EXACT SOLUTION OF THE EQUATIONS OF THE GENERAL CASCADE THEORY WITH COLLISION LOSS

BY K. S. K. IYENGAR

Received January 29, 1942

1. *Introduction*

THE Cascade Theory was first put forward by Bhabha and Heitler (1937) and Carlson and Openheimer (1937). It explains the behaviour of the soft component of cosmic radiation. For purposes of calculation, the above authors made approximations to the cross-sections given by Bethe and Heitler (1934) for radiation loss by electrons and pair-creation by quanta. Landau and Rumer (1938) have given the exact solution of the problem, when the cross-sections for radiation loss and pair-creation are given the limiting forms they assume for high energies, *i.e.*, when screening is complete. None of these authors has taken proper account of collision loss. Snyder (1938), taking a broad approximation for the cross-sections for radiation loss and pair-creation, and Serber (1938), with those for complete screening, have attempted to take account of collison loss quantitatively but their solutions do not satisfy the boundary conditions, either for the case of the electron-started cascade, or for the case of the photon-started one. Indeed, the proof supplied by these two authors that their solutions satisfy the boundary conditions approximately is not valid and, in view of this, the reliability of their numerical results is seriously open to doubt.

A solution for the case of complete screening, also taking into account collision loss and exactly satisfying the boundary conditions has been given by Bhabha[1] and Chakrabarty in a paper to be shortly published. In a recent paper (1941) Corben has given an approximate solution for the general problem, allowing for incomplete screening, by making empirical approximations to the Bethe-Heitler cross-sections and also taking collision loss into account. His method consists in transforming the original equations into a difference-differential equation, of the third order, of a very complicated

[1] I am much indebted to Dr. Bhabha for kindly showing me the manuscript of this paper.

type, which is at least as difficult to solve exactly as the original equations. An approximate solution is then obtained for this difference—differential equation, which leads to his approximate solution for the Cascade Problem. As remarked by Corben himself, his method does not lead to even a formal expression for the exact solution of the Cascade Problem (with his approximations for the cross-sections). Moreover, we have no exact measure, anywhere in his procedure, for the degree of accuracy of his solution, so that, although his differential equations of the cascade may be considered to be a slight extension of the form taken by the previous workers, his approximations cannot be shown to be any better than theirs. Also, his mode of taking collision loss into account, based on the method of Snyder and Serber, suffers from the same serious defects as theirs.

Summarising, the present position may be stated as follows:—(*a*) There has been no exact solution of the General Problem of the Cascade Theory in which the exact Bethe-Heitler cross-sections for pair-creation and radiation loss are used and collision loss is taken into account. (*b*) In the case of complete screening with collision loss, Bhabha and Chakrabarty have given a solution satisfying the boundary conditions exactly. While their solution can be rigorously proved to be exact, as I have indicated in a note in section 7, Bhabha and Chakrabarty's procedure for proving their solution is purely formal. However, their solution gives by taking the first term only a powerful approximation and hence is of great value for numerical calculation. As already remarked, the Snyder-Serber-Corben attempts are very seriously defective.

In this paper, we have solved the general problem rigorously when the exact expressions of the Bethe-Heitler cross-sections, valid over the whole range of energy, are used and collision loss is also taken into account. The solution for the particular case of complete screening is obviously deducible from that of the general case. An approximate solution for this particular case, in a form suitable for calculation, is deduced. Further, the exact solution for this particular case, satisfying the boundary conditions, is given in alternative forms. Also, Bhabha and Chakrabarty's solution is shown to be equivalent to these. It is worthy of note that the approximation yielded by using the first term only in Bhabha and Chakrabarty's solution agrees to a high degree of accuracy with our approximations of the same case, derived from the solution of the general problem.

In section 2, a brief statement is given of the general problem of the Cascade Theory, with collision loss and with the exact Bethe-Heitler cross-sections and the form of its solution. This solution is rigorously established in section 5. In section 3, we state briefly:—(i) the approximate solution for the particular case of complete screening derived from the solution of the general problem, in the form of a contour integral (ii) the exact solution for the same case in two equivalent forms. These are rigorously established in sections 6 and 7. In section 4 are given simple, analytical expressions approximating to the functions[2] which are solutions of the particular case of complete screening and with no collision loss. These are new and are established in App. 1 and 2.

2. *Statement of the General Problem and its Solution*

Let P (E, t) dE, Q (E, t) dE give the number of particles and photons in the Cascade Process in the energy range (E, E + dE) at a depth t, from the beginning of the layer. They satisfy the boundary conditions

$$P(E, 0) = \delta(E - E_0),\ Q(E, 0) = 0 \tag{1}$$

corresponding to a cascade started by an electron of energy E_0.

As shown by Bhabha and Chakrabarty, the differential equations of the General Cascade Theory with collision loss are

$$\frac{\partial P}{\partial t} - \beta\frac{\partial P}{\partial E} = 2\int_E^\infty \frac{R(E, U)}{U} . Q(U, t)\, dU$$

$$+ \left[\int_{E(1+\delta)}^\infty R(U, U - E) . \left(\frac{U - E}{U^2}\right) . P(U, t) . dU\right.$$

$$\left. - P(E, t) . \int_{E\delta}^{E} \frac{R(E, U)}{E^2} . U\, dU\right]_{\delta \to 0} \tag{2a}$$

and

$$\frac{\partial Q}{\partial t} + D(E) . Q = \int_E^\infty \frac{R(U, E)}{U^2} E . P(U, t) . dU. \tag{2b}$$

Here,

$$R(E, U) = \left(1 - \frac{4}{3}\frac{E}{U} + \frac{4}{3}\frac{E^2}{U^2}\right)(x_1 + x_2) - x_2; \tag{3a}$$

---

[2] The Mellin Transforms of these functions were first given by Landau and Rumer.

$$\left.\begin{aligned}
&x_1(\bar{\rho}),\ x_2(\bar{\rho}) \text{ are monotonically continuous differentiable functions defined in } 0 \leqslant \bar{\rho} \leqslant \infty.\\
&x_1(\bar{\rho}) \to 1,\ x_2(\bar{\rho}) \to \tfrac{3}{4}\,\alpha, \text{ as } \bar{\rho} \to 0,\\
&x_1(\bar{\rho}) \text{ and } x_2(\bar{\rho}) \to 0, \text{ as } \bar{\rho} \to \infty;\\
&\text{and}\\
&\bar{\rho} \equiv \frac{k.\,U}{E.\,|U-E|} \text{ with } k \equiv 100.\frac{mc^2}{Z^{\frac{1}{3}}};\\
&\text{and}\\
&\alpha = \frac{1}{9\log 183\ Z^{-1/3}},
\end{aligned}\right\} \tag{3b}$$

$$D(E) = \int_0^E \frac{R(U, E)}{E}\, dU \tag{3c}$$

Eliminating Q from (2*a*) and (2*b*), they become

$$\frac{\partial P}{\partial t} - \beta\frac{\partial P}{\partial E} = \int_0^t dt_1 \int_E^\infty \phi_1(V, E, t_1 - t).P(V, t_1).\,dV$$
$$+ \left[\int_{E(1+\delta)}^\infty \phi_2(V, E)\ \underset{\delta\to 0}{P}(V, t)\, dV - P(E, t)\int_{E\delta}^E \phi_3(E, V)\, dV\right], \tag{4}$$

where,

$$\left.\begin{aligned}
&\phi_1 = \frac{2}{V^2}\int_E^V R(E, U).\,R(V, U),\ e^{D(U)\cdot(t_1-t)}.\,dU,\\
&\phi_2 = R(V, V-E).\left(\frac{V-E}{V^2}\right),\\
&\phi_3 = \frac{R(E, V).\,V,}{E^2},\\
&\text{and}\\
&D(U) = \int_0^U \frac{R(V, U)}{U}\, dV.
\end{aligned}\right\} \tag{5}$$

In the case of complete screening, given by $\bar{\rho} \to 0$, the equation (4) takes the form

$$\frac{\partial P}{\partial t}-\beta\frac{\partial P}{\partial E}=\int_0^t dt_1\int_E^\infty \bar{\phi}_1(V, E, t_1-t).P(V, t_1).dV$$

$$+\left[\int_{E(1+\delta)}^\infty \bar{\phi}_2(V, E).P(V, t)\,dV-P(E, t).\int_{E\delta}^E \bar{\phi}_3(E, V)\,dV\right] \quad (6)$$

where

$$\left.\begin{aligned}
&\bar{\phi}_1(V, E, t_1-t)=\frac{2}{V^2}e^{D(t_1-t)}.\int_E^V [U^2+\alpha'(E^2-EU)]\,[U^2+\alpha'(V^2-VU)].\frac{dU}{U^4}.\\
&\bar{\phi}_2(V, E)=\frac{(V-E)^2+\alpha' VE}{V^2(V-E)},\\
&\bar{\phi}_3(V, E)=\frac{V^2+\alpha'(E-V)^!.E}{E^2V},\\
&\text{and}\\
&D=\frac{7}{9}-\frac{\alpha}{6}\\
&\text{with}\\
&\alpha'\equiv\frac{4}{3}+\alpha.
\end{aligned}\right\} \quad (7)$$

Let S (E, t)[3] be the solution of (6) when $\beta=0$, with the initial condition

$$S(E, 0)=\delta(E-E_0); \quad (8a)$$

let S_1 (E, t)[3] be the solution of the same equation, with the initial condition

$$S_1(E, 0)=\delta(E-1); \quad (8b)$$

and let

$$\mathscr{P}(E, t)=P(E-\beta t, t). \quad (8c)$$

Then, in section 5, I transform equation (4) into

$$\mathscr{P}(E, a)=S(E, a)+\int_0^a dt\int_E^\infty S_1\left(\frac{E}{U}, a-t\right).F(\mathscr{P}).\frac{dU}{U}, \quad (9)$$

where F is a functional involving $\mathscr{P}$ and other known functions, and show that the exact solution of (9) is given by

$$\mathscr{P}(E, a)=\sum_0^\infty Z_r(E, a), \quad (10a)$$

[3] The exact solution of (6) with $\beta=0$ has been given by Bhabha and Chakrabarty and by Landau and Rumer (*Proceedings of the Royal Society*, 1938).

where

$$Z_0 = S(E, a),\ Z_r = \int_0^a dt \int_E^\infty S_1\left(\frac{E}{U}, a-t\right).\ F(Z_{r-1}).\frac{dU}{U} \tag{10b}$$

for $r > 0$.

(10a) is obviously the formal solution of (9) and its exactness is established by proving the term by term integrability in (9) of $\sum_0^\infty Z_r$ in $0 \leqslant a < \frac{E_0}{\beta}$ and $\beta a < E \leqslant E_0$, so that[4]

$$P(E, a) = \sum_0^\infty Z_r(E + \beta a, a). \tag{11}$$

3. *Results for the Case of Complete Screening with Collision Loss*

(*a*) In the particular case of complete screening, *viz.*, of equation (6), the exact solution is a particular case of (9). Neglecting terms of order $\beta^2 a^3$, it is shown in section 6 that equation (6) is satisfied by

$$P(E, a) = \frac{1}{2\pi i E_0}\int_{\sigma - i\infty}^{\sigma + i\infty}\left(\frac{E_0}{E+\beta a}\right)^s \left\{f_1(s, a) + \frac{\beta a(s-1)}{E_0} f_1(s-1, a) - \frac{\beta(s-1)}{E_0}\int_0^a f_1(s-1, a).f_1(s, a-t)\,dt\right\} ds, \tag{12}$$

where

$$f_1(s, a) = \int_0^\infty S_1(E, a).\ E^{s-1}.\ dE, \text{ and } \sigma > 2,$$

$S_1(E, a)$ being defined in (8b)—(as the solution of (6) with $\beta = 0$). Between this approximation, *viz.*, (12), and Bhabha and Chakrabarty's approximation with their first term, there is complete parallelism to the same degree of accuracy.

