{D} -modules to this more general case, defining semistability, and constructing a moduli for these using GIT methods and Simpson's construction in [S] of moduli for semistable Λ -modules. Also, a moduli is constructed for the corresponding perverse sheaves, and the Riemann-Hilbert correspondence is represented by an analytic morphism having various good properties. We now give a quick overview of the contents of this paper.

Let X be a nonsingular variety of dimension d , and let $Y \subset X$ be a divisor with normal crossings (the irreducible components of Y can be singular). Let $S_d = X$, $S_{d-1} = Y$, and let S_i be the singular locus of S_{i+1} for $i < d - 1$. This defines a filtration of X by closed reduced subschemes

$$S_d \supset S_{d-1} \supset \dots \supset S_0$$

where each S_i is either empty or of pure dimension i . Let $S'_d = X - Y$ and for $i < d$ let $S'_i = S_i - S_{i-1}$. The stratification $X = \cup_i S'_i$ is called the **singularity stratification** of X induced by Y . Let T^*X be the total space of the cotangent bundle of X , and for $i \leq d$ let $N_i^* \subset T^*X$ be the locally closed subset which is the

conormal bundle of the closed submanifold S'_i of $X - S_{i-1}$. In particular, N_d^* is the zero section $X \subset T^*X$. Let $N^*(Y) \subset T^*X$ be defined to be the union

$$N^*(Y) = N_d^* \cup N_{d-1}^* \cup N_{d-2}^* \cup \dots$$

of all the N_i^* for $i \leq d$. Note that $N^*(Y)$ is a closed lagrangian subset of T^*X , and any irreducible component of $N^*(Y)$ is contained in the closure $\overline{N_i^*}$ for some i .

In this paper we consider those regular holonomic \mathcal{D} modules on X whose characteristic variety is contained in $N^*(Y)$. Equivalently (under the Riemann-Hilbert correspondence), we consider perverse sheaves on X which are cohomologically constructible with respect to the singularity stratification

$$X = \cup_{0 \leq i \leq d} S'_i$$

Such regular holonomic \mathcal{D} modules (equivalently, such perverse sheaves) form an abelian category, in which each object is of finite length. These will be called regular holonomic \mathcal{D} -modules (or perverse sheaves) on (X, Y) .

In section 2, we introduce the basic notation involving various morphisms which arise out of the singularity strata, their normalizations, and their étale coverings.

One of the problems we had to overcome was to give a convenient \mathcal{O} -coherent description of regular holonomic \mathcal{D} -modules on (X, Y) . This is done in section 3, by extending the notion of a pre- \mathcal{D} -modules from [N-S] to this more general setup.

In section 4, we explain the functorial passage from pre- \mathcal{D} -modules to \mathcal{D} -modules. In the case of a smooth divisor, this was only indirectly done, via Malgrange's presentation of \mathcal{D} -modules, in [N-S]. Here we do it more explicitly in our more general situation.

In section 5, we define a notion of (semi-)stability, and construct a moduli scheme for (semi-)stable pre- \mathcal{D} -modules on (X, Y) with prescribed numerical data (in the form of Hilbert polynomials). We have given here an improved quotient construction, which allows us to give a much more simplified treatment of stability when compared with [N-S].

In section 6, we represent perverse sheaves on (X, Y) with prescribed numerical data in *finite terms*, via a notion of 'Verdier objects', generalizing the notion of Verdier objects in [N-S]. We then construct a moduli scheme for these Verdier objects. This is a quotient of an affine scheme by a reductive group, so does not need any GIT. Though a more general moduli construction due to Gelfand, MacPherson and Vilonen exists in literature (see [G-M-V]), the construction here is particularly suited for the study the Riemann-Hilbert morphism (in section 7), which represents the de Rham functor.

In section 7 we show that the Riemann-Hilbert correspondence defines an analytic morphism from (an open set of) the moduli of pre- \mathcal{D} -modules to the moduli of perverse sheaves. We extend the rigidity results in [N] and [N-S] to this more general situation, which in particular means that this morphism is a local isomorphism at points with **good residual eigenvalues** as defined later.

Erratum I point out here some mistakes which have remained in [N] and [N-S], and their corrections.

(1) The lemma 2.9 of [N] is false as stated, and needs the additional hypothesis that A is reduced. This additional hypothesis of reducedness is satisfied in the part of proposition 2.8 of [N] where this lemma is employed.

(2) On page 58 of [N-S], in the construction of a local universal family, we take the action of $G_i = PGL(p_i(N))$ on Q_i . This should read $G_i = SL(p_i(N))$ and not $PGL(p_i(N))$.

(3) In Theorem 4.19.3 on page 63 of [N-S], the statement “The S-equivalence class of a semistable reduced module \mathbf{E} equals its isomorphism class if and only if \mathbf{E} is stable” should be corrected to read “The S-equivalence class of a semistable reduced module \mathbf{E} equals its isomorphism class if \mathbf{E} is stable”. The “... and only if...” should be removed. (The proof only proves the “if” part, ignores the “only if” part, and the “only if” statement is in fact trivially false.)

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2 Preliminaries on normal crossing divisors

In section 2.1 we define various objects naturally associated with a normal crossing divisor. The notation introduced here is summarized in section 2.2 for easy reference, and is used without further comment in the rest of the paper.

2.1 Basic definitions

Let X be a nonsingular variety of dimension d , Y a normal crossing divisor, and let closed subsets $S_i \subset X$ for $i \leq d$ be defined as in the introduction ($S_d = X$, $S_{d-1} = Y$, and by descending induction, S_{i-1} is the singularity locus of S_i). Let m be the smallest integer such that S_m is nonempty. We allow the possibility that m is any integer from 0 to d (for example, if $m = d$ then Y is empty, and if $m = d - 1$ when Y is smooth).

For $i \geq m$, let X_i be the normalization of S_i , with projection $p_i : X_i \rightarrow S_i$. This is a reduced, nonsingular, i -dimensional scheme of finite type over \mathcal{C} for $i \geq m$. This may not be connected (same as irreducible), and we denote its components by $X_{i,a}$, as a varies over the indexing set $\pi_0(X_i)$. Note that the fiber of $p_i : X_i \rightarrow S_i$ over a point $y \in S_i$ is the set of all branches of S_i which pass through y (where by definition a branch means a component in the completion of the local ring). Now let I be a nonempty subset of $\{m, \dots, d - 1\}$, and for any such I , let $m(I)$ denote the smallest element of I . To any such I , we now associate a scheme X_I and a morphism $p_I : X_I \rightarrow S_{m(I)}$ as follows. By definition, X_I is the finite scheme over $S_{m(I)}$ whose fiber over any point y of $S_{m(I)}$ consists of all nested sequences (x_j) of branches of S_j at the point y , for j varying over I . ‘Nested’ means for any two $j, k \in I$ with $j \leq k$, $x_k \in X_k$ is a branch of S_k containing the branch $x_j \in X_j$ of S_j . In particular when $I = \{i\}$, $p_{\{i\}} : X_{\{i\}} \rightarrow S_i$ is just the normalization $p_i : X_i \rightarrow S_i$ of S_i .

For nonempty subsets $I \subset J \subset \{m, \dots, d - 1\}$, we have a canonical forgetful map

$p_{I,J} : X_J \rightarrow X_I$. If $I \subset J \subset K$, then by definition we have the equality

$$p_{I,K} = p_{I,J} \circ p_{J,K} : X_K \rightarrow X_I$$

For $m \leq i \leq d-1$ we denote by Y_i^* the $d-i$ sheeted finite étale cover $p_{\{i\},\{i+1\}} : X_{\{i,i+1\}} \rightarrow X_i$ of X_i , which splits the branches of S_{i+1} which meet along S_i . Similarly, we denote by Z_i the C_2^{d-i} sheeted finite étale cover $p_{\{i\},\{i+2\}} : X_{\{i,i+2\}} \rightarrow X_i$ of X_i , which splits the branches of S_{i+2} which meet along S_i . We denote by Z_i^* the 2 sheeted finite étale cover $p_{\{i,i+2\},\{i,i+1,i+2\}} : X_{\{i,i+1,i+2\}} \rightarrow X_{\{i,i+2\}}$ of Z_i . Let $f_i : X_i \rightarrow X$ be the composite $X_i \xrightarrow{p_i} S_i \rightarrow X$.

Let N_i be the normal bundle of X_i in X . This is defined by means of the exact sequence

$$0 \rightarrow T_{X_i} \rightarrow f_i^* T_X \rightarrow N_i \rightarrow 0$$

Let $T(Y) \subset T_X$ be the closed subset of T_X consisting of vectors tangent to branches of Y . This gives a closed subset $f_i^* T(Y) \subset f_i^* T_X$. Let F_i be the closed subset of N_i which is the image of $f_i^* T(Y)$ under the morphism $f_i^* T_X \rightarrow N_i$ of geometric vector bundles. Similarly, let $N_{i,i+1}$ be the normal bundle to $Y_i^* = X_{i,i+1}$ in X , defined with respect to the composite morphism $Y_i^* \rightarrow X_i \rightarrow X$, and let $F_{i,i+1} \subset N_{i,i+1}$ be the normal crossing divisor in the total space of $N_{i,i+1}$, defined by vectors tangent to branches of Y .

Let $Y^* = Y_{d-1}^* = X_{d-1}$ be the normalization of Y . Let $h_i : Y_i^* \rightarrow Y^*$ be the canonical map, which associates to a point $(x_i, x_{i+1}) \in X_{\{i,i+1\}} = Y_i^*$ the unique branch $y_{d-1} \in Y^*$ of Y at $p_i(x_i) \in S_i$ such that

$$x_i = x_{i+1} \cap y_{d-1}$$

This defines a vector hyper subbundle $H_{i,i+1} \subset N_{i,i+1}$ (hyper subbundle means rank is less by 1) of $N_{i,i+1}$, which is given by vectors tangent to the branch of Y given by $h_i : Y_i^* \rightarrow Y^*$. Note that $H_{i,i+1}$ is a nonsingular irreducible component of $F_{i,i+1}$.

We now define some sheaves \mathcal{D}_i of algebras differential operators on X_i and \mathcal{D}_i^* on Y_i^* . (These are ‘split almost polynomial algebras’ of differential operators in the terminology of Simpson [S] as explained in later in section 5.1.) For this we need the following general remark:

Remark 2.1 Let V be nonsingular, $M \subset V$ a divisor with normal crossing, and $M' \subset M$ a nonsingular component of M , with inclusion $f : M' \hookrightarrow V$. Let $\Lambda = \mathcal{D}_V[\log M]$ be the subring of \mathcal{D}_V which preserves the ideal sheaf $I_M \subset \mathcal{O}_V$, and let $\Lambda' = \mathcal{O}_{M'} \otimes_{\mathcal{O}_V} \Lambda$ (we will write $\Lambda' = f^*(\Lambda)$ or $\Lambda' = \Lambda|_{M'}$ for brevity, though this abbreviated notation conceals that we have tensored with $\mathcal{O}_{M'}$ on the *left*, for it does not matter on what side we tensor). Then Λ' is naturally a split almost polynomial algebra of differential operators on M' (see section 5.1), and the category of Λ modules on V which are schematically supported on M' is naturally equivalent to the category of Λ' -modules on M' . This equivalence follows from the fact that Λ necessarily preserves the ideal sheaf of M' in V .

Now we come back to our given set up, where we apply the above remark with N_i (or $N_{i,i+1}$) as V , F_i (or $F_{i,i+1}$) as M , and the zero section X_i of N_i (or the zero section Y_i^* of $N_{i,i+1}$) as M' . As usual, let $\mathcal{D}_{N_i}[\log F_i]$ denote the subring of \mathcal{D}_{N_i} consisting of all operators which preserve the ideal sheaf of F_i in \mathcal{O}_{N_i} . Then we define \mathcal{D}_i to be the restriction (see the above remark) of $\mathcal{D}_{N_i}[\log F_i]$ to the zero section $X_i \subset N_i$, which is canonically isomorphic to $f_i^* \mathcal{D}_X[\log Y]$. Similarly, we define \mathcal{D}_i^* to be the restriction of $\mathcal{D}_{N_{i,i+1}}[\log F_{i,i+1}]$ to the zero section Y_i^* .

If $H \subset M$ is a nonsingular irreducible component of a normal crossing divisor M in a nonsingular variety V , then the **Euler operator along H in $\mathcal{D}_V[\log M]$** is the element $\theta_H \in (\mathcal{D}_V[\log M]|_H)$ which has the usual definition: if V has local analytic coordinates (x_1, \dots, x_d) with M locally defined by $x_1 \cdots x_{d-m} = 0$ and H by $x_1 = 0$, then the operator θ_H is given by the action of $x_1(\partial/\partial x_1)$.

For each $i \leq d-1$ we define the section θ_i of \mathcal{D}_i^* as the restriction to Y_i^* of the Euler operator along $H_{i,i+1}$ in $\mathcal{D}_{N_{i,i+1}}[\log F_{i,i+1}]$.

The above definitions work equally well in the analytic category. For the remaining basic definitions, we restrict to the analytic category, with euclidean topology (in particular, if T was earlier a finite type, reduced scheme over \mathcal{C} then now the same notation T will denote the corresponding analytic space). Vector bundles will denote their respective total analytic spaces.

Let U_i be the open subset $U_i = N_i - F_i$ of N_i , and let R_i be the open subset $N_{i,i+1} - F_{i,i+1}$ of $N_{i,i+1}$. Let $N_{i,i+2}$ be the normal bundle to Z_i in X , and let W_i be the open subset of $N_{i,i+2}$ which is the complement in $N_{i,i+2}$ of vectors tangent to branches of Y . Similarly, let $N_{i,i+1,i+2}$ be the normal bundle to Z_i^* in X . and let W_i^* be the open subset of $N_{i,i+1,i+2}$ which is the complement in $N_{i,i+1,i+2}$ of vectors tangent to branches of Y .

Finally, for $i \leq d-1$ we define some central elements $\tau_i(c)$ of certain fundamental groups, which are the topological counterparts of the operators θ_i . Let $Y_i^*(c)$, where c varies over $\pi_0(Y_i^*)$, be the connected components of Y_i^* , and let $N_{i,i+1}(c)$, $F_{i,i+1}(c)$, $H_{i,i+1}(c)$, and $R_i(c)$ be the restrictions of the corresponding objects to $Y_i^*(c)$. Let $\tau_i(c)$ be the element in the center of $\pi_1(R_i(c))$, (with respect to any base point) which is represented by a positive loop around $H_{i,i+1}(c)$. The fact that $\tau_i(c)$ is central, and is unambiguously defined, follows from the following lemma in the topological category.

Lemma 2.2 *Let S be a connected topological manifold, and $p : N \rightarrow S$ a complex vector bundle on S of rank r . Let $F \subset N$ be a closed subset such that locally over S^1 , the subset F is the union of r vector subbundles of N , each of rank $r-1$, in general position. Let $H \subset N$ be a vector subbundle of rank $r-1$ such that $H \subset F$. Let $U = N - F$ with projection $p : U \rightarrow S$, which is a locally trivial fibration with fiber $(\mathcal{C}^*)^r$. The fundamental group of any fiber is \mathbb{Z}^r , with a basis given by positive loops around the various hyperplanes. Let $u_0 \in U$ be a base point, and let $p(u_0) = s_0 \in S$. Let $\tau_H \in \pi_1(U, u_0)$ be represented by the positive loop around $H \cap p^{-1}(s_0)$ in the fiber U_{s_0} . Then we have the following:*

(1) *The element τ_H is central in $\pi_1(U, u_0)$.*

(2) Let $u_1 \in U$ be another base point, and let $\tau'_H \in \pi_1(U, u_1)$ be similarly defined. Let $\sigma : [0, 1] \rightarrow U$ be a path joining u_0 to u_1 , and let $\sigma^* : \pi_1(U, u_1) \rightarrow \pi_1(U, u_0)$ be the resulting isomorphism. Then $\sigma^*(\tau'_H) = \tau_H$.

