A NOTE ON NARASINGA RAO'S PROBLEM RELATING TO TETRAHEDRA.

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Received March 12, 1938.

§ 1. *Introduction.*

A. Narasinga Rao[1] has raised the question, "The areas of the four faces of a tetrahedron are α, β, γ, δ. Is the volume determinate? If not, between what limits does it lie?

It may be mentioned at once that an upper bound must exist as is evident from Schwarz's isoperimetric inequality[2]

$$O^3 - 36\pi V^2 \geqslant 0$$

for convex surfaces; or again from Steinitz's inequality[3] relating to the volume of a tetrahedron of given surface area. Narasinga Rao's problem, however, is quite distinct from these isoperimetric problems since what is given here is not the total surface area, but that of the individual faces.

In the first place, I notice that the problem could be easily generalised to n-dimensions, and would then become,

"Given $(n+1)$ positive numbers, does a simplex in n-dimensions exist having for the $(n-1)$-dimensional volumes of its $(n+1)$ faces, the $(n+1)$ given numbers? If so what are the limits between which the nth-dimensional volume of the simplex lies?"

The analysis for n-dimensions is so similar to that in 3 dimensions that I have given here in most places the solutions for 3 dimensions, and, wherever necessary, mentioned the corresponding analysis and result for n-dimensions.

The principal results obtained are as follows for 3 dimensions:—

$$\text{Let} \quad \Delta_1 \leqslant \Delta_2 \leqslant \Delta_3 \leqslant \Delta_4;$$

(1) The necessary and sufficient condition for the existence of one tetrahedron at least is that the sum of the lowest three areas must be greater than the greatest area, *i.e.*,

$$\Delta_1 + \Delta_2 + \Delta_3 > \Delta_4.$$

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[1] *The Mathematics Student*, June 1937, 5, (No. 2), 90.

[2] Blaschke, *Integral geometrie*, Bd. 2.

[3] Steinitz, *Ency. Math. Wiss.*, Bd. 3, Teil 1, 2, S. 139.

(2) The lower bound of the volumes of all the tetrahedra having for its areas the given numbers is zero.

(3) The tetrahedron having the maximum volume under the given conditions is an orthogonal one[4] (*i.e.*, having opposite edges orthogonal) and is unique.

(4) The upper bound of the volume is given by

$$V \leqslant \left(2/9\ \Delta_1\Delta_2\Delta_3\right)^{\frac{1}{2}} \left\{1 - \frac{\Omega^2}{3\,(\Delta_1^2\Delta_2^2 + \Delta_2^2\Delta_3^2 + \Delta_3^2\Delta_1^2)}\right\}.$$

where

$2\,\Omega = \Delta_4^2 - \Delta_1^2 - \Delta_2^2 - \Delta_3^2$, the equality holding if $\Omega = 0$,

For n-dimensions, (1) and (2) can be generalised in a straightforward way. (3) also holds, but I have not been able to prove, by the methods given in this paper, that the tetrahedron is unique, although it is likely to be so. (4) can be replaced by

$$V \leqslant \left\{\frac{(\underline{|n-1})^n}{(\underline{|n})^{n-1}}\ \Delta_1\Delta_2\cdots\Delta_n\right\}^{\frac{1}{n-1}} \cdot \left\{1 - \frac{(\Delta_{n+1}^2 - \sum_1^n \Delta_r^2)^2}{\frac{4\cdot n\,(n-1)}{2}\ (\Sigma\,\Delta_r^2\Delta_s^2)}\right\}$$

§ 2. *Fundamental Formulæ.*

Two formulæ easy to obtain are the most important in the following analysis and we will give them in the beginning.