(*b*) In this particular case (*i.e.*, of complete screening), the exact solution satisfying the boundary conditions, *viz.*,

$$P(E, 0) = \delta(E - E_0)$$

and

$$Q(E, 0) = 0$$

[4] That (10 a) is the solution of (9) is established by the classical method of dominant series in section 5 and Appendix 4.

can be put in the following equivalent forms:—

$$\text{(i) } P(E, t) = \frac{1}{2\pi i E_0} \int_C \left\{\frac{E_0}{E+\beta t}\right\}^s \left\{\sum_0^\infty \left(\frac{\beta}{E+\beta t}\right)^r \cdot \frac{\lfloor s+r}{\lfloor s} \cdot \theta_r(s, t)\right\} ds.$$

$$\text{(ii) } P(E, t) = \frac{1}{2\pi i E_0} \int_C \left(\frac{E_0}{E}\right)^s \left\{\sum_0^\infty \left(-\frac{\beta}{E}\right)^r \cdot \frac{\lfloor s+r}{\lfloor s} \cdot \psi_r(s, t)\right\} ds.$$

The equivalence of the above to Bhabha and Chakrabarty's solution is shewn in a note in 7.

[θ_r, ψ_r, C, etc., are defined in 7.]

4. *Approximations for the Case of Complete Screening neglecting Collision Loss*

In order to establish the validity of solution (10*a*), it was necessary to establish, abschatzungen for S (E, *t*) [defined in 8(*a*)] throughout its range of existence, as the approximations given by previous workers are valid only for certain regions. During the investigation, I obtained some new interesting results, which are established in App. 1 & 2, and are given herewith.

Let

$$\log \frac{E_0}{E} = y.$$

Then, for $0 \leqslant y \leqslant y_0$ and $t > 0$, where y_0 is some positive constant,

$$S(E, t) . E_0 = \frac{e^{-a''t}}{\lfloor a't} (y^{a't-1}) \left(1 - \frac{y}{2}\right) + e^{-k_2 t} \{M_1 y^2 + M_2 y^{1+a't}\}$$
$$+ \frac{2}{a'^2 D_3} e^{-Dt} \int_0^\infty \frac{1 - (yxD_3 + 1) e^{-D_3 yx}}{x^2 (\pi^2 + \log^2 x)} dx \tag{13}$$

where M_1 and M_2 are bounded functions of y and t in the region concerned, and k_2 a positive constant.

a' and D are as in (7) and

$a'' = a'(\gamma - 1) + \frac{1}{2}$, γ being the Eulerian constant,

$D_3 = e^{D/a'}$

For very small values of y, the last term of (13) is equal to

$$\frac{2}{a'^2} \cdot e^{-Dt} \cdot \frac{y}{\log^2 y} \cdot (1 + \epsilon); \tag{13$_1$}$$

Also,

$$S(E, t) . E_0 = y^{a't-1} . \frac{e^{-a''t}}{\lfloor a't} \left(1 - \frac{y}{2}\right) + (K - \epsilon(y)) y . t.$$
$$+ y\, t \{M_3 . y + M_4 . t\} \tag{14}$$

where

$$K = \frac{2}{\alpha'} \int_0^\infty \frac{dx}{x(\pi^2 + \log^2 x)},$$

E (y) being negligible for small y, and M_3 and M_4 being bounded functions of y and t in the region concerned. Formula (14) gives one a good picture of S (E, t) near ($y = 0$, $t = 0$) and formula (13) does the same for all t— (of course, for $y \leqslant y_0$).

When t is fixed and y is large,

$$S(E, t). E_0 = \frac{(t\,\rho)^{1/3}(1 + \epsilon(y))}{2^{4/3} . 3^{1/2} . \pi^{1/2} . y^{5/6}} \cdot e^{\{y + (3/2^{2/3})(t\,\rho)^{2/3} . y^{1/3}\}}, \tag{13}$$

where

$$\rho = \sqrt{(2\alpha' D)} \text{ and } \epsilon(y) \to 0 \text{ as } y \to \infty$$

Note.—On looking into Bhabha and Chakrabarty's paper, I find that (15) which has been obtained by the method of residues by me is contained in (32) of their paper, which they obtain by the method of steepest descent and there is an overlapping of (33) of their paper with the first term of (13). However, (32) of their paper is definitely invalid in two cases, *viz*., when, (i) y is large and $t \to 0$, (ii) y is very small for any t, as the authors themselves are aware. For the case (i), Bhabha and Chakrabarty themselves have given another approximation which is valid. For the case (ii), (13) and $(13)_1$ give the correct approximation. From (15), it is obvious that the omission of the non-exponential term leads to very serious errors when $\frac{y}{t}$ is fairly large and hence Landau and Rumer's calculations are very seriously defective when $\frac{y}{t}$ is large.

5. *Exact Solution of Equation* (4) *of Section II*

Consider now the equation (4) of the General Cascade Theory and the functions that occur in it.

Let

$$x_1 + x_2 = x_3.$$

Then ϕ_2 (V, E) can be written as

$$\left[\phi_2(V, E) = \frac{V - E}{V^2}\left\{x_3\left(1 - \frac{4}{3}\frac{V}{V - E}\right) - x_2\right\} + \frac{4}{3}\,\frac{\left\{x_3\left(k . \frac{V - E}{VE}\right) - x_3(0)\right\}}{V - E}\right] + \frac{4}{3}\frac{x_3(0)}{V - E}$$

$$= \xi_2(V, E) + \frac{\alpha'}{V - E} \tag{16}$$

where

$$\alpha' = \frac{4}{3}x_3(0) = \frac{4}{3} + \alpha \text{ by } (3b).$$

Similarly

ϕ_3 (V, E)

$$= \frac{V}{E_2} \left\{ \left(1 - \frac{4}{3}\frac{E}{V}\right) x_3 \left(\frac{kV}{E(E-V)} \right) - x_2 \left(\frac{kV}{E(E-V)} \right) \right\} + \frac{4}{3} \frac{\left\{ x_3 \left(\frac{kV}{E(E-V)} \right) - x_3(0) \right\}}{V} + \frac{4}{3} \frac{x_3(0)}{V}$$

$$= \xi_3 (V, E) + \frac{\alpha'}{V}. \tag{17}$$

Here ξ_2 and ξ_3 are continuous functions in $[0 < \bar{E} \leqslant E,\ 0 < \bar{E} \leqslant V]$, where $\bar{E}$ is some positive constant $< E_0$.

Equation (4) can now be put as

$$\frac{\partial P}{\partial t} - \beta \frac{\partial P}{\partial E} = \phi_4 (E) . P (E, t) + \alpha' \left[\int_{E(1+\delta)}^{\infty} \frac{P(U, t)}{U-E}\, dU + \log_{\delta \to 0} \delta . P (E, t) \right] + \int_{E}^{\infty} \xi_2 (U, E)\, P (U, t)\, dU$$

$$+ \int_0^t dt_1 \int_E^{\infty} \phi_1 (U, E, t_1 - t)\, P (U, t_1)\, dU, \tag{18}$$

where.

$$\phi_4 (E) = - \int_0^E \xi_3 (U, E)\, dU.$$

Let

$$\mathscr{P} (E, t) = P (E - \beta t, t).$$

Then, by putting $E - \beta t$ in place of E, it can be easily shown that equation (18) becomes

$$\frac{\partial \mathscr{P}}{\partial t} = \left\{ \alpha' \log \frac{E}{E - \beta t} + \phi_4 (E - \beta t) \right\} \mathscr{P} (E, t) + \alpha' \left[\int_{E(1+\delta)}^{\infty} \frac{\mathscr{P}(U, t)}{U-E}\, dU + \log_{(\delta \to 0)} \delta, \mathscr{P} (E, t) \right]$$

$$+ \int_E^{\infty} \xi_2 (U - \beta t, E - \beta t) . \mathscr{P} (U, t)\, dU$$

$$+ \int_0^t dt_1 \int_{E - \beta (t - t_1)}^{\infty} \phi_1 (U - \beta t_1, E - \beta t, t_1 - t)\, \mathscr{P} (U, t_1)\, dU. \tag{19}$$

It is to be noted here that the corresponding differential equation of the Cascade Theory for complete screening, when $\beta = 0$, derivable from (6) can be put as follows

$$\frac{\partial P}{\partial t} = \left(\alpha' - \frac{1}{2}\right) P(E, t) + \alpha' \left[\int_{E(1+\delta)}^{\infty} \frac{P(U, t)}{U - E} dU + \underset{\delta \to 0}{\log \delta}. P(E, t) \right]$$
$$+ \int_{E}^{\infty} \left\{ \frac{1-\alpha'}{U} - \frac{E}{U^2} \right\} \cdot P(U, t).\, dU + \int_{0}^{t} dt_1 \int_{E}^{\infty} \bar{\phi}_1 (U, E, t_1 - t).\, P(U, t_1).\, dU \quad (20)$$

[$\bar{\phi}_1$ is defined in (7)]

The exact solution of (20) has been given by Landau and Rumer, and also by Bhabha and Chakrabarty.

Now (19) can be written as

$$\frac{\partial \mathscr{P}}{\partial t} + \left(\frac{1}{2} - \alpha'\right) . \mathscr{P}(E, t) - \alpha' \left[\int_{E(1+\delta)}^{\infty} \frac{\mathscr{P}(U, t)}{U - E} dU + \underset{\delta \to 0}{\log \delta} . \mathscr{P}(E, t) \right]$$
$$- \int_{E}^{\infty} \left\{ \frac{1-\alpha'}{U} - \frac{E}{U^2} \right\} \cdot \mathscr{P}(U, t).\, dU - \int_{0}^{t} dt_1 \int_{E}^{\infty} \bar{\phi}_1 (U, E, t_1 - t); \mathscr{P}(U, t_1), dU$$
$$= \left[\alpha' \log \frac{E}{E - \beta t} + \phi_4 (E - \beta t) + \left(\frac{1}{2} - \alpha'\right) \right] \cdot \mathscr{P}(E, t)$$
$$+ \int_{E}^{\infty} \left\{ \xi_2 (U - \beta t, E - \beta t) - \frac{1-\alpha'}{U} + \frac{E}{U^2} \right\} \mathscr{P}(U, t).\, dU$$
$$+ \int_{0}^{t} dt_1 \int_{E - \beta(t - t_1)}^{E} \left\{ \phi_1 (U - \beta t_1, E - \beta t, t_1 - t) \right\} \mathscr{P}(U, t_1).\, dU$$
$$+ \int_{0}^{t} dt_1 \int_{E}^{\infty} \left\{ \phi_1 (U - \beta t_1, E - \beta t, t_1 - t) - \bar{\phi}_1 (U, E, t_1 - t) \right\} . \mathscr{P}(U, t_1) dU \quad (21)$$

Let L (E, t) stand for the left-hand side of equation (21), and F $(\mathscr{P})$ stand for the right-hand side of (21), F $(\mathscr{P})$ being obviously a functional in $\mathscr{P}$ and other known functions. Let S (E, t) be the solution of L $(E, t) = 0$ with initial condition S $(E, 0) = \delta (E - E_0)$ and $S_1 (E, t)$ be also the solution of $\alpha (E, t) = 0$ with initial condition $S_1 (E, 0) = \delta (E - 1)$. Then, I have established in App. 3,

$$\int_0^a dt \int_E^\infty S_1\left(\frac{E}{U}, a-t\right)\cdot L(U, t). \cdot \frac{dU}{U} = \mathscr{P}(E, a) - S(E, a), \tag{22}$$

so that equation (21) becomes

$$\mathscr{P}(E, a) = S(E, a) + \int_0^a dt \int_E^\infty S_1\left(\frac{E}{U}, a-t\right)\cdot F(\mathscr{P}) \frac{dU}{U}. \tag{23}$$