Proof (Sketch) Let S^1 be the unit circle with base point 1, and let $\gamma : S^1 \rightarrow U : 1 \mapsto u_0$ be another loop in U , based at u_0 . By pulling back the bundle N under the base change $p \circ \gamma : S^1 \rightarrow S$, we can reduce the statement (1) to the case that $S = S^1$. If base is S^1 , then as all complex vector bundles become trivial, the space U becomes a product $\mathcal{C}^* \times U'$ for some U' , and τ_H is the positive generator of the fundamental group of \mathcal{C}^* , so is central in this product. Similarly, (2) follows from a base change to the unit interval $[0, 1]$.

2.2 Summary of basic notation

X	= a nonsingular variety.
d	= the dimension of X .
Y	= a divisor with normal crossing in X .
S_d	= X
S_{d-1}	= Y

	By decreasing induction we define starting with $i = d - 2$,
S_i	= the singular locus of S_{i+1} for $i \leq d - 2$.
m	= the smallest i for which S_i is nonempty.
I	= any nonempty subset of $\{m, \dots, d - 1\}$
$m(I)$	= the smallest element of I .
$p_I : X_I \rightarrow S_{m(I)}$	= the finite scheme over $S_{m(I)}$ whose fiber over point of $S_{m(I)}$ consists of all nested sequences of branches of S_j at that point, for j varying over I . In particular when $I = \{i\}$, we have
$p_i : X_i \rightarrow S_i$	= the normalization of S_i .
	For nonempty subsets $J \subset I \subset \{m, \dots, d - 1\}$,
$X_{i,a}$	= connected components of X_i , as a varies over the indexing set $\pi_0(X_i)$
$p_{J,I} : X_I \rightarrow X_J$	= the canonical map. (All these commute.)

(Continued on next page)

	For $m \leq i \leq d-1$ we put
Y_i^*	$= X_{i,i+1}$, the $d-i$ sheeted finite étale cover of X_i which splits the branches of S_{i+1} which meet along S_i
	For $m \leq i \leq d-2$ we put
Z_i	$= X_{i,i+2}$, the C_2^{d-i} sheeted finite étale cover of X_i which splits the branches of S_{i+2} which meet along S_i
Z_i^*	$= X_{i,i+1,i+2}$, the 2 sheeted finite étale cover of Z_i
$f_i : X_i \rightarrow X$	$=$ the composite $X_i \xrightarrow{p_i} S_i \rightarrow X$
N_i	$=$ the normal bundle to X_i in X under f_i , defined by the exact sequence $0 \rightarrow T_{X_i} \rightarrow f_i^* T_X \rightarrow N_i \rightarrow 0$
$T(Y) \subset T_X$	$=$ the closed subset of T_X consisting of vectors tangent to branches of Y
F_i	$=$ the closed subset of N_i which is the image of $f^* T(Y)$ under the morphism $f^* T_X \rightarrow N_i$ of geometric vector bundles.
U_i	$= N_i - F_i$
\mathcal{D}_i	$=$ the restriction of $\mathcal{D}_{N_i}[\log F_i]$ to the zero section $X_i \subset N_i$, which is canonically isomorphic to $f_i^* \mathcal{D}_X[\log Y]$
Y^*	$= Y_{d-1}^* = X_{d-1}$, the normalization of Y
$N_{i,i+1}$	$=$ the normal bundle to $Y_i^* = X_{i,i+1}$ in X , defined with respect to the morphism $Y_i^* \rightarrow X_i \rightarrow X$
$F_{i,i+1}$	$=$ the normal crossing divisor in the total space of $N_{i,i+1}$, defined by vectors tangent to branches of Y .
$h_i : Y_i^* \rightarrow Y^*$	$=$ the canonical map, sending (x_i, x_{i+1}) to the branch y_{d-1} of Y^* which intersects the given branch x_{i+1} of S_{i+1} along given the branch x_i of S_i .
$H_{i,i+1}$	$=$ the hypersubbundle of $N_{i,i+1}$ contained in $F_{i,i+1}$, defined by vectors tangent to the branch of Y given by $h_i : Y_i^* \rightarrow Y^*$
\mathcal{D}_i^*	$=$ the restriction of $\mathcal{D}_{N_{i,i+1}}[\log F_{i,i+1}]$ to the zero section $Y_i^* \subset N_{i,i+1}$
θ_i	$=$ the Euler operator along $H_{i,i+1}$ in \mathcal{D}_i^* .
R_i	$=$ the open subset of $N_{i,i+1} - F_{i,i+1}$ of $N_{i,i+1}$ which is the complement in $N_{i,i+1}$ of vectors tangent to branches of Y
$R_i(c)$	$=$ connected components of R_i as c varies over $\pi_0(Y_i^*)$.
$\tau_i(c)$	$=$ the element in the center of $\pi_1(R_i(c))$, which is given by a positive loop around $H_{i,i+1}(c)$
$N_{i,i+2}$	$=$ the normal bundle to Z_i in X .
W_i	$=$ the open subset of $N_{i,i+2}$ which is the complement in $N_{i,i+1,i+2}$ of vectors tangent to branches of Y
$N_{i,i+1,i+2}$	$=$ the normal bundle to Z_i^* in X .
W_i^*	$=$ the open subset of $N_{i,i+1,i+2}$ which is the complement in $N_{i,i+1,i+2}$ of vectors tangent to branches of Y

3 Pre- \mathcal{D} -modules on (X, Y)

In this section, we first define the notion of a pre- \mathcal{D} -module. Then we consider the special case when X is a polydisk. Finally, we give some historical motivation.

3.1 Global definition

Let X is any smooth variety and Y a normal crossing divisor. We follow the notation introduced in section 2. The following definitnion works equally well in the algebraic or the analytic categories.

Definition 3.1 A pre- \mathcal{D} -module $\mathbf{E} = (E_i, t_i, s_i)$ on (X, Y) consists of the following.

(1) For each $m \leq i \leq d$, E_i is a vector bundle on X_i (of not necessarily constant rank) together with a structure of \mathcal{D}_i -module,

(Note that by (1), for each $m \leq i \leq d-1$, the pullbacks $E_{i+1}|Y_i^*$ and $E_i|Y_i^*$ under the respective maps $Y_i^* \rightarrow X_{i+1}$ and $Y_i^* \rightarrow X_i$ have a natural structure of a \mathcal{D}_i^* -module.)

(2) For each $m \leq i \leq d-1$, $t_i : (E_{i+1}|Y_i^*) \rightarrow (E_i|Y_i^*)$ and $s_i : (E_i|Y_i^*) \rightarrow (E_{i+1}|Y_i^*)$ are \mathcal{D}_i^* -linear maps, such that

$$\begin{aligned} s_i t_i &= \theta_i \text{ on } E_{i+1}|Y_i^* \\ t_i s_i &= \theta_i \text{ on } E_i|Y_i^* \end{aligned}$$

(3) Let $m \leq i \leq d-2$. Let $\pi : Z_i^* \rightarrow Z_i$ be the projection

$$p_{\{i, i+2\}, \{i, i+1, i+2\}} : X_{\{i, i+1, i+2\}} \rightarrow X_{\{i, i+2\}}$$

Let $E_{i+2}|Z_i$ and $E_i|Z_i$ be the pullbacks of E_{i+2} and E_i under respectively the composites $Z_i = X_{\{i, i+2\}} \rightarrow X_{i+2}$ and $Z_i = X_{\{i, i+2\}} \rightarrow X_i$. (Note that there is no object called $E_{i+1}|Z_i$.) We will denote the pullback of E_{i+1} under $Z_i^* = X_{\{i, i+1, i+2\}} \rightarrow X_{i+1}$ by $E_{i+1}|Z_i^*$. We will denote $\pi^*(E_{i+2}|Z_i)$ by $E_{i+2}|Z_i^*$ and $\pi^*(E_i|Z_i)$ by $E_i|Z_i^*$. Let

$$a_{i+2} : E_{i+2}|Z_i \rightarrow \pi_* \pi^*(E_{i+2}|Z_i) = \pi_*(E_{i+2}|Z_i^*)$$

$$a_i : E_i|Z_i \rightarrow \pi_* \pi^*(E_i|Z_i) = \pi_*(E_i|Z_i^*)$$

be adjunction maps, and let the cokernels of these maps be denoted by

$$q_{i+2} : \pi_*(E_{i+2}|Z_i) \rightarrow Q_{i+2}$$

$$q_i : \pi_*(E_i|Z_i) \rightarrow Q_i$$

Then we impose the requirement that the composite map

$$E_{i+2}|Z_i \xrightarrow{a_{i+2}} \pi_*(E_{i+2}|Z_i) \xrightarrow{\pi^*(t_{i+1}|Z_i^*)} \pi_*(E_{i+1}|Z_i^*) \xrightarrow{\pi_*(t_i|Z_i^*)} \pi_*(E_i|Z_i^*) \xrightarrow{q_i} Q_i$$

is zero.

(4) Similarly, we demand that for all $m \leq i \leq d - 2$ the composite map

$$Q_{i+2} \xleftarrow{q_{i+2}} \pi_*(E_{i+2}|Z_i^*) \xrightarrow{\pi_*(s_{i+1}|Z_i^*)} \pi_*(E_{i+1}|Z_i^*) \xrightarrow{\pi_*(s_i|Z_i^*)} \pi_*(E_i|Z_i^*) \xleftarrow{a_i} E_i|Z_i$$

is zero.

(5) Note that as $\pi : Z_i^* \rightarrow Z_i$ is a 2-sheeted cover, for any sheaf \mathcal{F} on Z_i the new sheaf $\pi_*\pi^*(\mathcal{F})$ on Z_i has a canonical involution coming from the deck transformation for $Z_i^* \rightarrow Z_i$ which transposes the two points over any base point. In particular, the bundles $\pi_*(E_{i+2}|Z_i^*) = \pi_*\pi^*(E_{i+2}|Z_i)$ and $\pi_*(E_i|Z_i^*) = \pi_*\pi^*(E_i|Z_i)$ have canonical involutions, which we denote by ν . We demand that the following diagram should commute.

Diagram III.

$$\begin{array}{ccccc} \pi_*(E_{i+1}|Z_i^*) & \xrightarrow{\pi_*(s_{i+1}|Z_i^*)} & \pi_*(E_{i+2}|Z_i^*) & \xrightarrow{\nu} & \pi_*(E_{i+2}|Z_i^*) \\ \pi_*(t_i|Z_i^*) \downarrow & & & & \downarrow \pi_*(t_i|Z_i^*) \\ \pi_*(E_i|Z_i^*) & \xrightarrow{\nu} & \pi_*(E_i|Z_i^*) & \xrightarrow{\pi_*(s_i|Z_i^*)} & \pi_*(E_{i+1}|Z_i^*) \end{array}$$

A **homomorphism** $\varphi : \mathbf{E} \rightarrow \mathbf{E}'$ of pre- \mathcal{D} -modules consists of a collection $\varphi_i : E_i \rightarrow E'_i$ of \mathcal{D}_i -linear homomorphisms which make the obvious diagrams commute.

Remark 3.2 As the adjunction maps are injective (in particular as a_i is injective), the condition (3) is equivalent to demanding the existence of a unique f which makes the following diagram commute.

Diagram I.

$$\begin{array}{ccccc} E_{i+2}|Z_i & & \xrightarrow{f} & & E_i|Z_i \\ a_{i+2} \downarrow & & & & \downarrow a_i \\ \pi_*(E_{i+2}|Z_i^*) & \xrightarrow{\pi_*(t_{i+1}|Z_i^*)} & \pi_*(E_{i+1}|Z_i^*) & \xrightarrow{\pi_*(t_i|Z_i^*)} & \pi_*(E_i|Z_i^*) \end{array}$$

Similarly, the condition (4) is equivalent to the existence of a unique homomorphism g which makes the following diagram commute.

Diagram II.

$$\begin{array}{ccccc} E_{i+2}|Z_i & & \xleftarrow{g} & & E_i|Z_i \\ a_{i+2} \downarrow & & & & \downarrow a_i \\ \pi_*(E_{i+2}|Z_i^*) & \xleftarrow{\pi_*(s_{i+1}|Z_i^*)} & \pi_*(E_{i+1}|Z_i^*) & \xleftarrow{\pi_*(s_i|Z_i^*)} & \pi_*(E_i|Z_i^*) \end{array}$$

Remark 3.3 The above definition is local, in the sense that

(1) if U_α is an open covering of X in the algebraic or analytic category, and \mathbf{E}_α is a collection of pre- \mathcal{D} -modules on $(U_\alpha, Y \cap U_\alpha)$ together with isomorphisms $\varphi_{\alpha,\beta} :$

$(\mathbf{E}_\beta|_{U_{\alpha,\beta}}) \rightarrow (\mathbf{E}_\alpha|_{U_{\alpha,\beta}})$ which form a 1-cocycle, then there exists a unique (upto unique isomorphism) pre- \mathcal{D} -module \mathbf{E} on (X, Y) obtained by gluing.

(2) If \mathbf{E} and \mathbf{E}' are two pre- \mathcal{D} -modules on (X, Y) , then a homomorphism $\varphi : \mathbf{E} \rightarrow \mathbf{E}'$ is uniquely defined by a collection of homomorphisms over U_α which match in $U_{\alpha,\beta}$.

Definition 3.4 Consider the action of θ_i on $E_{i+1}|_{Y_i^*}$, which is $\mathcal{O}_{Y_i^*}$ -linear. Note that in the global algebraic case, compactness of Y_i^* implies that the characteristic polynomial of the resulting endomorphism of $E_{i+1}|_{Y_i^*}$ is constant on each component of Y_i^* . We say that a **pre- \mathcal{D} -module $\mathbf{E} = (E_i, s_i, t_i)$ has good residual eigenvalues** if for each $i \leq d-1$ no two eigenvalues of $\theta_i \in \text{End}(E_{i+1}|_{Y_i^*})$, on any two components of Y_i^* which map down to intersecting subsets of X , differ by a non-zero integer.

Remark 3.5 Note that the above definition does not prohibit two eigenvalues of θ_i on $E_i|_{Y_i^*}$ from differing by non-zero integers. Also, note that the definition can involve more than one component of Y_i^* at a time: it is stronger than requiring that on each component of Y_i^* no two eigenvalues should differ by nonzero integers.

3.2 Restriction to a polydisk

There exists an open covering of X , where each open subset is a polydisk in \mathcal{C}^d with coordinates x_i , defined by $|x_i| < 1$, whose intersection with Y is defined by $\prod_{i \leq r} x_i = 0$ for a variable integer $r \leq d-m$. It is possible globally that the irreducible components of Y are singular. Moreover, it is possible that various branches of Y meeting at a point get interchanged as one moves around. This does not happen in a polydisk of the above kind, so the definition of a pre- \mathcal{D} -module becomes much simpler. We give it in detail in view of remark 3.3 above.

Let X be a polydisk in \mathcal{C}^d around the origin, with coordinates x_1, \dots, x_d , and let m be some fixed integer with $0 \leq m \leq d$. If $m \leq d-1$, let $Y \subset X$ be the normal crossing divisor defined by $\prod_{1 \leq i \leq d-m} x_i = 0$. If $m = d$, we take Y to be empty.

We will follow the notation summarised in section 2. We have a filtration $Y = S_{d-1} \supset \dots \supset S_m$ where each S_i is the singularity set of S_{i+1} for $m \leq i \leq d-2$, and S_m is nonsingular. Note that the irreducible components of S_i are as follows. For any subset $A \subset \{1, \dots, d-m\}$ of cardinality $d-i$, we have a component S_A of S_i defined by the ideal generated by all x_j for $j \in A$ (total C_{d-i}^{d-m} components). Then the normalization X_i of S_i is simply the disjoint union of all the S_A . Therefore $X_{i,A} = S_A$, which are polydisks of dimension $d - |A|$, are the components of X_i . Whenever $k \in A$, we have an inclusion $X_A \rightarrow X_{A-\{k\}}$. This is identified in our earlier notation with a component of $Y_{i-1}^* \rightarrow X_i$.