Let $O \cdot\cdot P_1 \cdot\cdot P_2 \cdot\cdot P_n$ be the $\overline{n+1}$ points of a simplex, O being the origin. Let the Vector $OP_1 = R_1\,(l_{11} \cdot\cdot l_{12} \cdot\cdot l_{13} \cdot\cdot l_{1n})$ where l_{1r} are the direction cosines of the line OP_1, etc. Let the $\overline{n-1}$ dimensional volume of the face $(O, P_1P_2 \cdots P_{n-1})$ be Δ_n. If we call OP_1 the vector A_1. Then Δ_n will be proportional to the absolute value of the vector product

$$[A_1\,A_2 \cdots A_{n-1}],\ i.e.,\ \underline{|n-1}\cdot\Delta_n = |\,[A_1, A_2 \cdots A_{n-1}]\,|,\quad \text{etc.}$$

Let the direction cosines of the vector $[A_1\,A_2 \cdot\cdot A_{n-1}]$ be $L_{n1}, L_{n2} \cdots L_{nn}$. We will, for the sake of convenience, put $\bar{A}_n = [A_1 \cdot A_2 \cdots A_{n-1}]$, etc. Then if V be the volume of the simplex

$$\text{Then}\qquad V^{n-1} = \frac{(\underline{|n-1})^n}{(\underline{|n})^{n-1}}\ \Delta_1\cdot\Delta_2\cdot\Delta_n\cdot \begin{vmatrix} L_{11}\ L_{12}\ L_{1n} \\ L_{21}\ L_{22}\ \cdots \\ \cdots\cdots \\ L_{n1} \cdots L_{nn} \end{vmatrix} \qquad \text{I}$$

where $\Delta_1\ \Delta_2 \cdots$ are the $\overline{n-1}$ dimensional volumes of the faces at O. Indicating the angle between the vectors $\bar{A}_1$ and $\bar{A}_2$ by $\bar{\xi}_{12}$,

[4] This theorem is due to Mr. K. V. Iyengar. See his paper elsewhere in this issue.

The formula (I) could be written as:

$$V^{n-1} = \frac{(\underline{|n-1})^n}{(\underline{|n})^{n-1}} \Delta_1 \Delta_2 \Delta_n \begin{vmatrix} 1 & \cos \xi_{12} & \cos \xi_{13} \cdots & \cos \xi_{1n} \\ \cos \xi_{12} & 1 & \cos \xi_{23} & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cos \xi_{n1} & \cdots & \cdots & 1 \end{vmatrix}^{\frac{1}{2}} \qquad \text{I}'$$

Let Δ_{n+1} be the $\overline{n-1}$ dimensional volume of the face opposite O, *i.e.*, of the points $(P_1, P_2, \cdots P_n)$. Then by projection or otherwise it is easy to prove

$$\sum_1^n \Delta_r^2 + \sum_1^n \sum_1^n 2\, \Delta_r\, \Delta_s \cos \xi_{rs} = \Delta^2_{n+1}. \qquad \text{(II)}$$

So that the problem becomes (1) does there exist a solution of (II) at all in n-dimensions and what are the bounds of the determinant in (I′) under condition (II). We will hereafter restrict our analysis to 3 dimensions.

§ 3. *Question of Existence.*

Equation (II) becomes

$$\Delta_1^2 + \Delta_2^2 + \Delta_3^2 + 2\Delta_2\Delta_3 \operatorname{Cos} \xi_{23} + 2\Delta_3\Delta_1 \operatorname{Cos} \xi_{31} + 2\Delta_1\Delta_2 \operatorname{Cos} \xi_{12} = \Delta_4^2. \qquad \text{(II}'\text{)}$$

We will so choose the vertex O such that $\Delta_1 \leqslant \Delta_2 \leqslant \Delta_3 \leqslant \Delta_4$ (*i.e.*, opposite the largest area).

We will call $\xi_{23} = x$, $\xi_{31} = y$, $\xi_{12} = z$. In order that a real tetrahedron may exist, we must have solution (II′) so that

$$\left.\begin{array}{ll} (1) & 0 \leqslant x \leqslant \pi,\ 0 \leqslant y \leqslant \pi,\ 0 \leqslant z \leqslant \pi. \\ (2) & x + y + z < 2\pi. \\ (3) & y + z > x,\ z + x > y,\ x + y > z. \end{array}\right\} \text{R (region).}$$

Let $E = \sum_1^3 \Delta_r^2 + \Sigma\, 2\Delta_2\Delta_3 \cos x$.