Let

$$Z_0 = S(E, a),\ Z_r = \int_0^a dt \int_E^\infty S_1\left(\frac{E}{U}, a-t\right)\cdot F(z_{r-1}). \frac{dU}{U}. \tag{24}$$

It is obvious, as pointed out in section 2, that

$$\sum_0^\infty Z_r(E, a) \tag{25}$$

is a formal solution of (23). The term by term integrability of $\sum\limits_0^\infty Z_r(E, a)$ in equation (23) in the range $\beta a < E \leqslant E_0$ and $0 \leqslant a < \frac{E_0}{\beta}$ is established by the classical method of dominant series in App. 4, thus establishing (25) as the solution of (23). From (25) it follows that

$$P(E, a) = \sum_0^\infty Z_r(E + \beta a, a) \tag{26}$$

which gives the exact solution of the General Cascade Equation (4) with the initial condition

$$P(E, 0) = \delta(E - E_0).$$

Note.—Since $Z_0 = S(E, a)$ and $S(E, a) = 0$ when $E > E_0$, the upper limit infinity in the right-hand integral of (23) could be replaced by E_0. If S and S_1 were throughout continuous since $F(\mathscr{P})$ is a continuous functional in the range $\beta\, a < E \leqslant E_0$, there would be no difficulty in establishing the exactness of the solution $\sum\limits_0^\infty z_r$ in $\beta\, a + \mathrm{E} \leqslant E \leqslant E_0$, where E is some positive constant. But we know that in $0 \leqslant a < \frac{1}{\alpha}$, both $S(E, a)$ and $S_1(E, a)$ have an infinite discontinuity at $E = E_0$ and $E = 1$ respectively. Hence there is need for establishing ΣZ_r as the solution of (23), which is carried out in Appendix 4.

6. *Solution of the Cascade Problem with Collision Loss when Screening is Complete*

The complete solution of the above problem *i.e.*, of equation (6) is obviously a particular case of (26), obtained by replacing the functions ϕ of (5) in (21) by ϕ's defined in (7). But we are here interested in calculating an approximation to (26) omitting terms of order $(\beta^2 a^3)$. We proceed herewith to calculate Z_r of (26) for this case.

Let $\frac{E_0}{\beta} > a > 0$, $\beta a + \bar{E} < U, V < E_0$ and $0 \leqslant t_1 \leqslant t \leqslant a$

($\bar{E}$ some positive constant)

Consider now $\bar{\phi}_1$ of (7). Then it is easy to see that

$$\bar{\phi}_1 (V, U, t_1 - t) = e^{D(t_1 - t)} \cdot \frac{1}{V} \cdot \psi\left(\frac{U}{V}\right) \tag{27}$$

where ψ is some function of the ratio $\frac{U}{V}$;

and

$$\begin{aligned}
&\bar{\phi}_1 (V - \beta t_1, U - \beta t, t_1 - t) - \bar{\phi}_1 (V, U, t_1 - t) \\
&= e^{D(t_1 - t)} \left\{ \psi\left(\frac{U - \beta t}{V - \beta t_1}\right) \frac{1}{V - \beta t_1} - \psi\left(\frac{U}{V}\right) \frac{1}{V} \right\} \\
&= \frac{e^{D(t_1 - t)} \cdot \beta}{V^2} \left\{ t_1 \left[\psi\left(\frac{U}{V}\right) + \frac{U}{V} \psi'\left(\frac{U}{V}\right) \right] - t\, \psi'\left(\frac{U}{V}\right) \right\} + 0\,(\beta^2 t^2) \qquad (28) \\
&= \theta_1 (V, U, t_1\, t), \qquad (29)
\end{aligned}$$

$$\frac{1 - \alpha'}{V - \beta t} - \frac{V - \beta t}{(V - \beta t)^2} - \frac{1 - \alpha'}{V} + \frac{U}{V^2} = \left(2 - \alpha' - \frac{2U}{V}\right) \frac{\beta t}{V^2} + 0\,(\beta^2 t^2) \tag{30}$$

$$= \theta_2 (V, U, t), \tag{31}$$

and

$$\log \frac{U}{U - \beta t} = \frac{\beta t}{U} + 0\,(\beta^2 t^2) \tag{32}$$

Further, it is obvious that $|\theta_1|$, $|\theta_2|$ and $\log \frac{U}{U - \beta t}$ are all of order βt in this region (33)

Now, in this particular case, from (24) we have

$$\begin{aligned}
Z_r = &\int_0^a dt \int_E^\infty S_1\left(\frac{E}{U}, a - t\right) \cdot \frac{dU}{U} \cdot \Bigg[\alpha'\left(\log \frac{U}{U - \beta t_1}\right) \cdot Z_{r-1} (U, t + {} \\
&\qquad \int_U^\infty \theta_2 (V, U, t)\, Z_{r-1} (V, t)\, dt \\
&+ \int_0^t dt_1 \int_{U - \beta(t - t_1)}^U \bar{\phi}_1 (V - \beta t_1, U - \beta t, t_1 - t) . Z_{r-1} (V, t_1)\, dV \\
&+ \int_0^t dt_1 \int_U^\infty \theta_1 (V, U, t_1, t)\, Z_{r-1} (V, t_1)\, dV \Bigg] \\
= &J_1 (Z_{r-1}) + J_2 (Z_{r-1}) + J_3 (Z_{r-1}) + J_4 (Z_{r-1}). \qquad (34)
\end{aligned}$$

On account of (33), we can easily establish that in this case $Z_r = 0\ (\beta^r t^{2r})$, from considerations similar to those of App. 4, (35)

so that for our degree of approximation, we need calculate a part of Z_1 only.

Now, $Z_1 = \sum_1^4 J_r(Z_0)$. Since $\bar{\phi}_1(U, U, t_1 - t) = 0$ and since in J_3 the range of V is of order βt,

$$J_3(Z_0) = 0\ (\beta^2 a^3),$$

so that we may neglect $J_3(Z_0)$ in the calculation. Utilizing (28) and (30) and (32), we therefore obtain, correct to $(\beta^2 a^3)$,

$$\begin{aligned}\mathscr{P}(E, a) = S(E, a) + \int_0^a dt \int_E^{E_0} S_1\left(\frac{E}{U}, a - t\right)\cdot\frac{dU}{U}\cdot\Bigg[\frac{a'\beta t}{U} S(U, t) + \\ \int_0^{E_0}\left(2 - a' - \frac{2U}{V}\right)\cdot\frac{\beta t}{V^2}\cdot S(V, t)\, dV \\ + \int_0^t e^{D(t_1 - t)}\, dt_1 \int_U \left\{t_1\left[\psi\left(\frac{U}{V}\right) + \frac{U}{V}\psi'\left(\frac{U}{V}\right)\right]\right. \\ \left. - t\,\psi'\left(\frac{U}{V}\right)\right\}\cdot\frac{\beta}{V^2}\cdot S(V, t_1)\, dV\Bigg] \qquad (36)\\ = S(E, a) + I_1 + I_2 + I_3. \qquad (37)\end{aligned}$$

It is to be noted that the integrals I_1, I_2, I_3 are faltung functions and are therefore more simply expressible in terms of their Mellin transforms.

Let

$$f_1(s, t) = \int_0^\infty S_1(E, t)\, E^{s-1}\cdot dE \qquad (38a)$$

and

$$f(s, t) = \int_0^\infty S(E, t)\, E^{s-1}\, dE. \qquad (38b)$$

It can be easily proved[5] that

$$f(s, t) = E_0^{\,s-1} f_1(s, t) \qquad (39)$$

Now we shall prove that the right-hand side of equation (36) is equal to

$$\begin{aligned}\frac{1}{2\pi i E_0}\int_{\sigma - i\infty}^{\sigma + i\infty}\left(\frac{E_0}{E}\right)^s \Bigg\{f_1(s, a) + \frac{\beta a}{E_0}(s - 1) f_1(s - 1, a) \\ - \frac{\beta(s-1)}{E_0}\int_0^a f_1(s - 1, a - t) f_1(s, t)\, dt\Bigg\}\, ds \qquad (40)\end{aligned}$$

$$(\sigma > 2)$$

Proof of (40):

Let A, B, C, D[5] be as defined in App. 1. The Mellin Transform of $I_1 + I_2$ can be easily shown to be equal to

$$\beta\, E_0^{s-2} \cdot \int_0^a t \cdot f_1(s-1,\, t)\, f_1(s,\, a-t) \cdot \left\{ a' + \frac{2-a'}{s} - \frac{2}{s+1} \right\}; \tag{41}$$

and the Mellin Transform of I_3 equals

$$\beta(1-s)\, E_0^{s-2} \int_0^a dt \int_0^t e^{D(t_1-t)} \cdot t_1 (B_s C_s)\, f_1(s,\, a-t)\, f_1(s-1,\, t_1)\, dt_1$$

$$-\beta(1-s)\, E_0^{s-2} \int_0^a t\, dt \int_0^t e^{D(t_1-t)} \cdot (B_{s-1} C_{s-1})\, f_1(s,\, a-t) \cdot f_1(s-1,\, t_1)\, dt_1 \tag{42}$$

$$= \beta(1-s)\, E_0^{s-2} (J_1 - J_2) \tag{43}$$

It can be easily proved by change in the order of integration that

$$J_1 = \int_0^a f_1(s-1,\, a-t)\,(a-t)\, dt \cdot \int_0^t e^{D(t_1-t)} \cdot (B_s\, C_s) \cdot f_1(s,\, t_1)\, dt_1 \tag{44}$$

We know from Landau and Rumer's work that

$$\frac{\partial}{\partial t} f_1(s,\, t) + A_s f_1(s,\, t) = \int_0^t e^{D(t_1-t)} \cdot (B_s\, C_s) \cdot f_1(s,\, t_1)\, dt_1, \tag{45}$$

so that

$$J_1 = \int_0^a f_1(s-1,\, a-t)\,(a-t) \cdot \left\{ \frac{\partial}{\partial t} f_1(s,\, t) + A_s \cdot f_1(s,\, t) \right\} dt \tag{46}$$

$$= A_s \cdot \int_0^a f_1(s-1,\, t) \cdot f_1(s,\, a-t) \cdot t\, dt - \int_0^a f_1(s-1, t) \cdot \frac{\partial}{\partial t} f_1(s,\, a-t) \cdot t \cdot dt \tag{47}$$

In exactly the same manner, it can be proved that

$$J_2 = A_{s-1} \int_0^a f_1(s-1,\, t) \cdot f_1(s,\, a-t) \cdot t\, dt + \int_0^a f_1(s,\, a-t) \cdot \frac{\partial}{\partial t} f_1'(s-1,\, t) \cdot t\, dt \tag{48}$$

[5] See Landau and Rumer's paper, *Proceedings of the Royal Society*, 1938.

so that the Mellin Transform of I_3 is equal to

$$\beta\,(1-s)\,E_0^{s-2}\Big(A_s - A_{s-1}\Big)\int_0^a f_1\,(s-1,\,t)\,.\,f_1\,(s,\,a-t).\,t\,dt$$

$$-\int_0^a t\,\frac{d}{dt}\left\{f_1\,(s,\,a-t)\,f_1\,(s-1,\,t)\right\}dt\Big]. \qquad (49)$$

Now from the definition of A in App. 1, we get

$$(1-s)\,(A_s - A_{s-1}) = -\left\{a'\left(1-\frac{1}{s}\right)+\frac{2}{s\,(s+1)}\right\}, \qquad (50)$$

so that from (41), (49) and (50), we obtain the Mellin Transform of $I_1 + I_2 + I_3$ to be

$$\beta\,(s-1)\;E_0^{s-2}\int_0^a t\cdot\frac{d}{dt}\left\{f_1\,(s,\,a-t)\,.\,f_1\,(s-1,\,t)\right\}dt \qquad (51)$$

$$= \beta\,(s-1)\;E_0^{s-2}\left\{a.\,f_1\,(s-1,\,a) - \int_0^a f_1\,(s,\,a-t)\,f_1\,(s-1,\,t)\,dt\right\}, \qquad (52)$$

since

$$f_1\,(s,\,o) \equiv 1$$

so that,

$$\mathscr{P}\,(E,\,a) = \frac{1}{2\pi i\,E_0}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E}\right)^s\left\{f_1\,(s,\,a) + \frac{\beta a}{E_0}\,(s-1)\,f_1\,(s-1,\,a)\right.$$

$$\left.-\frac{\beta\,(s-1)}{E_0}\int_0^a f_1\,(s,\,a-t)\,f_1\,(s-1,\,t)\,dt\right\}ds. \qquad (53)$$

Replacing E by $E + \beta a$, we obtain

$$P\,(E,\,a) = \frac{1}{2\,\pi\,i\,E_0}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E+\beta a}\right)^s\left\{f_1\,(s,\,a) + \frac{\beta a}{E_0}\,(s-1)\,f_1\,(s-1,\,a)\right.$$

$$\left.-\frac{\beta\,(s-1)}{E_0}\int_0^a f_1\,(s,\,a-t)\,f_1\,(s-1,\,t)\,dt\right\}ds \qquad (54)$$

thus proving (40) and (12).