It follows from its general definition that a pre- \mathcal{D} -module \mathbf{E} on (X, Y) consists of the following data.

(1) For each $A \subset \{1, \dots, d - m\}$, we are given a vector bundle E_A on S_A , together with the structure of a $\mathcal{D}_X[\log Y]$ -module. The E_A with $|A| = d - i$ are the restrictions of E_i to the components S_A of X_i .

(2) For any $k \in A \subset \{1, \dots, d - m\}$, we have $\mathcal{D}_X[\log Y]$ -linear homomorphisms $t_A^k : E_{A-k}|X_A \rightarrow E_A$ and $s_A^k : E_A \rightarrow E_{A-k}|X_A$ such that

$$s_A^k t_A^k = x_k \partial / \partial x_k \text{ on } E_{A-k}|X_A$$

$$t_A^k s_A^k = x_k \partial / \partial x_k \text{ on } E_A$$

In terms of earlier notation, the t_A^k (respectively, the s_A^k) with $|A| = d - i$ make up the t_i (respectively the s_i).

(3) Let $k \neq \ell$ such that $k, \ell \in A \subset \{1, \dots, d - m\}$. Then we must have

$$\begin{aligned} t_A^k t_{A-k}^\ell &= t_A^\ell t_{A-\ell}^k \\ s_{A-\ell}^k s_A^\ell &= s_{A-k}^\ell s_A^k \\ t_{A-\ell}^k s_{A-k}^\ell &= s_A^\ell t_A^k \end{aligned}$$

The above three equations respectively embody the conditions that diagrams I, II, and III in the definition of a pre- \mathcal{D} -module must commute.

3.3 Motivation for the definition

The definition of a pre- \mathcal{D} -module may be regarded as another step in the programme of giving concrete representations of regular holonomic \mathcal{D} modules and perverse sheaves. The earlier steps relevant to us are the following.

(1) Deligne's description (1982) of a perverse sheaf on a disk with singularity at the origin, in terms of pairs of vector spaces and linear maps and Malgrange's description of corresponding regular holonomic \mathcal{D} -modules.

(2) Verdier's functor of specialization (Asterisque 101-102), and his description of extension of a perverse sheaf across a closed subspace (Asterisque 130).

(3) Similar construction by Malgrange for regular holonomic \mathcal{D} -modules in place of perverse sheaves.

(4) Verdier's description of a perverse sheaf on the total space of a line bundle L on a smooth variety S , in terms of two local systems on $L - S$ (= the complement of the zero section) and maps between them (Asterisque 130, 1985).

(5) Definition of a pre- \mathcal{D} -module on (X, Y) when Y is nonsingular, which can be obtained by choosing compatible logarithmic lattices in a combination of Step 3 and Step 4 (see [N-S]).

(6) Description by Galligo, Granger, Maisonobe of perverse sheaves on a polydisk with coordinates (z_1, \dots, z_n) with respect to the smoothening stratification induced

by the normal crossing divisor $z_1 \cdots z_n = 0$, in terms of a hypercube of vector spaces and linear maps (1985).

The description (6) for a polydisk with a normal crossing divisor is local and coordinate dependent like the description (1) for the disk with a point. One first makes it coordinate free and globalizes it in order to have the equivalent of (4) (which gives us finite descriptions of perverse sheaves described in section 6 below), and then puts level structures generalizing (5) to arrive at the above definition of a pre- \mathcal{D} -module.

One of the problems in globalizing the local hypercube description is that one can not unambiguously label the branches of Y which meet at a point, because of twistedness of the divisor. This is taken care of by normalizing the closed strata S_i and going to the coverings $\pi : Z_i^* \rightarrow Z_i$.

The requirement that the various composites of s and t should give endomorphisms expressible in terms of Euler vector fields is present in the local hypercube description in much the same form.

The three commutative diagrams I, II, and III in the definition respectively embody the globalizations of the conditions in the hypercube description that

I: the two canonical maps C_x and C_y should commute,

II: the two variation maps V_x and V_y should commute, and

III: we should have $C_x V_y = V_y C_x$.

4 From pre- \mathcal{D} -modules to \mathcal{D} -modules

In section 4.1, we directly describe the \mathcal{D} -module associated to a pre- \mathcal{D} -module in the special case where Y is nonsingular. In section 4.2, we will associate a \mathcal{D} -module to a pre- \mathcal{D} -module when Y is normal crossing. This is done by first doing it on polydisks, and then patching up. Finally, in section 4.3 we show how to find a pre- \mathcal{D} -module with good residual eigenvalues over a given \mathcal{D} -module, proving that the functor from pre- \mathcal{D} -modules on (X, Y) with good residual eigenvalues to regular holonomic \mathcal{D} -modules on X whose characteristic variety is contained in $N^*(Y)$ is essentially surjective.

4.1 The case when Y is smooth

First we treat the case where the divisor Y is nonsingular. To a pre- \mathcal{D} -module (E, F, t, s) on (X, Y) , where $E = E_d$ is a logarithmic connection on (X, Y) , $F = E_{d-1}$ is a vector bundle on Y with structure of a $\mathcal{D}_X[\log Y]$ -module, and $t : E|Y \rightarrow F$ and $s : F \rightarrow E|Y$ are $\mathcal{D}_X[\log Y]$ -linear maps with $st = \theta_Y$ on $E|Y$, and $ts = \theta_Y$ on F , we will directly associate the following \mathcal{D} -module M on X . (This was indirectly described in [N-S]).

Let $M_0 = E$, and let $E \oplus_s F$ denote the subsheaf of $E \oplus F$ consisting of sections (e, f) such that $(e|Y) = s(f)$. Let $\mathcal{O}_X(Y)$ be the line bundle on X defined by the

divisor Y as usual, and let $M_1 = \mathcal{O}_X(Y) \otimes (E \oplus_s F)$. Let $M_0 \hookrightarrow M_1$ be the inclusion defined by sending a local section e of $M_0 = E$ to the local section $(1/x) \otimes (xe, 0)$, where x is a local generator for the ideal of Y in X (this can be readily seen to be independent of the choice of x).

We make $E \oplus_s F$ a $\mathcal{D}_X[\log Y]$ -module by putting for any local section ξ of $T_X[\log Y]$ and (e, f) of $E \oplus_s F$,

$$\xi(e, f) = (\xi(e), t(e|Y) + \xi(f))$$

The right hand side may again be checked to be in $E \oplus_s F$, using the relation $st = \theta_Y$ on $E|Y$. As $\mathcal{O}_X(Y)$ is naturally a $\mathcal{D}_X[\log Y]$ -module, this now gives the structure of a (left) $\mathcal{D}_X[\log Y]$ -module on the tensor product $M_1 = \mathcal{O}_X(Y) \otimes_{\mathcal{O}_X} (E \oplus_s F)$. Moreover, the inclusion $M_0 \hookrightarrow M_1$ defined above is $\mathcal{D}_X[\log Y]$ -linear.

We now define a connection $\nabla : M_0 \rightarrow \Omega_X^1 \otimes M_1$ by putting, for any local sections η of T_X and e of M_0 ,

$$\eta(e) = (1/x) \otimes ((x\eta)(e), \eta(x)t(e|Y))$$

where x is any local generator of the ideal of Y . The right hand side makes sense because $x\eta$ is a section of $T_X[\log Y]$, and so $(x\eta)(e)$ is defined by the logarithmic connection on E . It can be checked that the above formula is independent of the choice of x , is \mathcal{O}_X -linear in the variable η , and the resulting map $\nabla : M_0 \rightarrow \Omega_X^1 \otimes M_1$ satisfies the Leibniz rule. Moreover, the following diagram commutes, where the maps $M_i \rightarrow \Omega_X^1[\log Y] \otimes M_i$ (for $i = 0$ and for $i = 1$) are given by the $\mathcal{D}_X[\log Y]$ -module structure on M_i .

$$\begin{array}{ccc} M_0 & \rightarrow & \Omega_X^1 \otimes M_1 \\ \downarrow & & \downarrow \\ M_1 & \rightarrow & \Omega_X^1[\log Y] \otimes M_1 \end{array}$$

Now let M be the \mathcal{D}_X -module which is the quotient of $\mathcal{D}_X \otimes_{\mathcal{D}_X[\log Y]} M_1$ by the submodule generated by elements of the type $\eta \otimes e - 1 \otimes \eta(e)$ where e is a local section of $E = M_0 \subset M_1$ and η is a local section of T_X . Then M is the \mathcal{D}_X -module that we associate to the pre- \mathcal{D} -module (E, F, t, s) .

Relation with V -filtration

We now assume that the generalized eigenvalues of θ_Y on $E|Y$ do not differ by nonzero integers (it is actually enough to assume this along each connected component of Y , but for simplicity we will assume that Y is connected). Let μ the only possible integral eigenvalue (when there are more components in Y , there can be a possibly different μ along each component). Under this assumption, We now construct a V -filtration on M along the divisor Y . Put $V^\mu(M)$ to be the image of M_0 and $V^{\mu+1}(M)$ to be the image of M_1 in M . For $k \geq 1$ put $V^{\mu-k}(M)$ to be the image of $I_Y^k M_0 \subset M_0$ and $V^{\mu+k}(M)$ to be the image of $\mathcal{O}_X((k-1)Y) \otimes M_1$ in M . Then by definition each $V^k M$ for $k \in \mathbb{Z}$ is an \mathcal{O}_X -coherent $\mathcal{D}_X[\log Y]$ -module, with $\eta(V^k(M)) \subset V^{k+1}(M)$ and $I_Y V^{k+1}(M) \subset V^k(M)$ for all k .

Conversely, let M be a regular holonomic \mathcal{D} -module on X with $\text{car}(M) \subset N^*(Y)$. Let $V^k(M)$ be a V -filtration with μ the only integer in the fundamental domain

chosen for the exponential map $\mathcal{C} \rightarrow \mathcal{C}^*$). Then put $E = V^\mu(M)$, $F = N_{Y,X}^* \otimes (V^{\mu+1}(M)/V^\mu(M))$, $t : (E|Y) \rightarrow F$ is defined by putting

$$t(e|Y) = x \otimes (\partial/\partial x)e$$

where x is a local generator of I_Y (which is independent of the choice of x), and $s : F \rightarrow E|Y$ defined by simply the multiplication $I_Y \times V^{\mu+1}M \rightarrow V^\mu M$ (note for this that $N_{Y,X}^* = I_Y/I_Y^2$). Then we get a pre- \mathcal{D} -module (E, F, t, s) . Given a choice of a fundamental domain for the exponential map, the above two processes are inverses of each other.

4.2 General case of a normal crossing Y

Let $\mathbf{E} = (E_i, t_i, s_i)$ be a pre- \mathcal{D} -module on (X, Y) , where we now allow Y to have normal crossings. Let the sheaf F on X be the subsheaf of $\oplus(p_i)_*E_i$ whose local sections consist of all tuples (e_i) where $e_i \in (p_i)_*E_i$, such that $s_i(e_i|Y_i^*) = e_{i+1}|Y_i^*$. This is a $\mathcal{D}_X[\log Y]$ -submodule of $\oplus(p_i)_*E_i$ as may be seen. Let $G = \mathcal{O}_X(Y) \otimes F$. As $\mathcal{O}_X(Y)$ is naturally a $\mathcal{D}_X[\log Y]$ -module, G has a natural structure of a left $\mathcal{D}_X[\log Y]$ -module. The \mathcal{D}_X -module M that we are going to associate to the pre- \mathcal{D} -module \mathbf{E} is going to be a particular quotient of the \mathcal{D}_X -module $\mathcal{D}_X \otimes_{\mathcal{D}_X[\log Y]} G$.

The case of a polydisk

Let x_1, \dots, x_d be coordinates on the polydisk, let $0 \leq r \leq d$ and let Y be defined by the polynomial $P(x) = \prod_{1 \leq k \leq r} x_k$ (in particular $P = 1$ and therefore Y is empty if $r = 0$). A pre- \mathcal{D} -module \mathbf{E} on X has been described already in section 3.1 above.

For each k such that $1 \leq k \leq r$, we define a subsheaf $F_k \subset F$ (where F is the submodule of $\oplus(p_A)_*E_A$ defined above using the s_A) as follows.

$$F_k = x_k F \subset F$$

This defines an \mathcal{O}_X -coherent module, which is in fact a $\mathcal{D}_X[\log Y]$ -submodule of F as may be checked. We now put $G_k = \mathcal{O}_X(Y) \otimes F_k \subset \mathcal{O}_X(Y) \otimes F = G$. This is therefore a $\mathcal{D}_X[\log Y]$ -submodule of G . We now define the operator

$$\partial/\partial x_k : G_k \rightarrow G$$

as follows. If $e = (1/P) \otimes (e_A)$ is a section of G_k , we put $\partial_k(e) = (1/P) \otimes (f_B)$ where

$$f_B = (x_k \partial/\partial x_k)(e_B) + t_B^k(e_{B-k})$$

where by convention $t_B^k = 0$ whenever k does not belong to B .

Let K be the \mathcal{D}_X -submodule of $\mathcal{D}_X \otimes_{\mathcal{D}_X[\log Y]} G$ generated by elements of the form

$$\partial_k \otimes e - 1 \otimes \partial_k(e)$$

where $e \in G_k$.

Lemma 4.1 *The submodule K is independent of the choice of local coordinates, and restricts to the corresponding submodule on a smaller polydisk.*

Proof This is a local coordinate calculation, using the chain rule of partial differentiation under a change of coordinates. We omit the details.

Back to the global case

By the above lemma applied to an open covering of X by polydisks, we get a globally defined submodule K of $\mathcal{D}_X \otimes G$ (where the later is already defined globally). We now put M to be the quotient of $\mathcal{D}_X \otimes G$ by K . This is our desired \mathcal{D}_X -module.

4.3 The V filtration for a polydisk

Let X be a polydisk with coordinates x_1, \dots, x_d and Y be defined by $\prod_{k \in \Lambda} x_k$ for some initial subset $\Lambda = \{1, \dots, r\} \subset \{1, \dots, d\}$.

Let M be a regular holonomic \mathcal{D}_X -module on (X, Y) . We now describe a pre- \mathcal{D} -module \mathbf{E} such that M is associated to it. This is done via a V -filtration of M . We assume that we have fixed some fundamental domain Σ for $\exp : \mathcal{C} \rightarrow \mathcal{C}^* : z \mapsto e^{2\pi iz}$, in order to define the V -filtration. Let $\mu \in \Sigma$ be the only integer.

Note that the filtration will be by sub $\mathcal{D}_X[\log Y]$ -modules $V^{(n_1, \dots, n_r)}$ where $n_k \in \mathbb{Z}$, partially ordered as follows. If $(n_1, \dots, n_r) \leq (m_1, \dots, m_r)$ (which means $n_k \leq m_k$ for each $k \in \Lambda$), then $V^{(n_1, \dots, n_r)} \subset V^{(m_1, \dots, m_r)}$. In particular, there is a portion of the filtration indexed by the power set of Λ , where for any subset A of Λ , we put $V^A = V^{(n_1, \dots, n_r)}$ where $n_k = \mu + 1$ if $k \in A$ and $n_k = \mu$ otherwise. By this convention, note that $V^\phi = V^{(\mu, \dots, \mu)}$ for the empty set ϕ , and $V^\Lambda = V^{(\mu+1, \dots, \mu+1)}$.

Note that if $k \in A$ then multiplication by x_k defines a map

$$x_k : V^A M \rightarrow V^{A-k} M$$

and differentiation by $\partial/\partial x_k$ defines a map

$$\partial_k : V^{A-k} M \rightarrow V^A M$$

Let for any nonempty A ,

$$gr^A M = \frac{V^A M}{\sum_B V^B M}$$

where B varies over all proper subsets of A . All eigenvalues of $x_k \partial/\partial x_k$ on $gr^A M$ lie in Σ .