The maximum value of E in R is $(\sum_1^3 \Delta_r)^2$.

(C) $\therefore\ \sum_1^3 \Delta_r > \Delta_4$ equality cannot occur because then the only solution, then, will be $x = 0 = y = z$, which will not be a tetrahedron. Therefore condition (C) is necessary.

At $x = \pi,\ y = \pi,\ z = 0$

$E = (\Delta_1 + \Delta_2 - \Delta_3)^2$. In case $\Delta_1 + \Delta_2 > \Delta_3$

then, since $\Delta_1 \leqslant \Delta_2 \leqslant \Delta_3$.

$E < \Delta_1^2.\ \therefore\ < \Delta_4^2.$

In case $\Delta_3 > \Delta_1 + \Delta_2$

$E < \Delta_3^2.\ \therefore\ < \Delta_4^2.$

∴ If $\Delta_1 + \Delta_2 + \Delta_3 > \Delta_4$ there are always solution of (II′) in the interior of R. |C|

The region R is as a matter of fact a tetrahedron. We can join the point (000) to the point $(\pi, \pi, 0)$ by a space curve L lying entirely in R. Except for end points. Since $E(000) > \Delta_4^2 > E(\pi, \pi, 0)$.

∴ Since E is continuous there exists at least one solution on L in the interior of R. Since we can draw infinity of such curves L there are infinity of such tetrahedra. An entirely similar discussion holds for n-dimensions.

§ 4. *The Lower Bound.*

The points (000) and $(\pi, \pi, 0)$ are on the plane $z + x - y = 0$ in R.

$$E(000) = (\sum_1^3 \Delta_r)^2 \text{ and } E(\pi, \pi, 0) = (\Delta_1 + \Delta_2 - \Delta_3)^2.$$

∴ There are infinity of solutions of (II′) in the plane $z + x - y = 0$ in R. Consider the portion of the plane $z + x - y = \alpha$ in R, where α is a very small positive number. (We will call this α plane.) The maximum and minimum of E on this plane in the region R will be nearly the same as in the plane $z + x - y = 0$.

$$\text{Since } E(000) > \Delta_4^2 > E(\pi, \pi, 0).$$

We easily see that there will be infinity of solutions of (II′) on this plane in the region R.

Now the determinant in I′ for this case is

$$\begin{vmatrix} 1 & \cos x & \cos y \\ \cos x & 1 & \cos z \\ \cos y & \cos z & 1 \end{vmatrix} = 4 \sin\left(\frac{x+y+z}{2}\right) \cdot \sin\left(\frac{y+z-x}{2}\right) \cdot \sin\left(\frac{z+x-y}{2}\right) \cdot \sin\left(\frac{x+y-z}{2}\right)$$

$$\therefore \leqslant 4 \sin\left(\frac{z+x-y}{2}\right).$$

We therefore see that for all tetrahedra corresponding to solution in the plane (α) the $(\text{volume})^2 \leqslant 2/9\, \Delta_1 \cdot \Delta_2 \cdot \Delta_3 \cdot \sqrt{2\alpha}$. Therefore the lower bound of the volumes of the tetrahedra will be zero.

§ 5. *The Upper Bound.*

Coming now to the upper bound, we will take up the n-dimensional case. It is obvious that by the foregoing discussion, if (C) is satisfied there will be solutions for equation (II) in n-dimensions.

We therefore have to find the maximum of $\begin{vmatrix} 1 & \cos \xi_{12} & \cos \xi_{13} & \cdots \\ \cos \xi_{1n} & \cdots & & 1 \end{vmatrix} = D$

under $\sum_{1}^{n} \Delta_r^2 + 2\Delta_r\Delta_s \cos\bar{\xi}_{rs} = \Delta_{n+1}.^2$ Using the theory of Lagrangian Multipliers we get,

$$\frac{\begin{vmatrix} \cos\bar{\xi}_{12} & \cos\bar{\xi}_{23} & \cos\bar{\xi}_{24} \cdots \\ \cos\bar{\xi}_{31} & 1 & \cos\xi_{34} \\ \cos\bar{\xi}_{41} & \text{cis}\,\bar{\xi}_{43} & 1 \cdots \\ \cos\xi_{n1} & \cdots & 1 \end{vmatrix}}{\Delta_1\cdot\Delta_2} = \text{similar terms.}$$

there being nC_2 such terms.