7. *The Exact Solution of the Cascade Problem with Collision Loss (when Screening is Complete) in other Forms*

In this case, the form of the cross-section functions makes it possible to give the solution which satisfies the boundary conditions exactly in the form of contour integrals where the integrand can be defined in a simple form. The approximation given in § 6, for this case, besides being highly suitable for calculation, is highly suggestive that the exact solution of this case must be expressible in the form

$$\int_C \mathrm{F}\,(\mathrm{E}+\beta t,\, t,\, s,\, \beta,\, \mathrm{E}_0)\, ds. \tag{56}$$

We shall show here that it is so and that the exact solution satisfying the boundary conditions is actually given by

$$\mathrm{P}\,(\mathrm{E},\, t)=\frac{1}{2\,\pi\, i\, \mathrm{E}_0}\int_C \left\{\frac{\mathrm{E}_0}{\mathrm{E}+\beta t}\right\}^s \left\{\sum_0^\infty \left(\frac{\beta}{\mathrm{E}+\beta t}\right)^r \frac{|\underline{s+r}}{|\underline{s}}\cdot \theta_r\,(s,\, t)\right\} ds \tag{57}$$

where

$$\theta_r\,(s,\, t)=\left\{\psi_0\,(s,\, t)\cdot \frac{t^r}{|\underline{r}}-\psi_1\,(s,\, t).\,\frac{t^{r-1}}{|\underline{r-1}}+\cdots+(-1)^r\,\psi_r\,(s,\, t)\right\} \tag{58}$$

and $\psi_r\,(s,\, t)$ is defined by

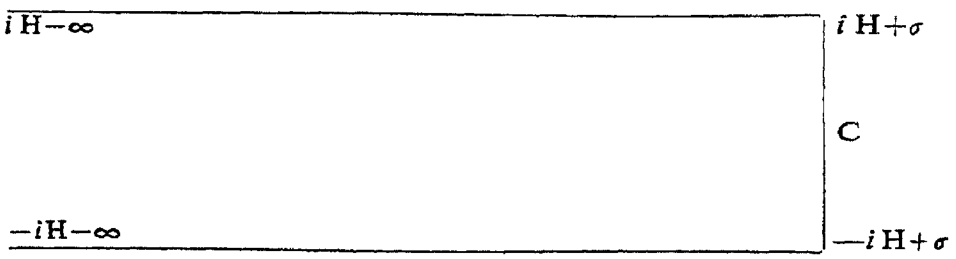
$$\left.\begin{aligned} &\psi_0\,(s,\, t)=f_1\,(s,\, t) \text{ as in } (38a) \\ &\text{and} \\ &\psi_r\,(s,\, t)=\int_0^t \psi_0\,(s,\, t-t_1)\,\psi_{r-1}(s+1,\, t_1)\, dt_1 \end{aligned}\right\} \tag{59}$$

(The contour C will be defined in the course of the argument.)

However, it is simpler to show that (57) is a solution of (6) by putting it in the following equivalent form (the equivalence is established in (65)):—

$$\mathrm{P}\,(\mathrm{E},\, t)=\frac{1}{2\,\pi\, i\,\mathrm{E}_0}\int_C \left(\frac{\mathrm{E}_0}{\mathrm{E}}\right)^s \left\{\sum_0^\infty \left(-\frac{\beta}{\mathrm{E}}\right)^r \frac{|\underline{s+r}}{|\underline{s}}\,\psi_r\,(s,\, t)\right\} ds, \tag{60}$$

where the contour C is as follows



(61)

H being fairly large and σ being > 1.

It is easy to show from the definition of ψ_r and ψ_0, that

$$\left(\frac{\partial^2}{\partial t^2}+(\mathrm{A}+\mathrm{D})_{s+r}\frac{\partial}{\partial t}+(\mathrm{AD}-\mathrm{BC})_{s+r}\right)\psi_r(s,t)=\left(\frac{\partial}{\partial t}+\mathrm{D}\right)\psi_{r-1}(s,t) \quad (62)$$

where $r \geqslant 1$ and A, B, C, D are as defined in App. 1. Making use of this, it is easy to show that for any fixed value of s, the integrand in (60) is a formal solution of the integro-differential equation (6) (*i.e.*, of the case of complete screening). Now, from App. 1, we can easily prove that

$$\psi_r(s,t) < (1+\epsilon_r)^r \cdot \frac{t^r}{\underline{|r}}, \quad (63\,a)$$

where s is a point on C.

Let $\beta t < \mathrm{E}$. (63*b*)

Then we can show that the series which forms the integrand in (60) is uniformly and absolutely convergent so that it becomes an exact solution of (6). We express the integrand of (60) as

$$\mathrm{E}_0^{s-1}\left\{\sum_0^\infty(-1)^r\cdot\beta^r\cdot\frac{\underline{|s+r}}{\underline{|s}}\cdot\frac{\psi_r(s,t)}{(\mathrm{E}+\beta t-\beta t)^{r+s}}\right\}, \quad (64)$$

and show by rearrangement that it is equal to the integrand

$$\mathrm{E}_0^{s-1}\left\{\sum_0^\infty \beta^r\cdot\frac{\underline{|s+r}}{\underline{|s}}\cdot\frac{\theta_r(s,t)}{(\mathrm{E}-\beta t)^{r+s}}\right\} \quad (65)$$

of (57). The justification for this transformation comes from the absolute convergence of the double series we get by expanding $(\mathrm{E}+\beta t-\beta t)^{-r-s}$ in (64), easily proved by making use of (63*a*) and (63*b*). In the form (65), the restriction $\beta t < \mathrm{E}$ should be replaced by the natural one $\mathrm{E} < 0$ [(65) being valid by the principle of analytic continuation for $\mathrm{E} > 0$]. Making use of the results in App. 1 in order to make the integral (57) on C uniformly convergent, it will be sufficient if

$$\log\frac{\mathrm{E}_0}{\mathrm{E}+\beta t} > \mathrm{K}\cdot\frac{\beta t}{\mathrm{E}+\beta t}, \quad (66)$$

where K is some positive constant depending upon H. So that, the integral (57) will be the exact solution of (6). It is to be noted that when t is very small, the solution (57) will be valid for a considerable region of energy. Further, since $f_1(s,0)=1$ when $t=0$

$$\mathrm{P}(\mathrm{E},0)=\delta(\mathrm{E}-\mathrm{E}_0) \quad (67)$$

so that, the boundary condition will be exactly satisfied by the solution (57).

Note.—The restrictions (i) that βt should be $< \mathrm{E}$ in order that the integrand in (60) should be convergent and (ii) that βt should be

$< \{(E+\beta t) \log (E_0/\overline{E+\beta t})\}/K$ so that the infinite integral of (57) (on the contour C) should exist are only sufficient conditions which I have deduced from rough calculations. Probably, with finer calculations, the restrictions may be liberalised. From these rough calculations, it is possible to show that when $t > \frac{1}{\alpha'}$, the infinite integral (57) will be uniformly convergent for all $0 < E < E_0$.

Note on the Bhabha-Chakrabarty Solution of the Same Problem

Just as we obtained (57) from (60), it is possible to obtain Bhabha and Chakrabarty's form from (60). Let $\frac{\psi_1(s, t)}{\psi_0(s, t)} = g\ (s, t)$. Then they have shown that for small t and large s, $g\ (s, t) \sim t$. In view of this, writing the integrand of (60) as

$$E_0^s \sum_0^\infty (-\beta)^r \cdot \frac{\underline{|s+r}}{\underline{|s}} \cdot \frac{\psi_r(s, t)}{(E+\beta g-\beta g)\ r+s} \tag{68}$$

and rearranging the expansion, we obtain,

$$\sum_0^\infty E_0^s\ \beta^r \cdot \frac{\underline{|s+r}}{\underline{|s}} \cdot \frac{\phi_r(s, t)}{(E+\beta g)^{r+s}}, \tag{69}$$

where

$$\phi_r(s, t) = \left(\psi_0(s, t) \cdot \frac{g^r}{\underline{|r}} - \psi_1(s, t) \cdot \frac{g^{r-1}}{\underline{|r-1}} + \cdots\right) \tag{70}$$

and Bhabha and Chakrabarty's form will be

$$\frac{1}{2\pi i E_0} \int \left(\frac{E_0}{E+\beta g}\right)^s \left\{\psi_0(s, t) + \frac{\beta^2\ \phi_2(s, t)}{(E+\beta g)^2} \frac{\underline{|s+2}}{\underline{|s}} + \cdots\right\} ds \tag{71}$$

I wish to thank Dr. Bhabha for suggesting the problem to me and for many helpful discussions.

Summary

A general solution is given of the integro-differential equations of the Cascade theory for given boundary conditions, using the exact expressions of the cross-sections for radiation loss and pair-creation as given by the quantum theory and also taking collision loss into account. In the particular case of complete screening, an approximate solution is derived in a form suitable for calculations. The exact solution for this case satisfying the boundary conditions is given in two other equivalent forms, and it is shown that the solution of Bhabha and Chakrabarty is completely equivalent to them.