The above defines maps

$$x_k : gr^A M \rightarrow gr^{A-k} M$$

and

$$\partial_k : gr^{A-k} M \rightarrow gr^A M$$

Then we can define a pre- \mathcal{D} -module $\mathbf{E} = (E_A, t_A^k, s_A^k)$ as follows: The $\mathcal{D}_X[\log Y]$ -modules E_A are defined by

$$E_A = \mathcal{O}_X \left(- \sum_{k \in A} Y^k \right) \otimes_{\mathcal{O}_X} gr^A M$$

where $Y^k \subset X$ is the divisor defined by $x_k = 0$. Let $k \in A$. To define t_A^k and s_A^k , note that the differentiation $\partial_k : gr^{A-k}M \rightarrow gr^A M$ induces a map $\mathcal{O}_X(Y^k) \otimes gr^{A-k}M \rightarrow gr^A M$, as ∂_k can be canonically identified with the section $1/x_k$ of $\mathcal{O}_X(Y^k)$. Tensoring this by the identity map on $\mathcal{O}_X(-\sum_{\ell \in A} Y^\ell)$, and restricting to X_A , we get $t_A^k : (E_{A-k}|X_A) \rightarrow E_A$. Also, the map $x_k : gr^A M \rightarrow gr^{A-k}M$ defines a homomorphism $\mathcal{O}_X(-Y^k) \otimes gr^A M \rightarrow gr^{A-k}M$ which after tensoring by the identity map on $\mathcal{O}_X(-\sum_{\ell \in A-k} Y^\ell)$ gives $s_A^k : E_A \rightarrow (E_{A-k}|X_A)$.

Then it can be checked that we indeed get a pre- \mathcal{D} -module \mathbf{E} , such that it has good residual eigenvalues, lying in Σ , such that M is the \mathcal{D} -module associated to it. It can again be checked that the above procedure over a polydisk is coordinate independent, so glues up to give such a correspondence globally over (X, Y) .

Remark 4.2 If we began with the \mathcal{D} -module M associated to a pre- \mathcal{D} -module \mathbf{E} with good residual eigenvalues lying in Σ , then the above will give back \mathbf{E} , as then $V^\phi M = E_d$ and for $A \subset \Lambda$ we will get

$$V^A M = \left(\prod_{\ell \in \Lambda - A} x_\ell \right) G \subset G$$

where $G = \mathcal{O}_X(Y) \otimes F$ where $F \subset \oplus_B E_B$ as above.

Remark 4.3 (The [G-G-M]-hypercube for a polydisk) : Let W_A be the fiber of E_A at the origin of the polydisk. For $k \in A$ let $t_A^k : W_{A-k} \rightarrow W_A$ again denote the restriction of $t_A^k : E_{A-k} \rightarrow E_A$ to the fiber at origin, and let $v_A^k : W_A \rightarrow W_{A-k}$ be defined by the following formula.

$$v_A^k = \frac{\exp(2\pi i \theta_k) - 1}{\theta_k} s_A^k$$

Then (W_A, t_A^k, v_A^k) is the hypercube description of M as given by Galligo, Granger, Maisonobe in [G-G-M].

5 Moduli for semistable pre- \mathcal{D} -modules

In this section we define the concepts of semistability and stability for pre- \mathcal{D} -modules, and construct a coarse moduli. The main result is Theorem 5.37 below.

5.1 Preliminaries on Λ -modules

Simpson has introduced a notion of modules over rings of differential operators which we first recall (see section 2 of [S]).

Let X be a complex scheme, of finite type over \mathcal{C} , and let Λ be a sheaf of \mathcal{O}_X -algebras (not necessarily non-commutative), together with a filtration by subsheaves of abelian groups $\Lambda_0 \subset \Lambda_1 \subset \dots \subset \Lambda$ which satisfies the following properties.

- (1) $\Lambda = \cup \Lambda_i$ and $\Lambda_i \cdot \Lambda_j \subset \Lambda_{i+j}$. (In particular, Λ_0 is a subring, and each Λ_i is a Λ_0 -bimodule.)
- (2) The image of the homomorphism $\mathcal{O}_X \rightarrow \Lambda$ is equal to Λ_0 . (In particular, each Λ_i is an \mathcal{O}_X -bimodule).
- (3) Under the composite map $\mathcal{C}_X \hookrightarrow \mathcal{O}_X \rightarrow \Lambda$, the image of the constant sheaf \mathcal{C}_X is contained in the center of Λ .
- (4) The left and right \mathcal{O}_X -module structures on the i th graded piece $Gr_i(\Lambda) = \Lambda_i/\Lambda_{i-1}$ are equal.
- (5) The sheaves of \mathcal{O}_X -modules $Gr_i(\Lambda)$ are coherent. (In particular, each Λ_i is bi-coherent as a bi-module over \mathcal{O}_X , and their union Λ is bi-quasicoherent.)
- (6) The associated graded \mathcal{O}_X -algebra $Gr(\Lambda)$ is generated (as an algebra) by the piece $Gr_1(\Lambda)$.
- (7) ('Split almost polynomial' condition) : The homomorphism $\mathcal{O}_X \rightarrow \Lambda_0$ is an isomorphism, the \mathcal{O}_X -module $Gr_1(\Lambda)$ is locally free, the graded ring $Gr(\Lambda)$ is the symmetric algebra over $Gr_1(\Lambda)$, and we are given a left- \mathcal{O}_X -linear splitting $\xi : Gr_1(\Lambda) \rightarrow \Lambda_1$ for the left- \mathcal{O}_X -linear projection $\Lambda_1 \rightarrow Gr_1(\Lambda)$.

Remark 5.1 The condition (7) is not necessary for the moduli construction, but allows a simple description (see lemma 2.13 of [S]) of the structure of Λ -module, just as the structure of a \mathcal{D}_X -module on an \mathcal{O}_X -module can be described in terms of the action of T_X .

The pair (Λ, ξ) is called a **split almost polynomial algebra of differential operators** on X . Simpson also defines this in the relative situation $X \rightarrow S$, and treats basic concepts such as base change, which we will assume.

A **Λ -module** will always mean a left Λ -module unless otherwise indicated. For any complex scheme T , a **family** E_T of Λ -modules parametrized by T has an obvious definition (see [S]).

The following basic lemma is necessary to parametrize families of pre- \mathcal{D} -modules.

Lemma 5.2 (Coherence and representability of integrable direct images)
Let Λ be an algebra of differential operators on X . Let E_T and F_T be a families of Λ -modules on X parametrised by a scheme T (which is locally noetherian and of finite type over the field of complex numbers).

- (i) *The sheaf $(\pi_T)_* \underline{Hom}_{\Lambda_T}(E_T, F_T)$ is a coherent sheaf of \mathcal{O}_T modules.*
- (ii) *Consider the contravariant functor from schemes over T to the category of abelian groups, which associates to $T' \rightarrow T$ the abelian group $Hom_{\Lambda_{T'}}(E_{T'}, F_{T'})$ where $E_{T'}$ and $F_{T'}$ are the pullbacks under $X \times T' \rightarrow X \times T$. Then there exists a linear scheme $V \rightarrow T$ which represents this functor.*

Proof The above lemma is a stronger version of lemma 2.7 in [N]. The first step in the proof is the following lemma, which is an application of the Grothendieck complex of semi-continuity theory.

Lemma 5.3 (EGA III 7.7.8 and 7.7.9) *Let $Z \rightarrow T$ be a projective morphism where T is noetherian, and let \mathcal{F} and \mathcal{G} be coherent sheaves on Z such that \mathcal{G} is flat over T . Consider the contra functor φ from the category of schemes over T to the category of abelian groups, which associates to any $T' \rightarrow T$ the abelian group of all $\mathcal{O}_{Z \times_T T'}$ -linear homomorphisms from $\mathcal{F}_{T'}$ to $\mathcal{G}_{T'}$. Then φ is representable by a linear scheme W over T .*

Now we prove lemma 5.2. Forgetting the structure of Λ -modules on E_T and F_T and treating them just as \mathcal{O} -modules, let $W \rightarrow T$ be the linear scheme given by the above lemma 5.3. Then W parametrizes a universal family of $\mathcal{O}_{X \times W}$ -linear morphism $u : E_W \rightarrow F_W$. The condition of Λ_W -linearity on u defines a closed linear subscheme V of W . By its construction, for any base change $T' \rightarrow T$, we have canonical isomorphism $Mor_T(T, V) = Hom_{\Lambda_{T'}}(E_{T'}, F_{T'})$ which proves the lemma 5.2.

5.2 Families of pre- \mathcal{D} -modules

We now come back to (X, Y) as before, and our earlier notation. Recall that X_i is the normalization of S_i , which is i -dimensional, nonsingular if non-empty. Let $X_{i,a}$, as a varies over the indexing set $\pi_0(X_i)$, be the connected components of X_i . We denote by $\mathcal{D}_{i,a}$ the restriction of \mathcal{D}_i to the component $X_{i,a}$ of X_i .

Lemma 5.4 *Each $\mathcal{D}_{i,a}$ satisfies the above properties (1) to (7) (where in (7), we take $\xi : (\mathcal{D}_{i,a})_1 \rightarrow \mathcal{O}_{X_{i,a}}$ to be induced by the splitting $(\mathcal{D}_V)_1 = \mathcal{O}_V \oplus T_V$ for any nonsingular variety V), so is a split almost polynomial algebra of differential operators on $X_{i,a}$.*

Definition 5.5 For any complex scheme T , a **family**

$$\mathbf{E}_T = (E_{i,T}, s_{i,T}, t_{i,T})$$

of pre- \mathcal{D} -modules parametrized by T is defined as follows. The $E_{i,T}$ are vector bundles on $X_i \times T$ with structure of $\mathcal{D}_{i,T}$ modules, where $\mathcal{D}_{i,T}$ are the relative versions of the \mathcal{D}_i . The morphisms $s_{i,T}$ and $t_{i,T}$ are the relative versions of the morphisms s_i and t_i in the definition of a pre- \mathcal{D} -module.

Given any morphism $f : T' \rightarrow T$ of complex schemes and a family \mathbf{E}_T of pre- \mathcal{D} -modules parametrized by T , the pullback family $f^*\mathbf{E}_T$ on T' has again the obvious definition. This therefore defines a fibered category over the base category of complex schemes. When we put the restriction that all morphisms in each fiber category be isomorphisms, we get a fibered category \mathcal{PD} of groupoids over $Schemes_{\mathbb{C}}$.

Proposition 5.6 *The fibered category of groupoid \mathcal{PD} of pre- \mathcal{D} -modules on (X, Y) is an algebraic stack in the sense of Artin.*

Proof (Sketch) We refer the reader to the notes of Laumon [L] for basic concepts and constructions involving algebraic stacks. As fpqc descent and fpqc effective descent is obviously satisfied by \mathcal{PD} , it follows that \mathcal{PD} is a stack. It remains to show that this stack is algebraic in the sense of Artin. For this, first note that if Λ is a split almost polynomial algebra of differential operators, then \mathcal{O} -coherent Λ -modules form an algebraic stack, for the forgetful functor (1-morphism of stacks) from Λ -modules to \mathcal{O} -modules is representable (as follows from the alternative description of the structure of a Λ -module given in lemma 2.13 of [S]), and coherent \mathcal{O} -modules form an algebraic stack in the sense of Artin. Now from the lemma 5.2 on coherence and representability of the integrable direct image functor it can be seen that the forgetful functor (1-morphism of stacks) from pre- \mathcal{D} -modules to the product of the stacks of its underlying \mathcal{D}_i -modules is representable (in fact, the details of this occur below in our construction of a local universal family for pre- \mathcal{D} -modules). Hence the result follows.

5.3 Filtrations and 1-parameter deformations

We first recall some standard deformation theory, for convenience of reference. Recall that in this paper a vector bundle means a locally free module (but not necessarily of constant rank), and a subbundle of a vector bundle will mean a locally free submodule such that the quotient is also locally free. Let E be a vector bundle on a complex scheme X together with an exhaustive increasing filtration E_p by vector subbundles, indexed by \mathbb{Z} . (The phrase **exhaustive** means that $E_p = 0$ for $p \ll 0$ and $E_p = E$ for $p \gg 0$.) Let $A^1 = \text{Spec } \mathcal{O}[\tau]$ be the affine line, and let $U = \text{Spec } \mathcal{O}[\tau, \tau^{-1}]$ be the complement of the origin with inclusion $j : U \hookrightarrow A^1$. Let $\pi_X : X \times A^1 \rightarrow X$ be the projection. Consider the quasi-coherent sheaf $(1_X \times j)_*(\pi_X^* E|_{X \times U})$ on $X \times A^1$, which is usually denoted by $E \otimes \mathcal{O}[\tau, \tau^{-1}]$. This has a subsheaf \overline{E} generated by all local sections of the type $\tau^p v_p$ where v_p is a local section of $\pi_X^* E_p$. It is common to write

$$\overline{E} = \sum_{p \in \mathbb{Z}} E_p \tau^p \subset E \otimes \mathcal{O}[\tau, \tau^{-1}]$$

Then we have the following basic fact:

Remark 5.7 It can be seen that \overline{E} is an $\mathcal{O}_{X \times A^1}$ -coherent submodule, which is in fact locally free, and $\overline{E}|_{X \times U}$ is just $\pi_X^* E$ where $\pi_X : X \times U \rightarrow X$ is the projection. On the other hand, the specialization of \overline{E} at $\tau = 0$ is canonically isomorphic to the graded object $E' = \bigoplus (E_p/E_{p-1})$ associated with E . So, the 1-parameter family E_τ is a deformation of E to its graded object E' .

Now let F vector bundle on X together with filtration F_p , and let $f : E \rightarrow F$ be an \mathcal{O}_X -homomorphism. We have an induced $\mathcal{O}_{X \times A^1}$ -homomorphism

$$\pi_X^* f : E \otimes \mathcal{O}[\tau, \tau^{-1}] \rightarrow F \otimes \mathcal{O}[\tau, \tau^{-1}]$$

Then we have the following basic fact :

Remark 5.8 The homomorphism $f : E \rightarrow F$ is filtered, that is, f maps each E_p into F_p , if and only if the above homomorphism $\pi^* f$ carries \overline{E} into \overline{F} . In that case, the induced map at $\tau = 0$ is the associated graded map $gr(f) : E' \rightarrow F'$.

As a consequence, we get the following :

Remark 5.9 Let E and F be vector bundles with exhaustive filtrations E_p and F_p indexed by \mathbb{Z} , and let \overline{E} and \overline{F} be the corresponding deformations parametrized by A^1 . Let $f : E \rightarrow F$ be an \mathcal{O}_X -homomorphism, and let $g : \overline{E} \rightarrow \overline{F}$ be an $\mathcal{O}_{X \times A^1}$ -homomorphism. Suppose that the restriction of g to $X \times U$ (where $U = A^1 - \{0\}$) is equal to the pullback $\pi_X^* f$ of f under $\pi_X : X \times U \rightarrow X$. Then f preserves the filtrations, and $g_0 : E' \rightarrow F'$ is the associated graded homomorphism $f' : E' \rightarrow F'$ (where E' and F' are the graded objects)

Definition 5.10 A **sub pre- \mathcal{D} -module** F of a pre- \mathcal{D} -module $E = (E_i, s_i, t_i)$ consists of the following data : For each i we are given an \mathcal{O}_{X_i} -coherent \mathcal{D}_i -submodule $F_i \subset E_i$ such that for each i ,

(i) the \mathcal{O}_{X_i} -modules F_i and E_i/F_i are locally free (but not necessarily of constant ranks over X_i). In other words, for each (i, a) , we are given a vector subbundle $F_{i,a} \subset E_{i,a}$ which is a sub $\mathcal{D}_{i,a}$ -module.

(ii) the maps s_i and t_i preserve F_i , that is s_i maps $F_i|Y_i^*$ into $F_{i+1}|Y_i^*$ and t_i maps $F_{i+1}|Y_i^*$ into $F_i|Y_i^*$.