Now let us designate the vector $L_{11}, L_{12}\cdots L_{1n}$ by L_1. Then it is easy to prove

$$\begin{vmatrix} \cos\bar{\xi}_{12} & \cos\bar{\xi}_{32} \cdots \\ \cos\bar{\xi}_{13} & 1 \end{vmatrix} = \text{scalar product of the two vector products given by}$$

$[L_1, L_3, L_4, \cdots L_n]$ and $[L_2, L_3, \cdots L_n]$ $\therefore$ equal to $= [L_1\cdot L_3\cdots L_n][L_2\cdot L_3\cdots L_n]$.

$$= \frac{[\bar{A}_1\bar{A}_3\cdots\bar{A}_n]\cdot[\bar{A}_2\cdot\bar{A}_3\cdots\bar{A}_n]}{\{|\bar{A}_1|\cdot|\bar{A}_2|\cdots|\bar{A}_n|\}^2}\cdot|\bar{A}_1|\cdot|\bar{A}_2|$$

where $|A|$ indicates the absolute value of the vector A.

Let us call $D_* =$ the determinant $|A_1, A_2, \cdots A_n|$ and $\vec{D}_* = |\bar{A}_1\cdots\bar{A}_n|$.

Then the vector $\quad [\bar{A}_1,\cdot\bar{A}_3\cdot\bar{A}_4\cdots\bar{A}_n] = D^{n-2}A_2$.

and $\quad [\bar{A}_2\cdot\bar{A}_3\cdots] = D^{n-2}A_1$.

$$\therefore \begin{vmatrix} \cos\bar{\xi}_{12} & \cos\bar{\xi}_{23} & \cdots \\ \cos\xi_{13} & 1 & \cdots \end{vmatrix} = \frac{D^{2n-4}|\bar{A}_1||\bar{A}_2|}{\{|\bar{A}_1||\bar{A}_2|\cdots|\bar{A}_n|\}^2}\cdot(A_1\cdot A_2)$$

$$= k\cdot\Delta_1\cdot\Delta_2\cdot(A_1\cdot A_2).$$

Therefore the equations now become

$$R_1 R_2\cdot(l_{11}l_{21} + l_{12}l_{22} + \cdots l_{1r}\cdot l_{2r} + \cdots)$$
$$= R_p R_q (\Sigma\, l_{pr}\, l_{qr}) \qquad \text{(III)}$$

It is easy now to verify that opposite sides of the simplex are orthogonal. Take OP_1 and any side opposite $P_r P_s$. $(r \neq 1\ s \neq 1.)$ The direction cosines of $P_r P_s$ are proportional to

$$(R_r l_{r1} - R_s l_{s1}), \quad (R_r l_{r2} - R_s l_{s2}), \text{ etc.}$$

Then for orthogonality we must have

$$l_{11}(R_r l_{r1} - R_s l_{s1}) + l_{12}(R_r l_{r2} - R_s l_{s2}) + \qquad = 0$$

i.e., $\sum_{k=1}^{n} R_r (l_{1k} l_{rk}) = \Sigma\, R_s (l_{1k} l_{sk})$, which is verified by equation (II).

Before we go to the discussion of the uniqueness of the orthogonal simplex having the specified volumes for its faces (the discussion being only for 3 dimensions), I should like here to obtain an expression for the upper bound of all the volumes of simplexes in n-dimensions.