APPENDIX I

Proof of the Results stated in § 4

Here, I propose proving (13), $(13)_1$ and (14). Following Landau and Rumer, and Bhabha and Chakrabarty, if

$$f(s, t) = \int_0^\infty S(E, t)\, E^{s-1}\, dE, \tag{72}$$

where S (E, t) is the solution of equation (6) with $\beta = 0$, then

$$\frac{\partial^2 f(s, t)}{\partial t^2} + (A + D)\frac{\partial f}{\partial t} + (AD - BC) f = 0, \tag{73}$$

where,

$$\left.\begin{aligned} A &= \alpha' \left\{ \frac{|\overline{s}'}{|\overline{s}} + \gamma - 1 + \frac{1}{s} \right\} + \frac{1}{2} - \frac{1}{s(s+1)}, \\ B &= 2\left\{ \frac{1}{s} - \frac{\alpha'}{(s+1)(s+2)} \right\}, \quad C = \frac{1}{s+1} - \frac{\alpha'}{s(s-1)}, \\ D &= \frac{7}{9} - \frac{\alpha}{6}, \quad \alpha' = \frac{4}{3} + \alpha, \quad \alpha = \frac{1}{9 \log 183\, Z^{-1/3}}. \end{aligned}\right\} \tag{74}$$

Then,

$$\begin{aligned} E_0\, S(E, t) &= \frac{1}{2\pi i}\left[\int_{\sigma - i\infty}^{\sigma + i\infty} e^{-\lambda t + ys}\, ds + \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{D - \mu}{\lambda - \mu}\left(e^{-\mu t} - e^{-\lambda t}\right) e^{ys}\, ds \right] \\ &= T_1(y, t) + T_2(y, t). \end{aligned}$$

Here, λ and μ[6] are given by $\dfrac{A + D \pm \sqrt{(A-D)^2 + 4\,BC}}{2}$ and $y = \log \dfrac{E_0}{E}$ and $\sigma > 1$

We shall here show that for $0 \leqslant y \leqslant y_0$ (some constant),

$$T_1(y, t) = y^{\,\alpha' t - 1} \left\{ \frac{e^{-\alpha'' t}}{|\overline{\alpha' t}} \left(1 - \frac{y}{2}\right) + \theta \cdot M \cdot e^{-k_2 t} \cdot t \cdot y^2 \right\}, \tag{76}$$

where $|\,\theta\,| \leqslant 1$, M and k_2 are some positive constants, and $\alpha'' = \alpha'(\gamma - 1) + \frac{1}{2}$; and that

$$T_2(y, t) = [k - \epsilon(y)]\, yt + yt\, \{M_1 y + M_2\}, \tag{77}$$

[6] The roots λ and μ, here, are the μ and λ of Bhabha and Chakrabarty.

where

$$K = \frac{2}{\alpha'} \int_0^\infty \frac{dx}{x(\pi^2 + \log^2 x)}, \quad \epsilon(y) \to 0 \text{ as } y \to 0, \tag{78}$$

and M_1 and M_2 are bounded functions of y and t.

We shall also prove that

$$T_2(y, t) = \frac{2}{\alpha'^2 D_3} e^{-Dt} \int_0^\infty \frac{1 - (y D_3 x + 1) e^{-y D_3 x}}{x^2 (\pi^2 + \log^2 x)} dx$$

$$+ M e^{-k_2 t} (\theta_1 y^{1 + \alpha' t} + \theta_2 y^2) \tag{79}$$

where $D_3 = e^{\frac{D}{\alpha'}}$, M and k_2 are some positive constants and $|\theta_1|, |\theta_2| < 1$ valid for $0 \leqslant y \leqslant y_0$ and $t > 0$; and for small y, the first term on the right-hand side of (79) is equal to

$$\frac{2}{\alpha'^2} \cdot e^{-Dt} \cdot \frac{y}{\log^2 y} \cdot [(1 + \epsilon(y)], \tag{80}$$

where $\epsilon(y) \to 0$ as $y \to 0$

Proof of (76).

Let σ be taken so great, that for $R(s) \geqslant \sigma$,

$$\left.\begin{aligned}
&BC = 0\left(\frac{1}{s^2}\right), \; A = \alpha'' + \alpha'\left(\log s + \frac{1}{2s}\right) + 0\left(\frac{1}{s^2}\right), \\
&\lambda = A + \frac{BC}{A - D} + 0\left(\frac{(BC)^2}{(A-D)^3}\right) = \alpha'' + \alpha'\left(\log s + \frac{1}{2s}\right) + 0\left(\frac{1}{s^2}\right) \\
&\mu = D - \frac{BC}{A - D} + 0\left(\frac{(BC)^2}{(A-D)^3}\right) = D + 0\left(\frac{1}{s^2 \log s}\right), \\
&\frac{D - \mu}{\lambda - \mu} = \frac{2}{\alpha'^2 s^2 \left(\log s - \frac{D}{\alpha'}\right)^2} + 0\left(\frac{1}{s^3 \log^3 s}\right).
\end{aligned}\right\} \tag{81}$$

Take σ so great that for all y in the given range $0 \leqslant y \leqslant y_0$, $\left|\log\left(\frac{s}{y}\right)\right|$ has a positive lower bound on $R(s) = \sigma$. Then, putting $y\, s = s_1$ in the integral for $T_1(y,t)$, we obtain

$$T_1(y, t) = \frac{1}{2\pi i y} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{s - t\left[\alpha'' + \alpha' \log \frac{s}{y} + \frac{\alpha'}{2} \frac{y}{s} + 0\left(\frac{y^2}{s^2}\right)\right]} \cdot ds \tag{82}$$

$$= y^{\alpha' t - 1} \cdot \frac{e^{-\alpha'' t}}{2\pi i} \int \frac{e^s}{s^{\alpha' t}} \cdot e^u \cdot ds, \tag{83}$$

where, $u = -t\left[\frac{\alpha' y}{2s} + 0\left(\frac{y^2}{s^2}\right)\right]$.

Now, we can put T_1 as follows:

$$T_1 = y^{\alpha' t-1}\cdot e^{-\alpha'' t}\left\{\frac{1}{2\pi i}\int \frac{e^s}{s^{\alpha' t}}(e^u - 1 - u)\,ds + \frac{1}{2\pi i}\int \frac{e^s}{s^{\alpha' t}}\left(1-\frac{t\alpha' y}{2s}\right)ds + \frac{1}{2\pi i}\int \frac{e^s}{s^{\alpha' t}}\cdot 0\left(\frac{t y^2}{s^2}\right)ds\right\} \tag{84}$$

$$= y^{\alpha' t-1}\cdot e^{-\alpha'' t}(T_3 + T_4 + T_5)$$

Since

$$|e^u - 1 - u| = \left|\int_0^u e^z (u-z)\,dz\right|,$$

and the real part of $u < 0$, we have

$$|e^u - 1 - u| \leqslant |u^2|, \tag{85}$$

so that,

$$|T_3| \leqslant M_1 \cdot e^{\sigma}\cdot t^2 \cdot y^2 \cdot \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{|ds|}{|s|^{2+\alpha' t}} \leqslant \frac{M\cdot t^2 \cdot y^2}{(1+\alpha' t)\,\sigma^{(1+\alpha' t)}}. \tag{86}$$

Similarly

$$|T_5| \leqslant \frac{M\cdot t\cdot y^2}{(1+\alpha' t)\,\sigma^{(1+\alpha' t)}}, \tag{87}$$

where M is a constant. Since σ can be taken as large as we like, we have

$$|(T_3 + T_5)\,e^{-\alpha'' t}| \leqslant M\cdot t\cdot y^2 \cdot e^{-k_2 t}, \tag{88}$$

M and k_2 being some positive constants.

By a deformation of the contour, it can be easily proved that

$$T_4 = \frac{y^{\alpha' t-1}}{|\underline{\alpha' t}}\left(1 - \frac{y}{2}\right). \tag{89}$$

From (88) and (89), we derive (76).

Proof of (77):

Let

$$T_2 = \frac{1}{2\pi i}\left[\int \frac{D-\mu}{\lambda-\mu}\,e^{-\mu t + ys}\,ds - \int \frac{D-\mu}{\lambda-\mu}\,e^{-\lambda t + ys}\,ds\right],$$

$$= P_3 - P_4 \tag{90}$$

Then, P_3 and P_4 can be put as follows:

$$P_3 = \frac{1}{2\pi i}\left[\int \frac{D-\mu}{\lambda-\mu} e^{ys}(e^{-\mu t} - 1 + \mu t)\, ds + \int \frac{D-\mu}{\lambda-\mu} e^{ys}(1-\mu t)\, ds\right]$$
$$= P_3' + P_3'' \tag{91}$$

and

$$P_4 = \frac{1}{2\pi i}\left[\int \frac{D-\mu}{\lambda-\mu} e^{ys}(e^{-\lambda t} - 1 + \lambda t)\, ds + \int \frac{D-\mu}{\lambda-\mu} e^{ys}(1-\lambda t)\, ds\right]$$
$$= P_4' + P_4'' \tag{92}$$

Let

$$P'_2 = P''_3 - P_4''$$
$$= \frac{t}{2\pi i}\int (D-\mu)\, e^{ys}\, ds. \tag{93}$$

By the relations (81), we can put

$$P'_2 = \frac{2t}{2\pi i a'}\int \frac{e^{ys}}{s^2 \log \dfrac{s}{D_3}}\, ds + t\int e^{ys}\cdot 0\left(\frac{1}{s^3 \log^2 s}\right) ds,$$
$$= P_2'' + P_2''', \quad (D_3 = e^{D/a'}) \tag{94}$$

By methods similar to those employed in proving (76) we can easily establish that

$$|P_3'|,\ |P_4'| \leqslant M\cdot y\cdot t^2, \tag{95}$$

and

$$|P_2'''| \leqslant M . y^2 . t, \tag{96}$$

where M is a positive constant. By putting $\dfrac{s}{D_3} = s_1$,

$$P''_2 = \frac{2t}{a' D_3 \cdot 2\pi i}\int \frac{e^{yD_3 s}}{s^2 \log s}\, ds \tag{97}$$

and

$$\frac{\partial P_2''}{\partial y} = \frac{2t}{a' . 2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{yD_3 s}}{s \log s}\, ds = \frac{2t}{a'}\int_0^\infty \frac{e^{-yD_3 x}}{x(\pi^2 + \log^2 x)}\, dx, \tag{98}$$

the latter result being obtained by a deformation of the contour of the first integral of (98), so that, since

$$P_2''(0, t) = 0$$

we have from (98),

$$P_2''(y, t) = y\, t\, [k - \epsilon(y)], \tag{99}$$

$$K = \frac{2}{a'}\int_0^\infty \frac{dx}{x(\pi^2 + \log^2 x)}, \text{ and } \epsilon(y) \to 0 \text{ as } y \to 0.$$

From (95), (96) and (99), we obtain (77)

Proof of (79) *and* (80):

Now, P_3 of (91) can, by use of relations (81), be put as

$$P_3=\frac{1}{2\pi i}\left[e^{-Dt}\int\frac{2\,e^{ys}}{s^2(A-D)^2}\,ds+e^{-Dt}\int\frac{2\,e^{ys}}{s^2(A-D)^2}\,(e^{\overline{D-\mu}t}\cdot-1)\,ds\right.$$
$$\left.+\int e^{-\mu t+ys}\cdot 0\left(\frac{1}{s^3}\right)\cdot ds\right]$$

$$=I_1+I_2+I_3. \tag{100}$$

Then,

$$I_1=\frac{2}{\alpha'^2}\frac{e^{-Dt}}{2\pi i}\left[\int\frac{e^{ys}}{s^2\log^2\dfrac{s}{D_3}}\,ds+\int e^{ys}\cdot 0\left(\frac{1}{s^3}\right)\cdot ds\right]$$

$$=I_1'+I_1'',\ (D_3=e^{D/\alpha'}.) \tag{101}$$

By methods similar to those employed in proving (76) we can establish that

$$|I_2|,\ |I_3| \text{ and } |I_1''|\leqslant M\,e^{-k_2 t}\cdot y^2 \tag{102}$$

where M and k_2 are some positive constants. As in the proof of (77), we can easily prove that

$$I'_1=\frac{2}{\alpha' D_3}\int_0^\infty\frac{1-(1+y\,D_3x)\,e^{-yD_3x}}{x^2(\pi^2+\log^2 x)}dx \tag{103}$$

and for small y,

$$I_1'=\frac{2}{\alpha^{2\prime}}\cdot\frac{y}{\log^2 y}\cdot[1+\epsilon(y)] \tag{104}$$

From (102), (103) and (104), we obtain (79) and (80). Combining (76), (77), (79) and (80), we obtain (13), $(13)_1$ and (14) of section 2.

APPENDIX 2

Here, I propose proving (15). The symbols A, B, C, D, λ, μ, S(E, t) etc., are the same as in App. 1.