Note that consequently, $F = (F_i, s_i^F, t_i^F)$ is also a pre- \mathcal{D} module, where s^F and t^F denote the restrictions of s and t . Also, the quotients E_i/F_i naturally form a pre- \mathcal{D} -module E/F which we call as the corresponding **quotient pre- \mathcal{D} -module**.

Remark 5.11 Given a pre- \mathcal{D} -module \mathbf{E} and a collection \mathbf{F} of subbundles $F_i \subset E_i$ which are \mathcal{D}_i -submodules, the job of checking whether these subbundles are preserved by the s_i and t_i is made easier by the following: it is enough to check this in the fiber of a point of each of the connected components of $Y_i^* - p^{-1}(S_{i-1})$ where $p : Y_i^* \rightarrow S_i$ is the projection. This is because the s_i and t_i are ‘integrable’ in a suitable sense, and if an integrable section σ of a vector bundle with an integrable connection has a value $\sigma(P)$ in the fibre at P of a subbundle preserved by the connection, then it is a section of this subbundle.

Definition 5.12 An **exhaustive filtration on a pre- \mathcal{D} -module** $\mathbf{E} = (E_i, s_j, t_j)$ consists of an increasing sequence \mathbf{E}_p of sub pre- \mathcal{D} -modules of \mathbf{E} indexed by \mathbb{Z} such that $\mathbf{E}_p = 0$ for $p \ll 0$ and $\mathbf{E}_p = \mathbf{E}$ for $p \gg 0$.) A filtration is **nontrivial** if \mathbf{E}_p is a nonzero proper sub pre- \mathcal{D} -module of \mathbf{E} for some p .

For a filtered pre- \mathcal{D} -module, each step $\mathbf{E}_p = ((E_i)_p, (s_i)_p, (t_i)_p)$ of the filtration, as well as the associated graded object $\mathbf{E}' = (E'_i, s'_i, t'_i)$ are pre- \mathcal{D} -modules.

Applying the remark 5.8 to filtrations of pre- \mathcal{D} -modules we get the following.

Remark 5.13 Let \mathbf{E} be a pre- \mathcal{D} -module, together with an exhaustive filtration \mathbf{E}_p . Then there exists a family $(\mathbf{E}_\tau)_{\tau \in A^1}$ of pre- \mathcal{D} -modules parametrized by the affine line $A^1 = \text{Spec } \mathcal{C}[\tau]$, for which the specialization at $\tau = 0$ is the graded object \mathbf{E}' while the family over $\tau_0 \neq 0$ is the constant family made from the original pre- \mathcal{D} -module \mathbf{E} defined as follows: put $\mathbf{E}_i = \sum_{p \in \mathbb{Z}} (E_i)_p \tau^p \subset E_i \otimes \mathcal{C}[\tau, \tau^{-1}]$.

5.4 Quot scheme and group action on total family

Let X be a projective scheme over a base S and \mathcal{V} a coherent \mathcal{O}_X -module. Let

$$G = \text{Aut}_{\mathcal{V}} : \text{Schemes}/S \rightarrow \text{Groups}$$

be the contrafunctor which associates to any $T \rightarrow S$ the group of all \mathcal{O}_{X_T} -linear automorphisms of the pullback \mathcal{V}_T of \mathcal{V} under $X_T = X \times_S T \rightarrow X$. Note that then G is in fact an affine group scheme over S , but this will not be relevant to us.

Let $Q = \text{Quot}_{\mathcal{V}/X/S}$ be the relative quot scheme of quotients of \mathcal{V} on fibers of $X \rightarrow S$. A T -valued point $y : T \rightarrow Q$ is represented by a surjective \mathcal{O}_{X_T} -linear homomorphism $q : \mathcal{V}_T \rightarrow F$ where F is a coherent sheaf on X_T which is flat over T . Two such surjections $q_1 : \mathcal{V}_T \rightarrow F_1$ and $q_2 : \mathcal{V}_T \rightarrow F_2$ represent the same point $y \in Q(T)$ if and only if either of the following two equivalent conditions is satisfied: (i) there exists an isomorphism $f : F_1 \rightarrow F_2$ such that $q_2 = f \circ q_1$, or (ii) the kernels of q_1 and q_2 are identical as a subsheaf of \mathcal{V}_T . Therefore, a canonical way to represent the point $y \in Q(T)$ is the quotient $\mathcal{V}_T \rightarrow \mathcal{V}_T/K_y$ where $K_y = \ker(q)$ depends only on y .

A natural group action $Q \times G \rightarrow Q$ over S is defined as follows: in terms of valued points $y \in Q(T)$ represented by $q : \mathcal{V}_T \rightarrow F$, and $(g : \mathcal{V}_T \rightarrow \mathcal{V}_T) \in G(T)$, the point $yg \in Q(T)$ is represented by $q \circ g : \mathcal{V}_T \rightarrow F$. In other words, if y is canonically represented by $\mathcal{V}_T \rightarrow \mathcal{V}_T/K_y$ then yg is canonically represented by $\mathcal{V}_T \rightarrow \mathcal{V}_T/g^{-1}(K_y)$. This means $K_{yg} = g^{-1}(K_y)$.

Let $q : \mathcal{V}_Q \rightarrow \mathcal{U}$ be the universal quotient family on X_Q . The action $Q \times G \rightarrow Q$ over S when pulled back under $X \rightarrow S$ gives an action $Q_X \times G_X \rightarrow Q_X$ over X (where $G_X = G \times_S X$ and $Q_X = Q \times_S X = X_Q$). The action $Q_X \times G_X \rightarrow Q_X$ has a natural lift to the sheaf \mathcal{U} on Q_X as follows. For $y \in Q(T)$ and $g \in G(T)$, the pull backs \mathcal{U}_y and \mathcal{U}_{yg} of the universal quotient sheaf under y and yg are canonically isomorphic to \mathcal{V}_T/K_y and $\mathcal{V}_T/g^{-1}(K_y)$ respectively, so $g : \mathcal{V}_T \rightarrow \mathcal{V}_T$ induces a canonical isomorphism $\varphi_g^y : \mathcal{U}_{yg} \rightarrow \mathcal{U}_y$. Hence we get an isomorphism $\varphi : g^*(\mathcal{U}) \rightarrow \mathcal{U}$ over X_Q . Since $\varphi_{gh}^y = \varphi_g^y \circ \varphi_h^{yg}$ for any $g, h \in G(T)$ and $t \in Q(T)$, we get

$$\varphi_{gh} = \varphi_g \circ g^*(\varphi_h)$$

Thus, φ is a ‘factor of automorphy’, and so defines the required lift.

Remark 5.14 By definition, if $y \in Q(T)$ is represented by the surjection $q : \mathcal{V}_T \rightarrow F$, then we get a canonical identification of F with $\mathcal{U}_y = \mathcal{V}_T/K_y$. For $g \in G(T)$ the point $yg \in Q(T)$ is represented by $q \circ g : \mathcal{V}_T \rightarrow F$, so we get another

canonical identification of F with $\mathcal{U}_{gy} = \mathcal{V}_T/g^{-1}(K_y)$. Under these identifications, the isomorphism $\varphi_g^y : \mathcal{U}_{yg} \rightarrow \mathcal{U}_y$ simply becomes the identity map $1_F : F \rightarrow F$. (This is so simple that it can sometimes cause confusion.) Hence if σ is a local section of \mathcal{U}_T represented by (q, s) , where $q : \mathcal{V}_T \rightarrow F$ and s is a local section of F , then the action of $g \in G(T)$ can be written as

$$(q, s) \cdot g = (q \circ g, s)$$

Remark 5.15 The central subgroup scheme $\mathbf{G}_m \subset G$ (where by definition $\lambda \in \mathbf{G}_m(T) = \Gamma(T, \mathcal{O}_T^\times)$) acts by scalar multiplication on \mathcal{V}_T acts trivially on Q but its action on the universal family \mathcal{U} is again by scalar multiplication so is non-trivial. In terms of the above notation have the equality

$$(q, s) \cdot \lambda = (q \circ \lambda, s) = (q, \lambda s)$$

Hence the induced action of $PG = G/\mathbf{G}_m$ on Q does not lift to \mathcal{U} , which is the basic reason why a Poincaré bundle does not in general exist in the kind of moduli problems we are interested in.

Remark 5.16 In the applications below, X will be in general a projective scheme over \mathcal{C} and the sheaf \mathcal{V} will be of the type $\mathcal{O}_X^{\oplus N} \otimes_{\mathcal{O}_X} \mathcal{W} = \mathcal{C}^N \otimes_{\mathcal{C}} \mathcal{W}$ where \mathcal{W} is some coherent sheaf over X . Then $GL(N)$ is naturally a subgroup scheme of $G = Aut_{\mathcal{V}}$, and we will only be interested in the resulting action of $GL(N)$.

5.5 Semistability and moduli for Λ -modules

We now recall the moduli construction of Simpson for Λ -modules (see section 2, 3 and 4 of [S] for details). Let X be projective, with ample line bundle $\mathcal{O}_X(1)$.

Let E be an \mathcal{O}_X -coherent Λ -module on X . Then Simpson defines E to be a **semistable Λ module** if

- (i) the \mathcal{O}_X -module E is pure dimensional, and
- (ii) for each non-zero \mathcal{O}_X -coherent Λ -submodule $F \subset E$, the inequality

$$\dim H^0(X, F(n))/\text{rank}(F) \leq \dim H^0(X, E(n))/\text{rank}(E)$$

holds for n sufficiently large (where $\text{rank}(F)$ for any coherent sheaf on $(X, \mathcal{O}_X(1))$ is by definition the leading coefficient of the Hilbert polynomial of F). If the Λ -module E is nonzero and moreover we can always have strict inequality in the above for $0 \neq F \neq E$, then E is called **stable**.

An **S-filtration** of a semistable Λ -module E is a filtration $0 = E_0 \subset E_1 \subset \dots \subset E_\ell = E$ by \mathcal{O}_X -coherent Λ -submodules, such that each graded piece E_i/E_{i-1} is a semistable Λ -module, with the same normalized Hilbert polynomial as that of E if non-zero (where ‘normalized Hilbert polynomial’ means Hilbert polynomial divided by its leading coefficient). It can be seen that an S-filtration on a non-zero E is maximal

(that is, can not be further refined) if and only if each graded piece E_i/E_{i-1} is stable. The associated graded object $\bigoplus_{1 \leq i \leq \ell} (E_i/E_{i-1})$ to a maximal S-filtration, after forgetting the gradation, is independent (upto isomorphism) of the choice of an S-filtration, and two non-zero semistable Λ -modules are called **S-equivalent** if they have S-filtrations with isomorphic graded objects (after forgetting the gradation). The zero module is defined to be S-equivalent to itself.

By a standard argument using Quot schemes (originally due to Narasimhan and Ramanathan), it can be seen that semistability is a Zariski open condition on the parameter scheme of any family of Λ -modules.

In order to construct a moduli for semi-stable Λ -modules, Simpson first shows that if we fix the Hilbert polynomial P , then all semi-stable Λ -modules whose Hilbert polynomial is P form a bounded set, and then shows the following:

Proposition 5.17 (Simpson [S]) : *Let $(X, \mathcal{O}_X(1))$ be a projective scheme, P a fixed Hilbert polynomial, and Λ a sheaf of differential operators on X . Then there exists a quasi-projective scheme C together with an action of $PGL(N)$ (for some large N), and a family E_C of semistable Λ -modules on X with Hilbert polynomial P parametrized by C such that*

(1) *the family E_C is a local universal family for semistable Λ -modules with Hilbert polynomial P ,*

(2) *two morphisms $f_1, f_2 : T \rightarrow C$ give isomorphic families $f_1^*(E_C)$ and $f_2^*(E_C)$ if and only if there exists a Zariski open cover $T' \rightarrow T$ (that is, T' is a disjoint union of finitely many open subsets of T whose union is T) and a T' -valued point $g : T' \rightarrow PGL(N)$ of $PGL(N)$ which carries $f_1|_{T'}$ to $f_2|_{T'}$.*

(3) *a good quotient ('good' in the technical sense of GIT) $C \rightarrow C//PGL(N)$ exists, and is (as a consequence of (1) and (2)) the coarse moduli scheme for S-equivalence classes of semistable Λ -modules with Hilbert polynomial P .*

(4) **Limit points of orbits** : *Let $\lambda : GL(1) \rightarrow GL(N)$ be a 1-parameter subgroup, and let $q \in C$ have limit $q_0 \in C$ under λ , that is,*

$$q_0 = \lim_{\tau \rightarrow 0} q \cdot \lambda(\tau)$$

Let E_τ be the pullback of E_C to $X \times A^1$ under the resulting morphism $\lambda : A^1 \rightarrow C$ on the affine line A^1 . Then there exists an exhaustive filtration F_p of E (where E is the module associated to $\tau = 1$) by Λ -submodules such that each F_p/F_{p-1} is semistable, and when non-zero it has the same reduced Hilbert polynomial as E , and the family E_τ is isomorphic to the family $\overline{E} = \sum_{p \in \mathbb{Z}} F_p \tau^p \subset E \otimes \mathcal{O}[\tau, \tau^{-1}]$ occurring in remark 5.7 above. In particular, the limit E_0 is the graded object corresponding to F_p , and so the closed points of the quotient are in bijection with the set of S-equivalence classes.

Remark 5.18 The following feature of the family E_C and the action of $GL(N)$ will be very important: C is a certain locally closed subscheme of the Quot scheme

$Quot_{\mathcal{O}^N \otimes \mathcal{W}/X/\mathcal{O}}$ of quotients

$$q : \mathcal{O}^N \otimes \mathcal{W} \rightarrow E$$

where W is some fixed coherent sheaf on X , such that C is invariant under $GL(N)$, and E_C is the restriction to C of the universal family \mathcal{U} on the Quot scheme, The action of $GL(N)$ on the Quot scheme and its lift to \mathcal{U} is as explained in remark 5.16 above.

5.6 Strong local freeness for semistable Λ -modules

Definition 5.19 We will say that a semistable Λ -module E on X is **strongly locally free** if for every S-filtration $0 = E_0 \subset E_1 \subset \dots \subset E_\ell = E$, the associated graded object $\bigoplus_{1 \leq i \leq \ell} (E_i/E_{i-1})$ is a locally free \mathcal{O}_X -module. (In particular, the zero module is strongly locally free.)

Remarks 5.20 (1) A nonzero semistable Λ -module E is strongly locally free if and only if there exists some maximal S-filtration such that the graded pieces are locally free \mathcal{O}_X -modules.

(2) A strongly locally free semistable Λ -module is necessarily locally free. However, a locally free semistable Λ -module is not necessarily strongly locally free.

Proposition 5.21 (1) Let E_C be the family of semistable Λ -modules parametrized by C . Then the condition that the associated graded object to any S-filtration F_k of E_q should be locally free defines a $GL(N)$ -invariant open subset C^o of C which is closed under limits of orbits, and hence has a good quotient under $GL(N)$, which is an open subscheme of the moduli of semistable Λ -modules.

(2) For any family E_T of semistable Λ -modules parametrized by a scheme T , the condition that E_t is strongly locally free is a Zariski open condition on T .

Proof Let $U \subset C$ be the open subscheme defined by the condition that E_q is locally free for $q \in U$ (this is indeed open as a consequence of the flatness of E_C over C). Note that U is $G = GL(N)$ -invariant. Let $\pi : C \rightarrow C//G$ be the good quotient. Then as $F = C - U$ is a G -invariant closed subset, $\pi(F)$ is closed in $C//G$ by properties of good quotient. Let $C^o = \pi^{-1}(C//G - \pi(F))$. Then C^o is the desired open subscheme of C , which proves the statement (1).

It follows from the relation between closure of orbits and S-filtrations that points of $C//G$ correspond to S-equivalence classes. Therefore the statement (2) now follows from (1) by the universal property of the moduli $C//G$.

5.7 Semistable pre- \mathcal{D} -modules — definition

We now come back to (X, Y) as before, and our earlier notation. We choose an ample line bundle on each $X_{i,a}$, and fix the resulting Hilbert polynomials $p_{i,a}(n)$ of the sheaf $E_{i,a}$ on $X_{i,a}$. By lemma 5.4 the theory of semistability and moduli for Λ -modules can be applied to $\mathcal{D}_{i,a}$ -modules.