Lemma :

$$\text{Let} \quad \Delta_n = \begin{vmatrix} 1 & \cos \xi_{12} & \cos \xi_{13} \cdots \cos \xi_{1n} \\ \cos \xi_{21} & 1 & \\ \cos \xi_{n1} & & 1 \end{vmatrix}$$

We will call the minor of 1st term in the 1st row of Δ_n, Δ_{n-1} which is

$$= \begin{vmatrix} 1 & \cos \xi_{23} \cdots \cos \xi_{2n} \\ \cos \xi_{23} & 1 \\ \cdot & \end{vmatrix} \text{ of order } (n-1).$$

We will call the minor of the last term in the last row of Δ_n as $\overline{\Delta}_{n-1}$

$$\text{which is} = \begin{vmatrix} 1 & \cos \xi_{12} \cdots \cos \xi_{1n-1} \\ \cos \xi_{12} & \\ \cos \xi_{1n-1} \cdots & 1 \end{vmatrix} \text{ of order } n-1.$$

We will call the minor of the 1st term in the first row of Δ_{n-1} as Δ_{n-2}

$$\text{Then} \quad \Delta_n = \begin{vmatrix} 1 - \dfrac{\Delta_{n-1}}{\Delta_{n-2}} & \cos \xi_{12} \cdots \cos \xi_{1n} \\ \cos \xi_{21} & \\ \cos \xi_{n1} & 1 \end{vmatrix} + \frac{\Delta_{n-1} \overline{\Delta}_{n-1}}{\Delta_{n-2}}.$$

Now in the first expression the minor of the last term in the last row is zero and hence it is easy to prove that it is $= -$ (square of an expression).

$$\therefore \quad \Delta_n \leqslant \frac{\Delta_{n-1} \overline{\Delta}_{n-1}}{\Delta_{n-2}}.$$

We wish to prove that

$$\Delta_n \leqslant (1 - \cos^2 \xi_{12}) \Delta_{n-1}.$$

This is obviously true when $n = 2$, we will therefore assume to be true for $\overline{n-1}$ for such determinants and prove for n by induction.

$$\text{Then} \quad \overline{\Delta}_{n-1} \leqslant (1 - \cos^2 \xi_{12}) \Delta_{n-2}.$$

$$\therefore \quad \Delta_n \leqslant (1 - \text{Cos}^2 \xi_{12}) \Delta_{n-1} \quad \text{by the above inequality.}$$

We therefore obtain $\Delta_n < \sin^2 \xi_{12} \cdots \sin^2 \xi_{23} \cdots \sin^2 \xi_{n-1 \cdot n}$.

There are nC_2 such terms. By suitably combining the terms by repeated operation of the above inequality, we get

$$\Delta_n^n \leqslant (\pi \sin^2 \xi_{rs})^2.$$

By (I) we have

$$V^{n-1} = k. \begin{vmatrix} 1 & \cos\xi_{12} & \cdots \\ \cdot & \cdot \cdot \cdot \cdot & \cdot \\ \cdot & \cdot \cdot \cdot \cdot & 1 \end{vmatrix}^{\frac{1}{2}}.$$

$$= k \cdot \Delta_n^{\frac{1}{2}}.$$

$$\therefore \quad V^{\frac{n(n-1)}{2}} = k^{\frac{n}{2}} \Delta_n^{\frac{n}{2}}.$$

$$\therefore \quad V = k^{\frac{1}{n-1}} \cdot \left(\Delta_n^{\frac{n}{2}}\right)^{\frac{1}{\frac{n(n-1)}{2}}}$$

$$\therefore \quad < k^{\frac{1}{n-1}} \{\pi \sin^2 \xi_{rs}\}^{\frac{1}{\frac{n(n-1)}{2}}}.$$

Since the geometric mean is less than the arithmetic mean

$$\therefore \quad V < k^{\frac{1}{n-1}} \left\{\frac{\Sigma \sin^2 \xi_{rs}}{N}\right\}, \qquad N = \frac{n(n-1)}{2}.$$

$$i.e. \quad < k^{\frac{1}{n-1}} \left\{1 - \frac{\Sigma \cos^2 \xi_{rs}}{N}\right\}.$$

Now $\Sigma\, \Delta_r \Delta_s \cos \xi_{rs} = \dfrac{\Delta^2_{n+1} - \sum_1^n \Delta_r^2}{2} = \Omega$ by equation (II).