We shall first show that

E_0 S(E, t) = the sum of the residues of the function

$$\left\{\frac{\lambda-D}{\lambda-\mu}\,e^{-\lambda t}+\frac{D-\mu}{\lambda-\mu}\,e^{-\mu t}\right\}e^{ys}=f_1(s,t)\,e^{ys} \tag{105}$$

at $s=1,0,-1,-2,\cdots,-n$, etc.,

and that for fixed t and large y, the significant part of E_0 S(E, t) is given by the residue at $s=1$

Let I = (imaginary part of) and R = (real part of).

Lemma (1).

Let

$|I(s)| \geqslant A > 0.$

Then,

$$\frac{|\overline{s}'}{|\overline{s}} = \log s + 0(1) \tag{106}$$

Proof of (106):

Let $2 > \sigma > 1$. Then we know that for $R(s) < \sigma$, we can find a positive integer p, such that

$\sigma \leqslant R(s+p) \leqslant \sigma + 1.$

Writing $s_1 = s + p$, we have

$$\frac{|\overline{s}'}{|\overline{s}} = \frac{|\overline{s_1 - p}'}{|\overline{s_1 - p}} = \frac{1}{1-s_1} + \frac{1}{2-s_1} + \cdots + \frac{1}{p-s_1} + \frac{|\overline{s_1}'}{|\overline{s_1}}. \tag{107}$$

Let $R(s_1) = \sigma_1$ and $I(s_1) = u_1$ where $|u_1| \geqslant A$. Then,

$$R\left(\sum_1^p \frac{1}{r-s_1}\right) = \sum_1^p \frac{r-\sigma_1}{(r-\sigma_1)^2 + u_1^2} = \frac{1}{2}\log\left[\frac{(p-\sigma_1)^2 + u_1^2}{(3-\sigma_1)^2 + u_1^2}\right] + 0(1) \tag{108}$$

and

$$I\left(\sum_1^p \frac{1}{r-s_1}\right) = -\sum_1^p \frac{u_1}{(r-\sigma_1)^2 + u_1^2} = 0(1), \tag{109}$$

and

$$\frac{|\overline{s_1}'}{|\overline{s_1}} = \frac{1}{2}\log\{\sigma^2{}_1 + u_1^2\} + 0(1), \tag{110}$$

so that

$$\frac{|\overline{s}'}{|\overline{s}} = \frac{1}{2}\log\{(p-\sigma_1)^2 + u_1^2\} + 0(1),$$

$$= \log s + 0(1), \tag{111}$$

thus proving Lemma (1)

Lemma (2).

Let $1 < \sigma < 2$ and $R(s) = -n + \sigma$, and $|I(s)| \leqslant A$, some positive constant. We shall prove that in this region

$$\frac{|\overline{s}'}{|\overline{s}} = \log n + 0(1) \tag{112}$$

Let

$s_1 = s + n.$

Then,

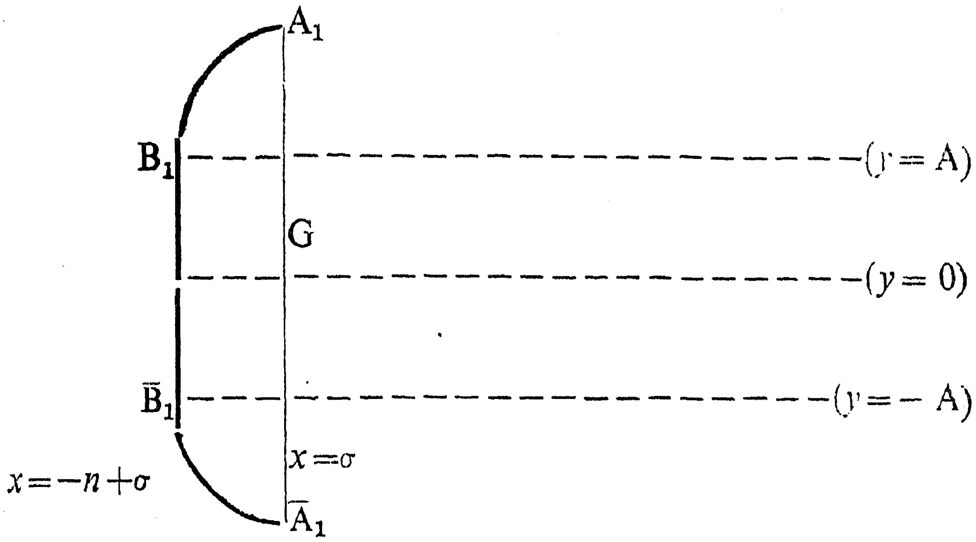
$$\frac{|s'}{|s}=\frac{|s_1'}{|s_1}+\frac{1}{1-s_1}+\frac{1}{2-s_1}+\cdots+\frac{1}{n-s_1}. \tag{113}$$

Since $|I\ (s_1)|\leqslant A$ and $R\ (s_1)=\sigma$, it follows that

$$\frac{|s'}{|s}=\log n+0\,(1) \tag{114}$$

in this region

Consider now the integral $\int f\,(s,\,t)\,e^{ys}\,ds$ along the following contour



(115)

Also let $1<\sigma<2$. Co-ordinates of $B_1=(-n+\sigma,\ A)$;

co-ordinates of $\bar{B}_1=(-n+\sigma,-A)$;

the arcs $B_1\ A_1$, $\bar{B}_1\ \bar{A}_1$ being arcs of circles with the origin as centre; $A_1G\ \bar{A}_1$ being the straight line $x=\sigma$ and $B_1\ \bar{B}_1$ being the straight line $x=-n+\sigma$.

Then, since by (81) of App. 1 and Lemma (1)

$$\left.\begin{aligned}\lambda&=\alpha'\log s+0\,(1)\\ \mu&=D+0\,(1)\end{aligned}\right\}\text{ for }|I\ (s)|\geqslant A, \tag{116}$$

and

$$\frac{\lambda-D}{\lambda-\mu}=0\,(1),\ \frac{D-\mu}{\lambda-\mu}=0\left(\frac{1}{s^2\log^2 s}\right) \tag{117}$$

we see at once that for $t>0$,

$$f_1\,(s,\,t)=0\left(\frac{1}{|s|^{\alpha' t}}\right)+0\left(\frac{e^{-Dt}}{s^2\log^2 s}\right), \tag{118}$$

so that, by the well-known Jordan's Lemma

$$\int_{A_1B_1} f_1(s, t)\, e^{ys}\, ds \text{ and } \int_{\bar{A}_1\bar{B}_1} f_1(s, t)\, e^{ys}\, ds \tag{119}$$

tend to zero as $n \to \infty$. Similarly we show that

$$\int_{B_1\bar{B}_1} f_1(s, t)\, e^{ys}\, ds \to 0 \textit{ as } n \to \infty \tag{120}$$

Hence, from (119) and (120), (105) follows.

From similar considerations, it also follows that if $0 < \sigma_1 < 1$ and $A > 0$,

$$E_0 \cdot S(E, t) = \frac{1}{2\pi i}\left[\int_{-iA-\infty}^{-iA+\sigma_1} + \int_{-iA+\sigma_1}^{iA+\sigma_1} + \int_{iA+\sigma_1}^{iA-\infty} f(s, t)\, e^{ys}\, ds\right] + \text{residue at } s = 1$$

$$= \quad I_1 + I_2 + I_3 \quad + \text{residue at } s = 1 \tag{121}$$

On the lines $y = \pm A$, $f(s, t)$ being bounded, we have

$$|I_1 + I_3| \leqslant M \cdot \int_{-\infty}^{\sigma_1} e^{yt}\, dt = \frac{M \cdot e^{y\sigma_1}}{y}, \tag{122}$$

and $|f_1(s, t)|$ being regular on $x = \sigma$, we have

$$|I_2| \leqslant e^{y\sigma_1} \int_{-A}^{A} |f_1(s, t)|\, ds \leqslant M \cdot e^{y\sigma_1}, \tag{123}$$

so that

$$E_0 \cdot S(E, t) = 0\,(e^{y\sigma_1}) + \text{residue at } s = 1 \qquad 124)$$

Estimate of residue at s = 1 (for fixed t and large y)

Writing $s = 1 + Z$.

Now, in the neighbourhood of $s = 1$, we have

$$(A - D)^2 + 4\,BC = \frac{8\alpha'\left(1 - \frac{\alpha'}{6}\right)}{Z} + \sum_{0}^{\infty} a_n Z^n, \tag{125}$$

so that

$$\frac{1}{2}\sqrt{\{(A - D)^2 + 4\,BC\}} = \frac{\rho}{\sqrt{Z}} + \sum_{1}^{\infty} \beta_n Z^{n-\frac{1}{2}}, \tag{126}$$

so that

$$\lambda - D = \frac{A-D}{2} + \frac{\rho}{\sqrt{Z}} + \sum_1^\infty \beta_n Z^{n-\frac{1}{2}},$$

$$\mu - D = \frac{A-D}{2} - \frac{\rho}{\sqrt{Z}} - \sum_1^\infty \beta_n Z^{n-\frac{1}{2}},$$

and

$$\lambda - \mu = \frac{2\rho}{\sqrt{Z}} + 2\sum_1^\infty \beta_n Z^{n-\frac{1}{2}} \qquad (127)$$

so that

$$\frac{\lambda - D}{\lambda - \mu} = \frac{1}{2} + \Sigma\, \gamma_n Z^{n/2},$$

where

$$\rho = \sqrt{2\alpha'\left(1 - \frac{\alpha'}{6}\right)} = \sqrt{2\alpha' D}$$

and $\Sigma\, \alpha_n Z^n$, $\Sigma\, \beta_n Z^{n-\frac{1}{2}}$, $\Sigma \gamma_n Z^{\frac{n}{2}}$ are all series convergent in a circle of some positive radius, so that $|\alpha_n|$, $|\beta_n|$, $|\gamma_n|$ are all $\leqslant k_1^n$ (k_1, some positive constant).

Now the residue of $f_1(s, t)\, e^{ys}$ at $s = 1$ is given by taking the integral of $\frac{1}{2\pi i}\left(e^{ys-\lambda t} \cdot \frac{\lambda - D}{\lambda - \mu}\right)$ twice along a small circle round the point $s = 1$ or $Z = 0$; *i.e.*, the residue at $s = 1$ (128)

$$= \frac{1}{2\pi i}\int \frac{1}{2}\left(1 + \Sigma\, \gamma_n Z^{n/2}\right) \cdot e^{-\frac{t\rho}{\sqrt{Z}} + t\,\Sigma\, \beta_n Z^{n/2} + y(1+Z)}\, dZ, \qquad (129)$$

(two rounds of the circle round $Z = 0$)

$$= \frac{e^y}{4\pi i}\int (1 + \Sigma\, \gamma'_n Z^{n/2}) \cdot e^{-\frac{t\rho}{\sqrt{Z}} + Zy} \cdot dZ, \qquad (130)$$

(two rounds)

where $\Sigma\, \gamma'_n Z^{n/2}$ converges inside some circle round $Z = 0$ larger than the contour circle. Putting $yZ = Z_1^2$, the integral becomes

$$\frac{e^y}{2\pi i \cdot y}\int \left(1 + \Sigma \frac{\gamma'_n}{y^{n/2}} Z_1^n\right) \cdot e^{-\frac{t\rho\sqrt{y}}{Z_1} + Z_1^2} \cdot Z_1 \cdot dZ_1 = \frac{e^y}{y} \cdot I(y, t). \qquad (131)$$

(one round)

Let $-t\rho\sqrt{y} = u$. Then

$$Z \cdot e^{\frac{u}{z} + z^2} = \sum_1^\infty \frac{g_n(u)}{Z^n} + \left\{\phi(z), \text{ which is regular}\right\}, \qquad (132)$$

where

$$g_n(u) = u^{n+1} \left\{ \sum_0^\infty \frac{u^{2p}}{\underline{|2p+n+1}\ \underline{|p}} \right\}$$

so that

$$I(y, t) = g_1(u) + \frac{\gamma_1'}{\sqrt{y}} g_2(u) + \cdots + \frac{\gamma'_n}{y^{n/2}} g_n(u) + \cdots \tag{133}$$

We shall here show that for fixed t and large y,

$$I(y, t) = g_1(u)(1+\epsilon) \tag{134}$$

* * * *

Lemma.