Definition 5.22 We say that a pre- \mathcal{D} -module is **semistable** if the following two conditions are satisfied.

- (i) Each of the $\mathcal{D}_{i,a}$ -modules $E_{i,a}$ is semistable in the sense of Simpson.
- (ii) Each $E_{i,a}$ is strongly locally free, that is, the associated graded object to $E_{i,a}$ under any S-filtration is again a locally free $\mathcal{O}_{X_{i,a}}$ -module (see definition 5.19 and its subsequent remarks).

Remark 5.23 The notion of stability is treated later.

Proposition 5.24 *A vector bundle $E_{i,a}$ on $X_{i,a}$ with the structure of a semistable $\mathcal{D}_{i,a}$ -module is strongly locally free if it has good residual eigenvalues as defined in definition 3.4.*

As a consequence, a pre- \mathcal{D} -module E such that each of the $\mathcal{D}_{i,a}$ -modules $E_{i,a}$ is semistable and has good residual eigenvalues is a semistable pre- \mathcal{D} -module.

Proof Let Z be the polydisk in \mathbb{C}^n defined by $|z_i| < 1$, and $W \subset Z$ the divisor with normal crossings defined by $\prod_{k \leq r} z_k = 0$ (if $r = 0$ then W is empty). Let there be given a set theoretic section (fundamental domain) for the exponential map $\mathbb{C} \rightarrow \mathbb{C}^* : z \mapsto \exp(2\pi iz)$. Then the Deligne construction, which associates to a local system \mathcal{L} on $Z - W$ an integrable logarithmic connection $E(\mathcal{L})$ on (Z, W) with residual eigenvalues in the given fundamental domain, has the following property: if $\mathcal{K} \subset \mathcal{L}$ are two local systems, then $E(\mathcal{K})$ is a subbundle of $E(\mathcal{L})$. Now suppose F is an \mathcal{O}_Z -coherent $\mathcal{D}_Z[\log W]$ -submodule of $E(\mathcal{L})$. Let \mathcal{K} be the local system on $Z - W$ defined by F , and let $E(\mathcal{K})$ be its associated logarithmic connection given by Deligne's construction. Then $F|(Z - W) = E(\mathcal{K})|(Z - W)$. As $E(\mathcal{K})$ is a vector subbundle of the vector bundle $E(\mathcal{L})$, it follows that $E(\mathcal{K})$ is just the \mathcal{O}_Z -saturation of F . Hence if E has good residual eigenvalues and if $F \subset E$ is an \mathcal{O}_Z -coherent and saturated $\mathcal{D}_Z[\log W]$ -submodule, then F is a vector subbundle.

Now we apply this to each $E_{i,a}$ as follows. Let $E = E_{i,a}$ be $\mathcal{D}_{i,a}$ -semistable, and let $F \subset E$ be a step in an S-filtration of E . Hence we must have

$$\frac{p_F}{\text{rank}(F)} = \frac{p_E}{\text{rank}(E)}$$

where p_F denotes the Hilbert polynomial of a sheaf F . Note that the $\mathcal{O}_{X_{i,a}}$ -saturation F' of F in E is again an \mathcal{O} -coherent $\mathcal{D}_{i,a}$ -submodule, which restricts to F' on a dense open subset of $X_{i,a}$, in particular, F' has the same rank as F . Hence with respect to any ample line bundle on $X_{i,a}$, the normalized Hilbert polynomials of F and F' satisfy the relation

$$\frac{p_F}{\text{rank}(F)} \leq \frac{p_{F'}}{\text{rank}(F')}$$

with equality only if $F = F'$. As $E = E_{i,a}$ is semistable, we have

$$\frac{p_{F'}}{\text{rank}(F')} \leq \frac{p_E}{\text{rank}(E)}$$

and hence $F = F'$. Hence we can assume that any step F in an S-filtration of $E = E_{i,a}$ is \mathcal{O} -saturated.

Hence the proposition would follow if we show that if for each $i \geq m + 1$, $E_{i,a}$ has good residual eigenvalues under θ_{i-1} (see definition 3.4), then any \mathcal{O} -coherent and saturated $\mathcal{D}_{i,a}$ -submodule F is a vector subbundle. This is a purely local question on X in the euclidean topology, so we may assume that X is a polydisk with local coordinates z_i and Y is defined by a monomial in the z_i . Then $X_{i,a}$ is just some

coordinate i -plane Z contained in Y , and $W = X_{i,a} \cap S_{i-1}$ is a normal crossing divisor in the polydisk Z , defined by a monomial in the coordinates. Using the local coordinates z_k , we get a structure of $\mathcal{D}_Z[\log W]$ -module on the vector bundle $E = E_{i,a}$ on Z with good residual eigenvalues, and F becomes an \mathcal{O}_Z -coherent and saturated $\mathcal{D}_Z[\log W]$ -submodule. (Note that these $\mathcal{D}_Z[\log W]$ -structures very much depend on the choice of local coordinates z_k). As E has good residual eigenvalues by assumption, it follows from the first part of our argument (involving Deligne constructions over polydisks) that F is a vector subbundle of $E_{i,a}$. This proves the proposition.

Remark 5.25 If $E_{i,a}$ has good residual eigenvalues under θ_{i-1} , and if F is a vector subbundle which is a \mathcal{D}_i -submodule, then it can be seen that the associated graded \mathcal{D}_i -module $F \oplus (E_{i,a}/F)$ also has good residual eigenvalues under θ_{i-1} .

The following example shows that if E is a vector bundle with an integrable logarithmic connection, and $F \subset E$ is an \mathcal{O} -coherent sub connection such that E/F is torsion free, then it can still happen that E/F is not locally free if E has bad residual eigenvalues.

Example 5.26 (Due to Hélène Esnault) Let X be a polydisk in \mathbb{C}^2 , with divisor Y defined by $xy = 0$. Let $E = \mathcal{O}_X^{\oplus 2}$ with basis $v_1 = (1, 0)$ and $v_2 = (0, 1)$. On E we define an integrable logarithmic connection $\nabla : E \rightarrow \Omega_X^1[\log Y] \otimes E$ by putting $\nabla(v_1) = (dx/x) \otimes v_1$ and $\nabla(v_2) = (dy/y) \otimes v_2$ (this has curvature zero, as $(-\log x)v_1$ and $(-\log y)v_2$ form a flat basis of $E|_{X-Y}$). Note that both 0 and 1 are eigenvalues of the residue of (E, ∇) , along any branch of Y . Let $\mathfrak{m} \subset \mathcal{O}_X$ be the ideal sheaf generated by x and y . This is a sub $\mathcal{D}_X[\log Y]$ -module of \mathcal{O}_X , which is torsion free but not locally free. Now define a surjective homomorphism $\varphi : E \rightarrow \mathfrak{m}$ sending $v_1 \mapsto x$ and $v_2 \mapsto y$, which can be checked to be $\mathcal{D}_X[\log Y]$ -linear, and put $F = \ker(\varphi)$. Then F is an \mathcal{O}_X -saturated subconnection (which is in fact locally free), but $E/F = \mathfrak{m}$ is not locally free.

5.8 Semistable pre- \mathcal{D} -modules — local universal family

We now construct a local universal family for semistable pre- \mathcal{D} -modules with given Hilbert polynomials. Let $C_{i,a}$ be the scheme for $(\mathcal{D}_{i,a}, p_{i,a})$ given by the proposition 5.17, with the action of $PGL(p_{i,a}(N_{i,a}))$ as in the proposition 5.17. Let $C_{i,a}^o \subset C_{i,a}$ be the open subset where $E_{i,a}$ is strongly locally free. By proposition 5.21, $C_{i,a}^o$ is an open subset of $C_{i,a}$ which is $PGL(p_{i,a}(N_{i,a}))$ -invariant and admits a good quotient for the action of $PGL(p_{i,a}(N_{i,a}))$, which is an open subscheme of $C_{i,a} // PGL(p_{i,a}(N_{i,a}))$.

Let $C_i^o = \prod_a C_{i,a}^o$ and let $C = \prod_i C_i^o = \prod_{i,a} C_{i,a}^o$. Let $E_{i,a}$ again denote the pullback of $E_{i,a}$ to $X_{i,a} \times C$ under the projection $C \rightarrow C_{i,a}^o$. Let $E_i|Y_i^*$ and $E_{i+1}|Y_i^*$ be regarded as families of \mathcal{D}_i^* -modules parametrized by C . These are again flat over C as the E_i are locally free on X_i . Hence by applying lemma 5.2 to the pair $E_{i+1}|Y_i^*$ and $E_i|Y_i^*$ of \mathcal{D}_i^* -modules parametrized by C , we get linear schemes A_i and B_i over C which parametrize \mathcal{D}_i^* -linear maps t_i and s_i in either direction between the specializations

of these two families. Let $H_i \subset A_i \times_C B_i$ be the closed subscheme defined by the conditions on t_i and s_i imposed by the definition of a pre- \mathcal{D} -module (it can be seen that the conditions indeed define a closed subscheme H_i). Finally, let H be the fibered product over C of all the H_i . By its construction, H parametrizes a natural family of pre- \mathcal{D} -modules on (X, Y) .

Let $G_i = \prod_a G_{i,a}$ and let $G = \prod_i G_i = \prod_{i,a} G_{i,a}$. Note that this is a reductive group. We define an action of G on H as follows. Any point $\underline{q}_{i,a}$ of $C_{i,a}^o$ is represented by a quotient $q_{i,a} : \mathcal{O}_{X_{i,a}}(-N_{i,a})^{p_{i,a}(N_{i,a})} \rightarrow E_{i,a}$ (which satisfies some additional properties) and a point of H over a point $(\underline{q}_{i,a}) \in C = \prod_{i,a} C_{i,a}^o$ is given by the additional data $s_j : (E_j|Y_j^*) \rightarrow (E_{j+1}|Y_j^*)$ and $t_j : (E_{j+1}|Y_j^*) \rightarrow (E_j|Y_j^*)$, and so a point of H is represented by the data (q_i, s_j, t_j) .

Remark 5.27 Note that two such tuples (q_i, s_j, t_j) and (q'_i, s'_j, t'_j) represent the same point of H if there exists an isomorphism $\phi : E \rightarrow E'$ of pre- \mathcal{D} -modules $E = (E_i, s_i t_i)$ and $E' = (E'_i, s'_i t'_i)$ such that $q'_i = \phi_i \circ q_i$ for each i .

Definition 5.28 (Right action of the group G on the scheme H .) In terms of valued points, we define this as follows. For any point h of H represented by (q_i, s_j, t_j) , and an element $g = (g_i) \in G = \prod_i G_i$, put

$$(q_i, s_j, t_j) \cdot g = (q_i \circ g_i, s_j, t_j)$$

Note that this is well defined with respect to the equivalence given by remark 5.27, and indeed defines an action of G on H lifting its action on C , as follows from remark 5.14.

It is clear from the definitions of H and this action that two points of H parametrise isomorphic pre- \mathcal{D} -modules if and only if they lie in the same G orbit.

The morphism $H \rightarrow C$ is an affine morphism which is G -equivariant, where G acts on C via $\prod_{i,a} G_{i,a}$. As seen before, the action of each $G_{i,a}$ on $C_{i,a}^o$ admits a good quotient in the sense of geometric invariant theory, and hence the action of G on C admits a good quotient $C//G$. A well known lemma of Ramanathan (see Proposition 3.12 in [Ne]) asserts that if G is a reductive group acting on two schemes U and V such that V admits a good quotient $V//G$, and if there exists an affine, G -equivariant morphism $U \rightarrow V$, then there exists a good quotient $U//G$. Applying this to the G -equivariant affine morphism $H \rightarrow C$, a good quotient $H//G$ exists, which by construction and universal properties of good quotients is the coarse moduli scheme of semistable pre- \mathcal{D} -modules with given Hilbert polynomials. By construction this is a separated scheme of finite type over \mathcal{C} , and is, in fact, quasiprojective.

Note that under a good quotient in the sense of geometric invariant theory, two different orbits can in some cases get mapped to the same point (get identified in the quotient). In the rest of this section, we determine what are the closed points of the quotient $H//G$.

5.9 Stability and points of the moduli

Let $T_{i,a} \subset G_{i,a}$ be its center. Let $T_i = \prod_a T_{i,a}$, and $T = \prod_i T_i = (\prod_{i,a} T_{i,a})$. Note that $T \subset G$ as a closed normal subgroup (which is a torus), which acts trivially on C . By definition of T we have a canonical identification

$$T = \prod_i \Gamma(X_i, \mathcal{O}_{X_i}^\times)$$

By definition 5.28 and remark 5.15, the action of $\lambda = (\lambda_i) \in T$ on $h = (q_i, s_i, t_i) \in H$ is given by

$$h \cdot g = (\lambda_i q_i, s_i, t_i) = (q_i, \frac{\lambda_i}{\lambda_{i+1}} s_i, \frac{\lambda_{i+1}}{\lambda_i} t_i)$$

In [N-S], the construction of the quotient $H//G$ was made in a complicated way in two steps: by Ramanathan's lemma, we can first have the quotient $R = H//T$, and then as the second step we have the quotient $H//G = R//(G/T)$. However, we now do it in a much simplified way, which gives a simplification also of [N-S].

Definition 5.29 A sub pre- \mathcal{D} -module \mathbf{F} of a semistable pre- \mathcal{D} -module \mathbf{E} will be called an **S-submodule** if each non-zero $F_{i,a}$ has the same normalized Hilbert polynomial $p_{i,a}$ as that of $E_{i,a}$. A filtration \mathbf{E}_p on a semistable pre- \mathcal{D} -module \mathbf{E} is an **S-filtration of the pre- \mathcal{D} -module** if each \mathbf{E}_p is an S-submodule, equivalently the given filtration on each $E_{i,a}$ is an S-filtration.

Remark 5.30 Given a semistable pre- \mathcal{D} -module \mathbf{E} , an S-submodule, the corresponding quotient module, and the graded pre- \mathcal{D} -module \mathbf{E}' associated with an S-filtration are again semistable pre- \mathcal{D} -modules. Moreover, \mathbf{E}' has the same Hilbert polynomials $p_{i,a}$ as \mathbf{E} . We will say that \mathbf{E} is (primitively) S-equivalent to \mathbf{E}' .

Definition 5.31 The equivalence relation on the set of isomorphism classes of all semistable pre- \mathcal{D} -modules generated by the above relation, under which the graded module \mathbf{E}' associated to an S-filtration of \mathbf{E} is taken to be equivalent to \mathbf{E} , will be called **S-equivalence for pre- \mathcal{D} -modules**.

Definition 5.32 We say that a semistable pre- \mathcal{D} -module is **stable** if it nonzero and does not admit any nonzero proper S-submodule.

Proposition 5.33 *Let \mathbf{E}_H denote the tautological family of pre- \mathcal{D} -modules parametrized by H . Let $\lambda : GL(1) \rightarrow G$ be a 1-parameter subgroup of $G = \prod G_i$, and let $h = (q_i, s_j, t_j) \in H$ be a point such that the limit $\lim_{\tau \rightarrow 0} h\lambda(\tau)$ exists in H . Let $\bar{\lambda} : A^1 \rightarrow H$ be the resulting morphism. Then there exists an S-filtration $(\mathbf{E}_h)_p$ of the pre- \mathcal{D} -module $(\mathbf{E}_h)_p$ such that the pullback of \mathbf{E}_H to A^1 under $\bar{\lambda} : A^1 \rightarrow H$ is isomorphic to the family constructed in remark 5.13*

Proof By the definition of the action of G on H , the family \mathbf{E}_τ satisfies the following properties:

- (i) The families $(E_{i,a})_\tau$ are of the necessary type by proposition 5.17.(4).
- (ii) Outside $\tau = 0$, the homomorphisms $(s_j)_\tau$ and $(t_j)_\tau$ are pull backs of s_j and t_j .