$\therefore$ The smallest value of $\Sigma \cos^2\xi_{rs}$ satisfying equation (II) is given by $\dfrac{\Omega^2}{\Sigma\, \Delta_r^2 \Delta_s^2}$.

$$\therefore \quad V \leqslant \left\{\frac{(\underline{|n-1})^n}{(\underline{|n})^{n-1}} \Delta_1 \Delta_2 \cdots \Delta_n\right\}^{\frac{1}{n-1}} \left\{1 - \frac{\Omega^2}{N \cdot \Sigma\, \Delta_r^2 \Delta_s^2}\right\}$$

and in 3 dimensions we have

$$V \leqslant \left\{\frac{2}{9} \Delta_1 \Delta_2 \Delta_3\right\}^{\frac{1}{2}} \cdot \left\{1 - \frac{\left(\dfrac{\Delta_1^2 + \Delta_2^2 + \Delta_3^2 - \Delta_4^2}{2}\right)^2}{3(\Delta_1^2 \Delta_2^2 + \Delta_1^2 \Delta_3^2 + \Delta_2^2 \Delta_3^2)}\right\}.$$

In case $\Delta_{n+1}^2 = \Delta_1^2 + \Delta_2^2 + \Delta_n^2$, we can easily prove that the upper bound is actually

$$\left\{\frac{(\underline{|n-1})^n}{(\underline{|n})^{n-1}} \Delta_1 \cdots \Delta_n\right\}^{\frac{1}{n-1}}.$$

§ 6. *Uniqueness of the Orthogonal Tetrahedron.*

Now we will prove the uniqueness of the orthogonal tetrahedron in *3 dimensions* (satisfying the given condition of having for the areas of its faces the given numbers).

Let us call the angle between OP_2 and OP_3 as λ, OP_3 and OP_1 as μ and OP_1 and OP_2 as ν. Then the orthogonality condition will reduce to the following by easy calculation,

$$\frac{\tan\lambda}{\Delta_1} = \frac{\tan\mu}{\Delta_2} = \frac{\tan\nu}{\Delta_3} = p \tag{IV}$$

and the equation (II) can be put as

$$\Delta_1\Delta_2\cos\xi_{12} + \cdots = \Omega = \frac{\Delta_4^2 - \Delta_1^2 - \Delta_2^2 - \Delta_3^2}{2}.$$

Now ξ_{12} is the angle between the normals to the plane OP_2P_3 and OP_1P_3. Therefore the actual dihedral angle will be $\pi - \xi_{12}$.

Let us call $\quad \pi - \xi_{12} = C,\ \pi - \xi_{23} = A,\ \pi - \xi_{31} = B.$

Then the spherical Δ with sides λ, μ, ν will have angles A, B, C. Equation (II) becomes

$$\frac{\cos A}{\Delta_1} + \frac{\cos B}{\Delta_2} + \frac{\cos C}{\Delta_3} = -\frac{\Omega}{\Delta_1\Delta_2\Delta_3}.$$

Now $\quad \cos A = \dfrac{\cos\lambda - \cos\mu\cos\nu}{\sin\mu\sin\nu}$

$\operatorname{Tan}\lambda = p\,\Delta_1. \quad \therefore \quad \cos\lambda = \dfrac{1}{\sqrt{1+p^2\Delta_1^2}},\ \sin\lambda = \dfrac{p\Delta_1}{\sqrt{1+p^2\Delta_1^2}}$, etc.