Let

$$f(x) = \Sigma\, a_n x^n = \Sigma \frac{x^n}{\underline{|2n}\ \underline{|n}}. \tag{135}$$

Then, for large $x > 0$,

$$f(x) = \frac{1}{2^{2/3}.3^{1/2}.\pi^{1/2}} \cdot e^{b x^{1/3}} \cdot x^{-1/6}(1+\epsilon) \tag{136}$$

where

$$b = 3 \cdot 2^{-2/3}$$

Method of Proof: By a method entirely analogous to that of 72, of page 12, of Polya and Szego's *Aufgaben and Lehrsatze*, Vol. II, by comparing the given series to $\Sigma \frac{(\alpha x)^n}{\underline{|3n}}$ where $\alpha = \frac{3^3}{2^2}$, we obtain the result.

* * * *

Let $f_1(x) = f(x^2)$; then it is easy to recognise that

$$g_1(x) = \int_0^x f_1(t) \cdot (x-t)\, dt \text{ and } g_r(x) = {}_0\!\int^x g_{r-1}(x)\, dx, \tag{137}$$

so that $g_1(x)$ can be proved to be

$$= M \cdot e^{b x^{2/3}} \cdot x^{1/3} \cdot (1 + \epsilon_x), \tag{138}$$

where

$$M = \frac{1}{2^{4/3}.3^{1/2}.\pi^{1/2}}.$$

We shall here show that,

$$g_r(x) \leqslant M_2 \cdot g_1(x) \cdot x^{\frac{r-1}{3}} \text{ (M}_2\text{ some positive constant)} \tag{139}$$

Now,

$$g_1(x) \leqslant M_1 \cdot e^{b x^{2/3}} . x^{1/3}, \tag{140}$$

so that

$$g_2 = \int_0^x g_1\, dt \leqslant M_1 \int_0^x e^{b.x^{2/3}} \cdot x^{1/3} \cdot dx$$

$$= M_1 \left\{ \frac{3}{2b} e^{b\,x^{2/3}} \cdot x^{2/3} - \frac{1}{b} \int_0^x e^{b.x^{2/3}} \cdot x^{-1/3}\, dx \right\}$$

$$\leqslant \frac{3\,M_1}{2b} \cdot e^{bx^{2/3}} \cdot x^{2/3}; \tag{141}$$

and by mathematical induction, we can easily deduce from (141) that

$$g_r \leqslant \left(\frac{3}{2b}\right)^{r-1} \cdot M_1 \cdot e^{bx^{2/3}} \cdot x^{r/3}. \tag{142}$$

Now $b = \frac{3}{2^{2/3}}$ so that $\frac{3}{2b} = \frac{1}{2^{1/3}} < 1$.

Hence

$$g_r \leqslant M_1 \cdot e^{b\,x^{2/3}} \cdot x^{r/3}, \tag{143}$$

so that for $x \geqslant$ some positive x_0,

$$g_r\,(x) \leqslant M_2 \cdot g_1 \cdot x^{\frac{r-1}{3}}, \tag{144}$$

thus establishing (139)

$$\text{Hence} \left| \frac{g_{n+1}\,(u)}{y^{n/2}} \right| \leqslant M \cdot \frac{g_1\,(u) \cdot (t\,\rho\,\sqrt{y})^{n/3}}{y^{n/2}} = \frac{M \cdot (t\,\rho)^{n/3}}{y^{n/3}} \cdot g_1\,(u) \tag{145}$$

and since $|\gamma'_n| \leqslant k^n$ (t is fixed), we see at once that for large y,

$$I\,(y,\,t) = g_1\,(u)\,(1 + \epsilon_y),$$

and therefore the residue at $s = 1$

$$= \frac{(t\,\rho)^{1/3}}{2^{4/3} . 2^{1/2} . \pi^{1/2}} \cdot \frac{e^{y + \frac{3}{2^{2/3}} (t\,\rho)^{2/3} . y^{1/3}}}{y^{5/6}} \cdot (1 + \epsilon_y) \tag{146}$$

Combining (124) and (146), we obtain the result (15) of section 2.

APPENDIX 3

Proof of the Transformation (22) *of Section* 5.

(22) of section 5 could be established by the usual trick of change in the order of integration. But it is far more elegant to prove it by using Mellin Transforms—which we propose to do now.

$$\text{Let} \int^{\infty} L\,(U,\,t) \cdot U^{s-1} \cdot dU = \bar{L}\,(s,\,t) \tag{147}$$

$$\text{and} \int_0^\infty S_1(U, t)\, U^{s-1} \cdot dU = f_1(s, t) \tag{148}$$

the latter as defined in (12) of § 3. So that the Mellin Transform of

$$\int_0^a dt \int_E^\infty S_1\left(\frac{E}{U}, a-t\right) L(U, t) \cdot \frac{dU}{U}$$

is equal to

$$\int_0^a \bar{L}(s, t)\, f_1(s, a-t)\,dt. \tag{149}$$

Following Landau and Rumer, it can be easily proved that the Mellin Transform of $L(\mathscr{P})$ {L being the left-hand side operator of (21·)} is equal to

$$\frac{\partial \mathscr{P}}{\partial t} + A_s \cdot \mathscr{P} - \int_0^t e^{D(t_1 - t)} \cdot (B_s\, C_s)\, \mathscr{P}(s, t_1)\, dt_1 = \bar{L}(s, t) \tag{150}$$

where A_s, B_s, C_s and D are as in App. 1, so that

$$\begin{aligned}\int_0^a \bar{L}(s, t) \cdot f_1(s, a-t)\, dt = & \int_0^a f_1(s, a-t) \cdot \frac{\partial \mathscr{P}(s, t)}{\partial t} \cdot dt \\ & + \int_0^a A_s \mathscr{P}(s, t) \cdot f_1(s, a-t) \cdot dt \\ & - (B_s\, C_s) \cdot \int_0^a f_1(s, a-t) \cdot dt \int_0^t e^{D(t_1 - t)}\, \mathscr{P}(s, t_1)\, dt_1, \\ & = I_1 + I_2 - I_3\end{aligned} \tag{151}$$

Then, integrating by parts,

$$I_1 = f_1(s, 0)\, \mathscr{P}(s, a) - f_1(s, a)\, \mathscr{P}(s, 0) + \int_0^a \mathscr{P}(s, a-t) \cdot \frac{\partial f_1(s, t)}{\partial t} \cdot dt \tag{152}$$

Also it can be easily proved (by a change in the order of integration) that

$$I_3 = (B_s\, C_s) \int_0^a \mathscr{P}(s, a-t)\, dt \cdot \int_0^t e^{D(t_1 - t)} \cdot f_1(s, t_1)\, dt_1, \tag{153}$$

and

$$I_2 = A_s \cdot \int \mathscr{P}(s, a-t) \cdot f_1(s, t)\, dt \tag{154}$$

Since initially $f_1(s, 0) = 1$ and $\mathscr{P}(s, 0) = E_0^{s-1}$, we have

$$I_1 + I_2 - I_3 = \mathscr{P}(s, a) - E_0^{s-1} \cdot f_1(s, a) + \int_0^a \mathscr{P}(s, a-t) \left\{ \frac{\partial f_1}{\partial t} + A_s f_1 - \int_0^t e^{D(t_1 - t)} \cdot f_1(s, t_1)\, dt_1 \right\} dt$$

$$= \mathscr{P}(s, a) - E_0^{s-1} \cdot f_1(s, a),$$

because

$$\frac{\partial f_1}{\partial t} + A_s \cdot f_1 = \int_0^t e^{D(t_1 - t)} \cdot f_1(s, t)_1\, dt_1{}^7 \qquad (155)$$

we have, therefore

$$I_1 + I_2 - I_3 = \mathscr{P}(s, a) - \text{Mellin Transform of } S(E, a) \qquad (156)$$

Hence

$$\int_0^a dt \int_E^\infty S_1\left(\frac{E}{U}, a-t\right) \cdot L(U, T) \cdot \frac{dU}{U} = \mathscr{P}(E, a) - S(E, a) \qquad (157)$$

thus proving (22)

APPENDIX 4

Proof of the term-by-term integrability of $\sum_0^\infty |Z_r(E, a)|$ of (25), in the integral equation (23), where

$$Z_r(E, a) = \int_0^a dt \int_E^{E_0} S_1\left(\frac{E}{U}, a-t\right) \cdot F(Z_{r-1}, U, t) \cdot \frac{dU}{U},$$

$$= \int_0^a dt \int_E^{E_0} S_1\left(\frac{E}{U}, a-t\right) \cdot \frac{dU}{U} \cdot \Big[q_1(U, t) \cdot Z_{r-1}(U, t) + \int_0^\infty q_2(V, U, t) \cdot Z_{r-1}(V, t)\, dV$$

$$+ \int_0^t dt_1 \int_{U-\beta(t-t_1)}^{U} q_3(V, U, t, t_1) \cdot Z_{r-1}(V, t_1)\, dV + \int_0^t dt_1 \int_U^{E_0} q_4(V, U, t, t_1) \cdot Z_{r-1}(V, t_1)\, dV \Big],$$

$$= J_1(Z_{r-1}) + J_2(Z_{r-1}) + J_3(Z_{r-1}) + J_4(Z_{r-1}) \qquad (158)$$

[7] This result is given in Landau and Rumer's paper.

where, from equation (21), we have

$$\left.\begin{aligned} q_1(\mathrm{U}, t) &= \alpha' \log \frac{\mathrm{U}}{\mathrm{U}-\beta t} + \phi_4(\mathrm{U}-\beta t) + \frac{1}{2} - \alpha', \\ q_2(\mathrm{V}, \mathrm{U}, t) &= \xi_2(\mathrm{V}-\beta t, \mathrm{U}-\beta t) - \frac{1-\alpha'}{\mathrm{V}} + \frac{\mathrm{U}}{\mathrm{V}^2}, \\ q_3(\mathrm{V}, \mathrm{U}, \mathrm{T}, t_1) &= \phi_1(\mathrm{V}-\beta t, \mathrm{U}-\beta t, t_1 - t), \\ q_4(\mathrm{V}, \mathrm{U}, t.\ t_1) &= \phi_1(\mathrm{V}-\beta t_1, \mathrm{U}-\beta t, t_1 - t) - \phi_1(\mathrm{V}, \mathrm{U}, t_1 - t) \end{aligned}\right\} \tag{159}$$

Let a and E be two positive numbers $\left(a < \frac{\mathrm{E}_0}{\beta}, \mathrm{E} < \mathrm{E}_0\right)$, and let $\mathrm{E}_1 = \mathrm{E} + \beta a$ and $\log \frac{\mathrm{E}_0}{\mathrm{U}} = y$. Then, we shall prove that when $0 \leqslant t \leqslant a$ and $\mathrm{E}_1 \leqslant \mathrm{U} \leqslant \mathrm{E}_0$,

$$|\mathrm{Z}_r(\mathrm{U}, t)| \leqslant \left[1 + \frac{y^{\alpha' t - 1}}{|\underline{\alpha' t}}\right] \cdot \mathrm{G}^{r+1} \cdot \frac{t^r}{|\underline{r}},\tag{160}$$

where G is some constant. So that, from (160), the term-by-term integrability of $\Sigma\, \mathrm{Z}_r(\mathrm{E}, t)$ in (23) follows, thus establishing that $\Sigma\, \mathrm{Z}_r(\mathrm{E}, t)$ is the solution of equation (23).