Therefore now the proposition follows from remark 5.9.

The following lemma, whose proof is obvious, is necessary to show that stability in an open condition on the parameter scheme T of a family \mathbf{E}_T of pre- \mathcal{D} -modules.

Lemma 5.34 *Let \mathbf{E}_T be a family of pre- \mathcal{D} -modules parametrized by T . Let there be given a family \mathbf{F}_T of sub vectorbundles $F_{i,T} \subset E_{i,T}$ which are $\mathcal{D}_{i,T}$ -submodules. Then there exists a closed subscheme $T_o \subset T$ with the following universal property. Given any base change $T' \rightarrow T$, the pullback $\mathbf{F}_{T'}$ is a sub pre- \mathcal{D} -module of $\mathbf{E}_{T'}$ if and only if $T' \rightarrow T$ factors through T_o .*

Using the above lemma, the ‘quot scheme argument for openness of stability’ can now be applied to a family of pre- \mathcal{D} -modules, to give

Proposition 5.35 *Stability is a Zariski open condition on the parameter scheme T of any family \mathbf{E}_T of semistable pre- \mathcal{D} -modules.*

Proof Let $\pi : P \rightarrow T$ be the projective scheme, which is closed subscheme of a fibered product over T of relative quot schemes of the $E_{i,T}$, which parametrizes families of sub vector bundles F_i which are \mathcal{D}_i -submodules with the same reduced Hilbert polynomials as those of \mathbf{E} . (Actually, the Quot scheme parametrizes coherent quotients flat over the base, and the condition of \mathcal{D}_i -linearity gives a closed subscheme. Now by the assumption of strong local freeness on the $E_{i,a}$, it follows that the quotients are locally free). Now by the above lemma, P has a closed subscheme P_o where \mathbf{F}_{P_o} is a family of sub pre- \mathcal{D} -modules with the same normalized Hilbert polynomials, and every such sub pre- \mathcal{D} -module of a \mathbf{E}_t for $t \in T$ occurs among these. Hence $T - \pi(P_o)$ is the desired open subset of T .

Remark 5.36 As semistability is itself a Zariski open condition on any family of pre- \mathcal{D} -modules, it now follows that stability is a Zariski open condition on the parameter scheme T of any family \mathbf{E}_T of pre- \mathcal{D} -modules.

Now all the ingredients are in place for the following main theorem, generalizing theorem 4.19 in [N-S].

Theorem 5.37 *Let X be a non-singular variety with a normal crossing divisor Y . Let a numerical polynomial $p_{i,a}$ and an ample line bundle on $X_{i,a}$ be chosen for each $X_{i,a}$. Then we have the following.*

- (1) *There exists a coarse moduli scheme \mathcal{M} for semistable pre- \mathcal{D} -modules \mathbf{E} on (X, Y) where $E_{i,a}$ has Hilbert polynomial $p_{i,a}$. The scheme \mathcal{M} is quasiprojective, in particular, separated and of finite type over \mathbb{C} .*

- (2) The points of \mathcal{M} are S -equivalence classes of semistable pre- \mathcal{D} -modules.
- (3) The S -equivalence class of a stable pre- \mathcal{D} -module equals its isomorphism class.
- (4) \mathcal{M} has an open subscheme \mathcal{M}^s whose points are the isomorphism classes of all stable pre- \mathcal{D} -modules. This is a coarse moduli for (isomorphism classes of) stable pre- \mathcal{D} -modules.

Proof The statement (1) is by the construction of $\mathcal{M} = H//G$ and properties of a good quotient.

The statement (2) follows from remark 5.13 and proposition 5.33.

Let $x \in H$ and let $x_0 \in H$ be a limit point of the orbit Gx . Then by properties of GIT quotients, there exists a 1-parameter subgroup $\lambda : GL(1) \rightarrow G$ such that $x_0 = \lim_{\tau \rightarrow 0} x \cdot \lambda(\tau)$. Any such limit is of the type given by proposition 5.33, made from an S -filtration of the corresponding pre- \mathcal{D} -module \mathbf{E}_x . If $x \in H$ be stable (means corresponds to a stable pre- \mathcal{D} -module), then it has no non-trivial S -filtration, so the orbit of x is closed. As stability is an open condition on H by proposition 5.35, if the orbit of a point y in H has a limit x which is stable, then the point y must itself be stable. So by the above, the orbit of y must be closed, so $x \in Gy$. Hence a stable point x is not the limit point of any other orbit. Hence (3) follows.

Finally, the statement (4) follows from (1), (2), (3), and proposition 5.35. This completes the proof of the theorem.

Note: the statement (3) in theorem 4.19 of [N-S] has a mistake - the ‘if and only if’ should be changed to ‘if’, removing the ‘only if’ part.)

6 Perverse Sheaves on (X, Y)

In this section, we give a finite description (in terms of a finite quiver of finite dimensional vector spaces and linear maps) of perverse sheaves on (X, Y) , that is, perverse sheaves on X that are cohomologically constructible with respect to the stratification $X = \cup_i (S_i - S_{i-1})$, which closely parallels our definition of pre- \mathcal{D} -modules. This enables us to describe the perverse sheaf associated to the \mathcal{D} -module associated to a pre- \mathcal{D} -module directly in terms of the pre- \mathcal{D} -module. In turn, this allows us to deduce properties of the analytic morphism from the moduli of pre- \mathcal{D} -modules to the moduli of such quivers (which is the moduli of perverse sheaves with given kind of singularities).

More general finite descriptions of perverse sheaves in terms of quivers exist in literature (see for example MacPherson and Vilonen [M-V]), where the requirement of normal crossing singularities is not needed, and the resulting moduli space has been constructed by Brylinski, MacPherson, and Vilonen [B-M-F]). We cannot use their moduli directly, as it is too general for the specific purpose of describing the Riemann-Hilbert morphism by a useable formula (where, for example, we can see the differential of the map).

6.1 The specialization functor

In this section, a vector bundle will usually mean a geometric vector bundle in the analytic category. That is, if E is a locally free sheaf (but not necessarily of constant rank) on a reduced scheme X of finite type over \mathcal{C} , then when we refer to the **vector bundle** E , what we mean is the analytic space (with euclidean topology) associated to the scheme $\text{Spec}_X \text{Sym}(E^*)$.

The replacement by Verdier (see [V1] and [V2]) of the not so canonical operation of *restriction to a tubular neighbourhood* by *specialization to normal cone (or bundle)* is used below in a somewhat more general set up as follows.

Let M be a complex manifold, $T \subset M$ be a divisor with normal crossings, and for some integer k let T_k be a union of components of the k -dimensional singularity stratum of T . Let $C \rightarrow T_k$ be a union of components of the normalization of T_k , and $f : C \rightarrow M$ the composite map $C \rightarrow T_k \rightarrow M$. Let N_f be the normal bundle to $f : C \rightarrow M$, and let $U \subset N_f$ be the open subset which is the complement of the normal crossing divisor F_f in N_f defined by vectors tangent to branches of T . Note that in particular F_f contains the zero section of N_f . Then we have a functor from local systems on $M - T$ to local systems on U defined as follow.

For each $x \in C$, there exists an open neighbourhood V_x of $f(x)$ in M such that the restricted map $f_x : C_x \rightarrow M$ where $C_x = f^{-1}(V_x)$ and $f_x = f|_{C_x}$ is a closed imbedding of the manifold C_x into V_x . Then note that the normal bundle N_{f_x} of $f_x : C_x \rightarrow V_x$ is the restriction of N_f to $C_x \subset C$. Let $F_{f_x} = F_f \cap N_{f_x}$, and $U_x = N_{f_x} - F_{f_x}$. The usual functor of specialization (see [V1] and [V2]) now associates a local system on U_x to a local system on $V_x - T$. These glue together to define our desired functor. Given a local system \mathcal{E} on M_T , we denote its specialization by $\mathcal{E} \parallel U$, which is a local system on U .

More generally, the above method gives a definition of a specialization functor between the derived categories of cohomologically bounded constructible complexes of sheaves of complex vector spaces on M and N_f . If the complex \mathcal{F} is cohomologically constructible with respect to the singularity stratification of (M, T) , then its specialization $\mathcal{F} \parallel N_f$ to N_f is cohomologically constructible with respect to the singularity stratification of (N_f, F_f) . This functor carries perverse sheaves to perverse sheaves.

Remark 6.1 For a topological manifold M which is possibly disconnected, choose a base point in each component M_a , and let Γ_a^M denote the indexed set of the fundamental groups of the components of M with respect to the chosen base points, indexed by $a \in \pi_0(M)$. We have an equivalence categories between local systems on T and an indexed collection of group representations $\rho_a : \Gamma_a^T \rightarrow GL(n_a)$. In the above situation, we have the groups Γ_a^U and Γ_b^{M-T} . Then there exists a map $\gamma : \pi_0(U) \rightarrow \pi_0(M - T)$ and a group homomorphism

$$\psi_a : \Gamma_a^U \rightarrow \Gamma_{\gamma(a)}^{M-T}$$

for each $a \in \pi_0(U)$, such that the above functor of specialization from local systems on $M - T$ to local systems on U is given by associating to a collection of representations $\rho_b : \Gamma_b^{M-T} \rightarrow GL(n_b)$ where b varies over $\pi_0(M - T)$ the collection composite representations $\rho_{\gamma(a)} \circ \psi_a$. In this sense, specialization is like pullback.

6.2 Finite representation

We now return to (X, Y) and use our standard notation (see section 2). We will apply the above specialization functor to the following cases.

Case(1): M is X , T is Y , and $k = d - 1$, so $C = X_{d-1} = Y^*$ is the normalization of Y . In this case, starting from a local system \mathcal{E}_d on $X - Y$ we get a local system on U_{d-1} , which we denote by $\mathcal{E}_d \parallel U_{d-1}$.

Case(2): M is N_{i+1} (= the normal bundle to $f_i : X_i \rightarrow X$), T is the divisor F_{i+1} in N_{i+1} defined by vectors tangent to branches of Y , $k = i$, T the inverse image of S_i under the map $p_{i+1} : X_{i+1} \rightarrow S_{i+1}$, and $C = Y_i^*$. In this case, starting from a local system \mathcal{E}_{i+1} on U_{i+1} , we get a local system on R_i which we denote by $\mathcal{E}_{i+1} \parallel R_i$.

Case(3): Apply this with $M = N_{i+2}$, $T = F_{i+2}$, $C = Z_i$ and $C \rightarrow M$ the composite

$$Z_i \xrightarrow{P_{\{i+2\}, \{i, i+2\}}} X_{i+2} \hookrightarrow N_{i+2}$$

In this case, starting from a local system \mathcal{E}_{i+2} on U_{i+2} , we get a local system on W_i which we denote by $\mathcal{E}_{i+2} \parallel W_i$.

Case(4): Apply this with $M = N_{i+1}$, $T = F_{i+1}$, $C = Z_i^*$ and $C \rightarrow M$ the composite

$$Z_i^* \xrightarrow{P_{\{i+1\}, \{i, i+1, i+2\}}} X_{i+1} \hookrightarrow N_{i+1}$$

In this case, starting from a local system \mathcal{E}_{i+1} on U_{i+1} , we get a local system on W_i^* which we denote by $\mathcal{E}_{i+1} \parallel W_i^*$.

Case(5): Apply this with $M = N_{i+1, i+2}$, $T = F_{i+1, i+2}$, $C = Z_i^*$, and $C \rightarrow M$ the composite

$$Z_i^* \xrightarrow{P_{\{i+1, i+2\}, \{i, i+1, i+2\}}} Y_{i+1}^* \hookrightarrow N_{i+1, i+2}$$

In this case, starting from a local system \mathcal{F}_{i+1} on R_{i+1} , we get a local system on W_i^* which we denote by $\mathcal{F}_{i+1} \parallel W_i^*$.

On the other hand, note that the derivative of the covering projection $p_{\{i\}, \{i, i+1\}} : Y_i^* \rightarrow X_i$ is a map $dp : N_{i, i+1} \rightarrow N_i$ under which $F_{i, i+1} \subset N_{i, i+1}$ is the inverse image of $F_i \subset N_i$. Hence dp induces a map $R_i \rightarrow U_i$. If \mathcal{E}_i is a local system on U_i , then we denote its pullback under this map by $\mathcal{E}_i | R_i$, which is a local system on R_i .

Similarly, for the covering projection $p_{\{i\}, \{i, i+2\}} : Z_i \rightarrow X_i$ the derivative induces a map $W_i \rightarrow U_i$. If \mathcal{E}_i is a local system on U_i then we denote its pullback under this map by $\mathcal{E}_i | W_i$ which is a local systems on W_i .

Note that the derivative of the 2-sheeted covering projection $X_{\{i, i+1, i+2\}} \rightarrow X_{\{i, i+2\}}$ induces a 2 sheeted covering projection $\pi : W_i^* \rightarrow W_i$. We will denote the pullbacks of $\mathcal{E}_i | W_i$ and $\mathcal{E}_{i+2} \parallel W_i$ under $\pi : W_i^* \rightarrow W_i$ by $\mathcal{E}_i | W_i^*$ and $\mathcal{E}_{i+2} \parallel W_i^*$ respectively. Note

that the same $\mathcal{E}_{i+2}\|W_i^*$ could have been directly defined by specializing, similar to case 4 above.

In summary, for a collection of local systems \mathcal{E}_i on U_i , we have various pullbacks or specializations associated with it follows:

- (i) On R_i we have local systems $\mathcal{E}_i|R_i$ and $\mathcal{E}_{i+1}\|R_i$, for $i \leq d-1$
- (ii) On W_i we have local systems $\mathcal{E}_i|W_i$ and $\mathcal{E}_{i+2}\|W_i$, for $i \leq d-2$
- (iii) On W_i^* we have local systems $\mathcal{E}_i|W_i^*$, $\mathcal{E}_{i+1}\|W_i^*$, and $\mathcal{E}_{i+2}\|W_i^*$, for $i \leq d-2$.

Remark 6.2 On W_i^* , we have a canonical identification between the sheaf $\mathcal{E}_{i+1}\|W_i^*$ (defined as in case (4) above) and the sheaf $(\mathcal{E}_{i+1}\|R_{i+1})\|W_i^*$ (defined as in case (5) above).

Remark 6.3 We have earlier defined central elements $\tau_i(c)$ in the fundamental group of each connected component $R_i(c)$ of R_i (see section 2). Given a linear system \mathcal{F} on R_i , we denote by τ_i the automorphism of \mathcal{F} induced by the monodromy action of the central element $\tau_i(c)$ on $R_i(c)$.

Definition 6.4 A **Verdier object** $(\mathcal{E}_i, C_i, V_i)$ on (X, Y) consists of the following.

- (1) For each $m \leq i \leq d$, \mathcal{E}_i is a local system on U_i (the ranks of the local systems are not necessarily constant.)
- (2) For each $m \leq i \leq d-1$, $C_i : (\mathcal{E}_{i+1}\|R_i) \rightarrow (\mathcal{E}_i|R_i)$ and $V_i : (\mathcal{E}_i|R_i) \rightarrow (\mathcal{E}_{i+1}\|R_i)$ are homomorphisms of local systems, such that

$$\begin{aligned} V_i C_i &= 1 - \tau_i \text{ on } \mathcal{E}_{i+1}\|R_i \\ C_i V_i &= 1 - \tau_i \text{ on } \mathcal{E}_i|R_i \end{aligned}$$

- (3) Let $m \leq i \leq d-2$. Let $\pi : W_i^* \rightarrow W_i$ be the covering projection induced by $\pi : Z_i^* \rightarrow Z_i$. Let

$$a_{i+2} : \mathcal{E}_{i+2}\|W_i \rightarrow \pi_* \pi^*(\mathcal{E}_{i+2}\|W_i) = \pi_*(\mathcal{E}_{i+2}\|W_i^*)$$

$$a_i : \mathcal{E}_i|W_i \rightarrow \pi_* \pi^*(\mathcal{E}_i|W_i) = \pi_*(\mathcal{E}_i|W_i^*)$$

be adjunction maps, and let the cokernels of these maps be denoted by

$$q_{i+2} : \pi_*(\mathcal{E}_{i+2}\|W_i) \rightarrow Q_{i+2}$$

$$q_i : \pi_*(\mathcal{E}_i|W_i) \rightarrow Q_i$$

Then we impose the requirement that the composite map

$$\mathcal{E}_{i+2}\|W_i \xrightarrow{a_{i+2}} \pi_*(\mathcal{E}_{i+2}\|W_i) \xrightarrow{\pi_*(C_{i+1}\|W_i^*)} \pi_*(\mathcal{E}_{i+1}\|W_i^*) \xrightarrow{\pi_*(t_i|W_i^*)} \pi_*(\mathcal{E}_i|W_i^*) \xrightarrow{q_i} Q_i$$

is zero.