the radical having the same sign for the 3 angles by equation (IV). Substituting for cos A, etc., we have

$$\Sigma \frac{\dfrac{1}{\sqrt{1+p^2\Delta_1^2}} - \dfrac{1}{\sqrt{1+p^2\Delta_2^2}\,\sqrt{1+p^2\Delta_3^2}}}{\dfrac{p^2\Delta_2\Delta_3\Delta_1}{\sqrt{1+p^2\Delta_2^2}\cdot\sqrt{1+p^2\Delta_3^2}}} = \frac{-\Omega}{\Delta_1\Delta_2\Delta_3}$$

$$i.e.,\quad \Sigma\left\{\frac{(1+p^2\Delta_2^2)(1+p^2\Delta_3^2)}{1+p^2\Delta_1^2}\right\}^{\frac{1}{2}} = 3 - p^2\Omega$$

$$\therefore \quad \{\Sigma(1+p^2\Delta_2^2)(1+p^2\Delta_3^2)\}^2 = (3-p^2\Omega)^2 \prod_{r=1}^{3}(1+p^2\Delta_r^2).$$

We will now call
$$\sum_1^3 \Delta_r^2 = A_1$$
$$\sum_1^3 \Delta_r^2\,\Delta_s^2 = A_2$$
$$\Delta_1^2\,\Delta_2^2\,\Delta_3^2 = A_3.$$

$\therefore$ L.H.S. $= (3 + 2A_1p^2 + A_2p^4)^2$

R.H.S. $= (9 - 6p^2\Omega + p^4\Omega^2)(1 + p^2A_1 + p^4A_2 + p^6A_3)$.

After simplification and putting $p^2 = t$ and removing the root $t = p^2 = 0$, we have R.H.S. − L.H.S. =

$$A_3\,\Omega^2 t^4 + (A_2\Omega^2 - 6\Omega\,A_3 - A_2^2)\,t^3 + (A_1\,\Omega^2 - 6\,\Omega\,A_2 + 9A_3 - 4\,A_1\,A_2)\,t^2 + (\Omega^2 - 6A_1\,\Omega + 3A_2 - 4A_1^2 - A)\,t - (6\,\Omega + 3A_1)$$

$$= a_0\,t^4 + a_1\,t^3 + a_2\,t^2 + a^3\,t + a_4.$$

$$\Omega = \frac{\Delta_4^2 - \Delta_1^2 - \Delta_2^2 - \Delta_3^2}{2} = \frac{\Delta_4^2 - A_1}{2}.$$

$$\therefore\quad a_4 = -\{3\,(\Delta_2^4 - A_1) + 3A_1\} = -3\,\Delta_4^2$$

$$a_3 = \Omega^2 + 3A_2 - 4A_1^2 - 3A_1\,(\Delta_4^2 - A_1) = \Omega^2 - A_1^2 + 3\,(A_2 - A_1\,\Delta_4^2).$$

Now, $\Delta_1 \leqslant \Delta_2 \leqslant \Delta_3 \leqslant \Delta_4 < \Delta_1 + \Delta_2 + \Delta_3$.

$$\therefore\quad \Delta_4^2 \leqslant 3\,(\Delta_1^2 + \Delta_2^2 + \Delta_3^2)$$

Now
$$\frac{\Delta_4^2 - \Delta_1^2 - \Delta_2^2 - \Delta_3^2}{2} \leqslant \Delta_1^2 + \Delta_2^2 + \Delta_3^2 = A_1.$$

In case $\Delta_4^2 \leqslant A_1$. Then $|\Omega| < A_1$

In case $\Delta_4^2 \geqslant A_1$. Then by above $|\Omega| \leqslant A_1$

$$\therefore\quad \Omega^2 - A_1^2 \leqslant 0$$

Since $\Delta_4 \geqslant \Delta_3 \geqslant \Delta_2 \geqslant \Delta_1$

$$A_2 \leqslant A_1.\ \Delta_4^2.$$

$$\therefore\quad a_3 \text{ is also} \leqslant 0$$

$$a_2 = A_1\,\Omega^2 + 9A_3 - 4\,A_1A_2 - 6A_2\,\Omega.$$

Case I.—Let $\Omega \leqslant 0$, *i.e.*, $\Delta_4^2 \leqslant \Delta_1^2 + \Delta_2^2 + \Delta_3^2$

Then $a_2 = A_1\,\Omega^2 + 9A_3 - 4A_1A_2 - 3A_2\,(A_1 - \Delta_4^2)$

$= A_1\,(\Omega^2 - A_2) + 3\,(3A_3 - A_2\,\Delta_4^2)$.