Preliminaries :

When $\mathrm{E}_0 \geqslant \mathrm{U} \geqslant \mathrm{E}_1$, $\mathrm{E}_0 \geqslant \mathrm{V} \geqslant \mathrm{U} - \beta(t - t_1) \geqslant \mathrm{E} > 0$ and $0 \leqslant t_1 \leqslant t \leqslant a$, it is obvious that

$|q_1(\mathrm{U}, t)|$, $|q_2(\mathrm{V}, \mathrm{U}, t)|$, $|q_3(\mathrm{V}, \mathrm{U}, t_1, t)|$ and $|q_4(\mathrm{V}, \mathrm{U}, t_1, t)|$

are all bounded. We shall indicate the upper bound of these functions in this region by λ. (161)

Now $\mathrm{S}(\mathrm{U}, t) = \mathrm{Z}_0(\mathrm{U}, t)$ and $\mathrm{S}_1(\mathrm{U}, t)$ are both solutions of equation (6) with $\beta = 0$[8] and because of their initial conditions,

$$\mathrm{S}_1(\mathrm{U}, t) = \mathrm{E}_0\, \mathrm{S}(\mathrm{E}_0 \mathrm{U}, t) \tag{162}$$

Let $\log \frac{1}{\mathrm{E}_0} = \mathrm{Z}_1$. By (13) and (14) of Section 4 we have

$$\mathrm{S}(\mathrm{U}, t) = \mathrm{M}_0(y, t) + \mathrm{N}_0(y, t) \cdot \frac{y^{\alpha' t - 1}}{|\underline{\alpha' t}},\tag{163}$$

where M_0 and N_0 are continuous functions in the whole range $0 \leqslant t$, $0 \leqslant y$. When $0 \leqslant y \leqslant \log \frac{\mathrm{E}_0}{\mathrm{E}_1}$, M_0 and N_0 are bounded. Also from (163),

$$\mathrm{S}_1(\mathrm{U}, t) = \mathrm{E}_0 \left\{\mathrm{M}_0(y + \mathrm{Z}_1, t) + \mathrm{N}_0(y + \mathrm{Z}_1, t) \cdot \frac{(y + \mathrm{Z}_1)^{\alpha' t - 1}}{|\underline{\alpha' t}}\right\} \tag{164}$$

[8] S and S_1, as stated before, have been given by Landau and Bhabha and Chakrabarty.

Let K be the upper bound of $E_0\,|M_0(y+Z_1,t)|$ and $E_0\,|N_0(y+Z_1,t)|$, when $0\leqslant t\leqslant a$ and $0\leqslant y\leqslant \log\frac{E_0}{E_1}$. Also, let the maximum value of $\left(y,\frac{y^{\alpha' t}}{|\underline{\alpha' t+1}}\right)$ when $0\leqslant y\leqslant \log\frac{E_0}{E}$ be $\left(\frac{p}{3}\right)$. (165)

* * * *

Let us assume (160) to be true for r. We shall prove it to be true for $(r+1)$,

$$\left.\begin{aligned}&\text{Now, } Z_{r+1}(E,t)=J_1(Z_r)+J_2(Z_r)+J_3(Z_r)+J_4(Z_r),\\&\text{and}\\&J_1(Z_r)=\int_0^t dt_1\int_E^{E_0} S_1\left(\frac{E}{U},t-t_1\right)\cdot q_1(U,t_1)\cdot Z_r(U,t_1)\cdot\frac{dU}{U}.\end{aligned}\right\}\quad(166)$$

Changing the variable in J_1 by putting $\log\frac{E_0}{U}=y$ and $\log\frac{E_0}{E}=y_0$, and using inequalities (160), (161) and (165), we have

$$|J_1(Z_r)|\leqslant\lambda\cdot K\cdot\int_0^t G^{r+1}\cdot\frac{t_1^{\,r}}{|\underline{r}}\cdot dt_1\cdot\int_0^{y_0}\left(1+\frac{(y_0-y)^{\alpha'(t-t_1)-1}}{|\underline{\alpha'(t-t_1)}}\right)\left(1+\frac{y^{\alpha' t_1-1}}{|\underline{\alpha' t_1}}\right)dy=$$

$$=\lambda K\cdot\frac{G^{r+1}}{|\underline{r}}\int_0^t t_1^{\,r}\left(y_0+\frac{y_0^{\alpha'(t-t_1)}}{|\underline{\alpha'(t-t_1)+1}}+\frac{y_0^{\alpha' t_1}}{|\underline{1+\alpha' t_1}}+\frac{y_0^{\alpha' t-1}}{|\underline{\alpha' t}}\right)dt_1\quad(167)$$

which, by (165),

$$\leqslant\lambda K\cdot\frac{G^{r+1}}{|\underline{r}}\int_0^t t_1^{\,r}\left(p+\frac{y_0^{\alpha' t-t}}{|\underline{\alpha' t}}\right)dt_1=$$

$$=\lambda K\cdot\frac{G^{r+1}}{|\underline{r+1}}\cdot t^{r+1}\cdot\left(p+\frac{y_0^{\alpha' t-1}}{|\underline{\alpha' t}}\right).\quad(168)$$

Similarly, changing the variable in J_2 by putting $\log\frac{E_0}{E}=y_0$, $\log\frac{E_0}{U}=y$ and $\log\frac{E_0}{V}=y'$, we obtain

$$|J_2(Z_r)|\leqslant\lambda K\,\frac{G^{r+1}}{|\underline{r}}\cdot\int_0^t dt_1\int_0^{y_0}\left(1+\frac{(y_0-y)^{\alpha'(t-t_1)-1}}{|\underline{\alpha'(t-t_1)}}\right)dy\int_0^y\left(1+\frac{y'^{\alpha' t_1-1}}{|\underline{\alpha' t_1}}\right)t_1^{\,r}E_0\cdot e^{-y'}.\,dy'$$

$$\leqslant\lambda K\,E_0\cdot\frac{G^{r+1}}{|\underline{r}}\int_0^t t_1^{\,r}\,dt_1\int_0^{y_0}\left(1+\frac{(y_0-y)^{\alpha'(t-t_1)-1}}{|\underline{\alpha'(t-t_1)}}\right)\left(y+\frac{y^{\alpha' t_1}}{|\underline{1+\alpha' t_1}}\right)dy\quad(169)$$

which, by (165),

$$\leqslant \lambda \, K \, E_0 \cdot \frac{G^{r+1}}{\underline{|r}} \cdot \frac{2p}{3} \cdot \int_0^t t_1^{\,r} \, dt_1 \int_0^{y_0} \left(1 + \frac{(y_0 - y)^{\alpha'(t-t_1)-1}}{\underline{|\alpha'(t-t_1)}}\right) dy \qquad (170)$$

which, again by (165),

$$\leqslant \lambda \, K \, E_0 \cdot \frac{G^{r+1}}{\underline{|r+1}} \cdot \left(\frac{2p}{3}\right)^2 \cdot t^{r+1}. \qquad (171)$$

Similarly, making the same transformations as above in J_3, we have

$$|J_3 (Z_r)| \leqslant \lambda \, K \cdot \frac{G^{r+1}}{\underline{|r}} \int_0^t dt_1 \int_0^{y_0} dy \left(1 + \frac{(y_0 - y)^{\alpha'(t-t_1)-1}}{\underline{|\alpha'(t-t_1)}}\right) \cdot$$

$$\int_0^{t_1} dt_2 \int_y^{\log\left(\frac{E_0}{U-\beta\,(t_1-t_2)}\right)} t_2^{\,r} \left\{1 + \frac{y'^{\,\alpha' t-1}}{\underline{|\alpha' t}}\right\} E_0 \, e^{-y'} \, dy' \qquad (172)$$

which, by (165),

$$\leqslant \lambda K \, E_0 \cdot \frac{G^{r+1}}{\underline{|r}} \int_0^t dt_1 \int_0^{y_0} 1 + \frac{(y_0 - y)^{\alpha'(t-t_1)-1}}{\underline{|\alpha'(t-t_1)}}\Bigg) \, dy \int_0^{} \left(\frac{2p}{3}\right) t^{r}{}_2 \, dt_2 \; (173)$$

$$\leqslant \lambda \, K \, E_0 \, \frac{G^{r+1}}{\underline{|r+1}} \cdot \left(\frac{2p}{3}\right) \cdot \int_0^t t_1^{\,r+1} \, dt_1 \int_0^{y_0} \left(1 + \frac{(y_0 - y)^{\alpha'(t-t_1)-1}}{\underline{|\alpha'(t-t_1)}}\right) dy \; (174)$$

which, again by (165),

$$\leqslant \lambda \, K \, E_0 \, \frac{G^{r+1}}{\underline{|r+2}} \cdot \left(\frac{2p}{3}\right)^2 t^{r+2}. \qquad (175)$$

In exactly the same manner, we prove that

$$|J_4 (Z_r)| \leqslant \lambda \, K \, E_0 \, \frac{G^{r+1}}{\underline{|r+2}} \left(\frac{2p}{3}\right)^2 \cdot t^{r+2} \qquad (176)$$

So that, from (168), (171), (175) and 176) we have

$$|Z_{r+1} (E, t)| \leqslant \frac{G^{r+1} \cdot t^{r+1}}{\underline{|r+1}} \left\{\lambda K \, \frac{y_0^{\,\alpha' t-1}}{\underline{|\alpha' t}} + \lambda \, K \, p + \lambda \, K \, E_0 \left(\frac{2p}{3}\right)^2\right.$$

$$\left. + \, 2 \, \lambda \, K \, E_0 \cdot \left(\frac{2p}{3}\right)^2 \cdot \frac{t}{r+2}\right\} \qquad (177)$$

Let G be taken initially so great that

$$G > |M_0 (y, t)| \text{ and } |N_0 (y, t)| \text{ of (163),} \qquad (178)$$

$$\left.\begin{aligned} & G > \lambda \, K. \\ & \text{and} \\ & G > \lambda \, K \, p + \lambda \, K \, E_0 \left(\frac{2p}{3}\right)^2 + 2 \, \lambda \, K \, E_0 \left(\frac{2p}{3}\right) \cdot \frac{a}{2} \end{aligned}\right\} \qquad (179)$$

Then

$$|Z_{r+1}(E, t)| \leqslant \frac{G^{r+1} \cdot t^{r+1}}{\underline{|r+1}} \left[\left\{\lambda K \cdot \frac{y_0^{\alpha' t - 1}}{\underline{|\alpha' t}}\right\} + G\right]$$

$$\leqslant \frac{G^{r+2} \cdot t^{r+1}}{\underline{|r+1}} \left\{1 + \frac{y_0^{\alpha' t - 1}}{\underline{|\alpha' t}}\right\}. \tag{180}$$

Thus inequality (160) is proved for $(r+1)$

Also, by (178),

$$Z_0(E, t) \leqslant G\left(1 + \frac{y_0^{\alpha' t - 1}}{\underline{|\alpha' t}}\right) \tag{181}$$

So that, (160) is universally true.

REFERENCES

Bhabha and Heitler .. *Proc. Roy. Soc.*, 1937, A **159**, 422.

Carlson and Openheimer .. *Phys. Rev.*, 1937, **51**, 220.

Landau and Rumer .. *Proc. Roy. Soc.*, 1938, A **166**, 213.

Snyder .. *Phys. Rev.*, 1938, **53**, 960.

Serber .. *Ibid.*, 1938, **54**, 317.

Corben .. *Ibid.*, 1941, **60**, No. 6, 435.