(4) Similarly, we demand that for all $m \leq i \leq d - 2$ the composite map

$$Q_{i+2} \xleftarrow{a_{i+2}} \pi_*(\mathcal{E}_{i+2}\|W_i^*) \xleftarrow{\pi_*(V_{i+1}\|W_i^*)} \pi_*(\mathcal{E}_{i+1}\|W_i^*) \xleftarrow{\pi_*(V_i|W_i^*)} \pi_*(\mathcal{E}_i|W_i^*) \xleftarrow{a_i} \mathcal{E}_i|W_i$$

is zero.

(5) Note that as $\pi : W_i^* \rightarrow W_i$ is a 2-sheeted cover, for any sheaf \mathcal{F} on W_i the new sheaf $\pi_*\pi^*(\mathcal{F})$ on W_i^* has a canonical involution coming from the deck transformation for $W_i^* \rightarrow W_i$ which transposes the two points over any base point. In particular, the local systems $\pi_*(\mathcal{E}_{i+2}\|W_i^*) = \pi_*\pi^*(\mathcal{E}_{i+2}\|W_i)$ and $\pi_*(\mathcal{E}_i|W_i^*) = \pi_*\pi^*(\mathcal{E}_i|W_i)$ have canonical involutions, which we denote by ν . We demand that the following diagram should commute.

Diagram III.

$$\begin{array}{ccccc} \pi_*(\mathcal{E}_{i+1}\|W_i^*) & \xrightarrow{\pi_*(V_{i+1}\|W_i^*)} & \pi_*(\mathcal{E}_{i+2}\|W_i^*) & \xrightarrow{\nu} & \pi_*(\mathcal{E}_{i+2}\|W_i^*) \\ \pi_*(C_{i+1}\|W_i^*) \downarrow & & & & \downarrow \pi_*(C_i|W_i^*) \\ \pi_*(\mathcal{E}_i|W_i^*) & \xrightarrow{\nu} & \pi_*(\mathcal{E}_i|W_i^*) & \xrightarrow{\pi_*(V_i|W_i^*)} & \pi_*(\mathcal{E}_{i+1}\|W_i^*) \end{array}$$

Remark 6.5 As the adjunction maps are injective (in particular as a_i is injective), the condition (3) is equivalent to demanding the existence of a unique f which makes the following diagram commute.

Diagram I.

$$\begin{array}{ccccc} \mathcal{E}_{i+2}\|W_i & & \xrightarrow{f} & & \mathcal{E}_i|W_i \\ a_{i+2} \downarrow & & & & \downarrow a_i \\ \pi_*(\mathcal{E}_{i+2}\|W_i^*) & \xrightarrow{\pi_*(C_{i+1}\|W_i^*)} & \pi_*(\mathcal{E}_{i+1}\|W_i^*) & \xrightarrow{\pi_*(C_i|W_i^*)} & \pi_*(\mathcal{E}_i|W_i^*) \end{array}$$

Similarly, the condition (4) is equivalent to the following: there must exist a unique homomorphism g which makes the following diagram commute.

Diagram II.

$$\begin{array}{ccccc} \mathcal{E}_{i+2}\|W_i & & \xleftarrow{g} & & \mathcal{E}_i|W_i \\ a_{i+2} \downarrow & & & & \downarrow a_i \\ \pi_*(\mathcal{E}_{i+2}\|W_i^*) & \xleftarrow{\pi_*(V_{i+1}\|W_i^*)} & \pi_*(\mathcal{E}_{i+1}\|W_i^*) & \xleftarrow{\pi_*(V_i|W_i^*)} & \pi_*(\mathcal{E}_i|W_i^*) \end{array}$$

It can be seen that the above definition of a Verdier object on (X, Y) reduces in the case of a polydisk to the hypercube description of perverse sheaf on a polydisk. As Verdier objects, perverse sheaves, and the specialization functors are all local in nature, we get the following by gluing up.

Proposition 6.6 *There is an equivalence of categories between the category of Verdier objects and the category of perverse sheaves on (X, Y) .*

6.3 Moduli for perverse sheaves

As the various fundamental groups are finitely generated, the definition of a Verdier object has an immediate translation in terms of **quivers**, that is, diagrams of finite dimensional vector spaces and linear maps, by means of remark 6.1. We define a **family of Verdier objects** parametrized by some space T as a family of such quivers over T , in which vector spaces are replaced by vector bundles over T and linear maps (or group representations) are replaced by endomorphisms of the bundles. These obviously form an algebraic stack in the sense of Artin, if we work in the category of schemes over \mathcal{C} .

When we fix the ranks $n_{i,a}$ of the restrictions of local systems \mathcal{E}_i on connected components $X_{i,a}$ of X_i , and go modulo the conjugate actions of the various $GL(n_a)$ (this is exactly as in the section 6 of [N-S] so we omit the details), we get an affine scheme of finite type over \mathcal{C} as the moduli of Verdier objects with given ranks. The points of this moduli space are Jordan-Holder classes (that is, semisimplifications) of Verdier objects.

The above definition of families and construction of moduli is independent (upto isomorphism) of the choices of base points and generators for the various fundamental groups.

We define an **algebraic (or holomorphic) family of perverse sheaves on (X, Y)** to be an algebraic (or holomorphic) family of Verdier objects, parametrized by a complex scheme (or a complex analytic space) T . Therefore, we have

Proposition 6.7 *There exists a coarse moduli scheme \mathcal{P} for perverse sheaves on (X, Y) of fixed numerical type. The scheme \mathcal{P} is an affine scheme of finite type over \mathcal{C} , and points of \mathcal{P} correspond to Jordan-Holder classes of perverse sheaves.*

7 The Riemann-Hilbert morphism

In section 7.1 we define an analytic morphism \mathcal{RH} from the stack (or moduli) of pre- \mathcal{D} -modules to the stack (or moduli) of Verdier objects, which represents the de Rham functor. Note that even if both sides are algebraic, the map is only analytic in general, as it involves integration in order to associate to a connection its monodromy.

Next (in section 7.2) we prove some properties of the above Riemann-Hilbert morphism \mathcal{RH} , in particular that it is a local isomorphism at points representing pre- \mathcal{D} -modules which have good residual eigenvalues. This generalizes the rigidity results in [N] and [N-S].

7.1 Definition of the Riemann-Hilbert morphism

The following allows us to go from pre- \mathcal{D} -modules to perverse sheaves.

Proposition 7.1 *Let X be a nonsingular variety, $Y \subset X$ a divisor with normal crossing, and let $Y^* \rightarrow Y$ the normalization of Y , with $f : Y^* \rightarrow X$ the composite map. Let N be the normal bundle to $f : Y^* \rightarrow X$, and let $F \subset N$ be the closed subset of the total space of N defined by vectors tangent to branches of Y (in particular, this includes the zero section Y^* of N). Let $\pi : N_f \rightarrow Y$ be the bundle projection. Then we have*

(1) *If E is a vector bundle on Y^* together with the structure of a $\mathcal{D}_N[\log F]|Y^*$ -module, then π^*E is naturally a $\mathcal{D}_N[\log F]$ -module.*

(2) *Let E be a vector bundle on X together with the structure of a $\mathcal{D}_X[\log Y]$ -module, and let $E|Y^*$ be its pullback to Y^* , which is naturally a module over $f^*\mathcal{D}_X[\log Y]$. Then $\pi^*(E|Y^*)$ is naturally a $\mathcal{D}_N[\log F]$ -module.*

(3) *If E is as in (2) above and if the residual eigenvalues of E do not differ by non-zero integers on any component of Y^* , then the local system $(\pi^*(E|Y^*))^\nabla$ on $N - F$ of integrable sections of $\pi^*(E|Y^*)$ is canonically isomorphic to the specialization of the local system $(E|X - Y)^\nabla$ on $X - Y$ of integrable sections of E .*

(4) *If E_T is a holomorphic family of vector bundles with integrable logarithmic connections on (X, Y) parametrized by a complex analytic space T , such that each E_t has good residual eigenvalues, then the corresponding local systems on $N - F$ given by (3) form a holomorphic family of local systems on N_F parametrized by T .*

Proof The statement (1) is a special case of the following more general statement. Let S be any nonsingular variety, $\pi : N \rightarrow S$ any geometric vector bundle on S , and $F \subset N$ a normal crossing divisor in the total space of N such that analytic (or étale) locally F is the union of r vector subbundles of N of rank $r - 1$ where r is the rank of N_S . Then for any vector bundle E on S together with the structure of a $\mathcal{D}_N[\log F]$ -module, the vector bundle $\pi^*(E)$ on N has a natural structure of a $\mathcal{D}_N[\log F]$ -module. This can be seen by choosing analytic local coordinates $(x_1, \dots, x_m, y_1, \dots, y_r)$ on N where (x_i) are local coordinates on S and y_i are linear coordinates on the fibers such that F is locally defined by $\prod_i y_i = 0$, and defining a logarithmic connection on $\pi^*(E)$ in terms of the actions of $\partial/\partial x_i$ and $y_i \partial/\partial y_i$ given by

$$\begin{aligned} \nabla_{\partial/\partial x_i}(g(y) \otimes_{\mathcal{O}_S} e) &= g(y) \otimes_{\mathcal{O}_S} \nabla_{\partial/\partial x_i} e \\ \nabla_{y_i \partial/\partial y_i}(g(y) \otimes_{\mathcal{O}_S} e) &= (y_i(\partial/\partial y_i)g(y)) \otimes_{\mathcal{O}_S} e \end{aligned}$$

The statement (2) follows from the canonical isomorphism between $f^*(\mathcal{D}_X[\log Y])$ and $\mathcal{D}_N[\log f]|Y^*$. The statements (3) and (4) follows over polydisks from the relation between V-filtrations and specializations, and we have the global statements by gluing.

Remark 7.2 As we have to integrate in order to associate its monodromy representation to an integrable connection, even if T was associated to an algebraic variety and the original family (E_T, ∇_T) was algebraic, the associated family of local systems will in general only be an analytic family.

Remark 7.3 It is sometimes erroneously believed that if E is a locally free logarithmic connection on (X, Y) , and Y is nonsingular, then the restriction $E|_Y$ has a natural connection on it. This is false in general, and is correct in some special case if the exact sequence $0 \rightarrow \mathcal{O}_Y \rightarrow T_X[\log Y]|_Y \rightarrow T_Y$ has a natural splitting in some special case under consideration.

The above proposition allows us to directly associate a Verdier object $(\mathcal{E}_i, C_i, V_i)$ to a pre- \mathcal{D} -module (E_i, s_i, t_i) which has good residual eigenvalues in the sense of definition 3.4, as follows.

Definition 7.4 We put \mathcal{E}_d to be the local system on $X - Y$ given by E_d , and for $i \leq d - 1$ we define \mathcal{E}_i to be the local system on U_i associated to the logarithmic connection on (N_i, F_i) associated to E_i by proposition 7.1(1). For $m \leq j \leq d - 1$ we put C_j to be the map induced (using the statements (2) and (3) in proposition 7.1 above) by $\pi_j^*(t_j)$ where $\pi_j : N_{j,j+1} \rightarrow Y_j^*$ is the bundle projection, and define V_j by the formula

$$V_j = \frac{\exp(2\pi i \theta_j) - 1}{\theta_j} \pi_j^*(s_j)$$

It is immediate from the definitions that this gives a Verdier object starting from a pre- \mathcal{D} -module. By proposition 7.1(4), the above association is well behaved for analytic families, and gives rise to an analytic family of Verdier objects when we apply it to an analytic family of pre- \mathcal{D} -modules. This gives the **Riemann-Hilbert morphism** \mathcal{RH} at the level of analytic stacks.

Remark 7.5 As a consequence of remark 7.2, the above morphism \mathcal{RH} of stacks is holomorphic but not algebraic.

Remark 7.6 By its definition, the Verdier object $\mathcal{E} = (\mathcal{E}_i, C_i, V_i)$ associated by definition 7.4 to a pre- \mathcal{D} -module $\mathbf{E} = (E_i, t_i, s_i)$ with good residual eigenvalues defines the perverse sheaf associated by the de Rham functor to the \mathcal{D} -module M associated to \mathbf{E} in section 4.2 above, as follows locally from the hypercube descriptions of these objects restricted to polydisks.

If a semistable pre- \mathcal{D} -module has good residual eigenvalues, then the graded object associated to any S-filtration again has good residual eigenvalues by remark 5.25. It follows that the condition that the residual eigenvalues be good defines an analytic open subset \mathcal{M}_o of the moduli space \mathcal{M} by theorem 5.37(2). It can be proved as in Lemma 7.3 of [N-S] and its following discussion, using analytic properties of a GIT quotient proved by Simpson in [S], that our association of a Verdier object to a pre- \mathcal{D} -module with good residual eigenvalues now descends to an **analytic morphism** $\mathcal{RH} : \mathcal{M}_o \rightarrow \mathcal{P}$ from \mathcal{M}_o to the moduli \mathcal{P} of Verdier objects. This is the Riemann-Hilbert morphism at the level of the moduli spaces.

7.2 Properties of the Riemann-Hilbert morphism

This section contains material which is a straightforward generalization of [N-S], so we omit the details.

In the reverse direction, can construct a pre- \mathcal{D} -module with good residual eigenvalues over a given Verdier object by using repeatedly the **Deligne construction**. This gives the **surjectivity** of the Riemann-Hilbert morphism. The exponential map $M(n, \mathcal{C}) \rightarrow GL(n, \mathcal{C})$ is a submersion at points where the eigenvalues do not differ by $2\pi i$ times a nonzero integer. Using this, we can extend the Deligne construction to families of local systems parametrized by Artin local rings (e.g., $\mathcal{C}[\epsilon]/(\epsilon^2)$) to get families of logarithmic connections with good residual eigenvalues. From this it follows that the Riemann-Hilbert morphism is **surjective at tangent level** at points above having good residual eigenvalues.

Proposition 5.3 of [N] shows that for a meromorphic connection M on X with regular singularities on a normal crossing divisor Y , any locally free logarithmic lattice whose residual eigenvalues do not differ by positive integers is infinitesimally rigid. In Proposition 8.6 of [N-S], this is extended to pre- \mathcal{D} -modules on (X, Y) when Y is smooth, by analyzing the derivative of a map of the form

$$(s, t) \mapsto \left(s, t \frac{e^{st} - 1}{st}\right)$$

for matrices s and t (lemma 3.10 of [N-S]). By a similar proof applied to the formula given in definition 7.4, we have the following when Y is normal crossing.

Proposition 7.7 (Infinitesimal rigidity): *A pre- \mathcal{D} -module on (X, Y) with good residual eigenvalues does not admit any nontrivial infinitesimal deformations such that the associated \mathcal{D} module (or perverse sheaf) on (X, Y) is constant.*

This shows that the Riemann-Hilbert morphism is a **tangent level isomorphism** of stacks at points above with good residual eigenvalues.

The above properties, which are for the Riemann-Hilbert morphism as a morphism of analytic stacks, are valid by 5.37 and 6.7 for the morphism $\mathcal{RH} : \mathcal{M}_o \rightarrow \mathcal{P}$ at the level of the two moduli spaces at stable points of \mathcal{M}_o which go to simple Verdier objects.

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