Since $\Delta_4 \geqslant \Delta_3 \geqslant \Delta_2 \geqslant \Delta_1$, $3A_3 < A_2\,\Delta_4^2$

and
$$4\,(\Omega^2 - A_2) = (\Delta_1^2 + \Delta_2^2 + \Delta_3^2 - \Delta_4^2)^2 - 4\sum_1^3 \Delta_2^2\,\Delta_3^2$$

$$= [\Delta_1^4 + \Delta_2^4 + \Delta_3^4 - \Delta_4^2\,(\Delta_1^2 + \Delta_2^2 + \Delta_3^2)] + \Delta_4^2\,(\Delta_4^2 - \Delta_1^2 - \Delta_2^2 - \Delta_3^2) - 2\sum_1^3 \Delta_2^2\,\Delta_3^2$$

Since $\Delta_4 \geqslant \Delta_3 \geqslant \Delta_2 \geqslant \Delta_1$ and since $\Omega \leqslant 0$, each of the expressions on the R.H.S. is negative.

$$\therefore\quad a_2 \leqslant 0.$$

Case II.—$\Omega > 0$

$$a_2 = A_1\left(\Omega - \frac{3A_2}{A_1}\right)^2 - \left\{4A_2 - \frac{9A_3}{A_1}\right\} - \frac{9A_2^2}{A_1^2}.$$

We notice that $3A_1 A_2 > 9A_3$. (Easily provable.)

$\therefore$ Since $\Omega > 0$

If a_2 is to be > 0.

Then Ω should be $> \dfrac{3A_2}{A_1} + \sqrt{\dfrac{9A_2^2}{A_1^2} + A_2 + \left(3A_2 - \dfrac{9A_3}{A_1}\right)}$

Consider $a_1 = A_2 \Omega^2 - 6A_3 \Omega - A_2^2$

$$= A_2\left(\Omega - \frac{3A_3}{A_2}\right)^2 - A_2 - \frac{9A_3^2}{A_2}\bigg)$$

$$= A_2\left\{\Omega - \frac{3A_3}{A_2} - \sqrt{A_2 + \frac{9A_3^2}{A_2^2}}\right\}\left\{\Omega - \frac{3A_3}{A_2} + \sqrt{A_2 + \frac{9A_3^2}{A_2^2}}\right\}$$

Now $\dfrac{3A_2}{A_1} + \sqrt{\dfrac{9A_2^2}{A_1^2} + A_2 + \left(3A_2 - \dfrac{9A_3}{A_1}\right)}$

$$> \frac{3A_2}{A_1} + \sqrt{A_2 + \frac{9A_2^2}{A_1^2}}.$$

Since $\dfrac{A_2}{A_1} > \dfrac{3A_3}{A_2}$ and much more so $\dfrac{A_3}{A_2}$.

(easily provable)

$$\therefore \quad \frac{3A_2}{A_1} + \sqrt{A_2 + \frac{9A_2^2}{A_1^2}} > \frac{3A_3}{A_2} + \sqrt{A_2 + \frac{9A_3^2}{A_2^2}}.$$

$\therefore$ Whenever under Case II ($\Omega > 0$)

$a_2 \geqslant 0$

then $a_1 \geqslant 0$

a_0 being > 0. The coefficients a_0, a_1, a_2, a_3, a_4 will have the following signs Case I + ? — — —. Whatever be the sign of a_1, by Descartes' rule the equation will have exactly one positive root.

Case II.—$a_2 > 0$. Then the coefficients a_0 a_1 a_2 a_3 a_4 will have the following signs + + + — —.
We therefore see that the equation has in this case also one and only one positive root.

We have therefore established that in all cases the equation will have one positive root only. Hence the uniqueness of the orthogonal tetrahedron having the maximum volume and the given numbers for the areas of its faces is proved.

The same method of investigation for proving the uniqueness or otherwise of the maximum orthogonal tetrahedron cannot be followed for the n-dimensional case.