NEW CONVERGENCE AND SUMMABILITY TESTS FOR FOURIER SERIES

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1. Let $f(x)$ be a periodic summable function and

$$\psi_{x_0}(t) = | f(x_0 + t) + f(x_0 - t) - 2f(x_0) |. \quad (1 \cdot 1)$$

Theorem A of this note is a generalization of the FeJer-Lebesgue-Hardy theorem which (latter) is:—

$$\frac{1}{u}\int_0^u \psi_{x_0}(t)\, dt \to o \text{ as } u \to o \text{ implies the (C, } r)\cdot(r > 0)\cdot\text{convergence of} \quad (1\cdot 2)$$

the Fourier series of $f(x)$ at x_0 to $f(x_0)$. [Zygmund[1], FeJer[2], Lebesgue[3], Hardy[4].] The subsidiary theorem B of this note discusses the relation between the various forms of the hypothesis of theorem A, and in particular their relation to the (F. L. H.) hypothesis in (1·2); and with the examples given below it in 3, establishes that the (F. L. H.) theorem is a particular case of theorem A.

Theorem C is a generalization of convergence test of the Fourier series of $f(x)$ at a point x_0 due to Hardy and Littlewood[5] (1932), the latter test being as follows:

$$(1)\ \psi_{x_0}(t) \log t \to 0\ t \to 0. \quad (2) \text{ the Fourier coefficients of } f(x) \leqslant An^{-\delta} \quad (1\cdot 3)$$

$(A > 0,\ \delta > 0)$. Hardy and Littlewood have established their test by using the Tauberian theorem associated with Valeron[6] (1917) means. In a recent paper[7] (1943) I have derived this test by proving

(*a*) (1) of (1·3) implies that $\dfrac{1}{\operatorname{Log} n} \sum_{r=0}^{n} \dfrac{S_r(x_0)}{n+1-r} \to f(x_0)$ as $n \to \infty$.

(*b*) (2) of (1·3) is exactly the Tauberian condition associated with convergence by the means in (*a*). Proceeding along the lines of my paper,[7] I have derived theorem C, from theorem A and a new Tauberian theorem in another paper[8] of mine to be published shortly. It is noteworthy that the form of theorem A permits us to replace (1) in the Hardy-Littlewood test by the slightly broader one $(1)' \ \frac{1}{u}\int_0^u \psi_{x_0}(t)\,\mathrm{Log}\,t\,dt \to 0, u \to 0$

or the still broader one $(1)'' \ \frac{1}{\mathrm{Log}\,u}\int_u^\delta \psi_{x_0}(t)\,\frac{\mathrm{Log}\,t}{t}\,dt \to 0\ u \to 0$

In 7 I have discussed particular cases of interest of A and C.

[*Note.*—In what follows "F.L.H. hypothesis" will stand for the condition in (1·2) namely $\frac{1}{u}\int_0^u \psi_{x_0}(t)\,dt \to 0$ as $u \to 0$ and F.L.H. theorem, the theorem of (1·2).]

2. *Notations and Definitions.*—The b_n, b_n', B_n, B_n'. $\psi_{x_0}(t)$, B_t, b_t, $S_n(x_0)$, are defined as follows :

(1) $b_n > 0.$, (2) $b_0 \geqslant b_1 \geqslant b_2 .. \geqslant b_n$;

(3) $B_n = \sum_{r=0}^{n} b_r$ (and $\to \infty$ as $n \to \infty$). (4) $b_t = b_{[t]}$, $B_t = B_{[t]}$, b_n', B_n', etc., being restricted by the same conditions;

(5) $\psi_{x_0}(t)$ defined as in (1·1); (6) $S_n(x_0) = \frac{1}{2}a_0 + \sum_{1}^{n}(a_r \cos r\, x_0 + + b_r \sin r x_0)$ a_r, b_r, being the Fourier coefficients of $f(x)$, $f(x)$ being as in (1·1).

3. *Statement of theorems A and B and remarks.*—

THEOREM A: If $\frac{1}{B_x}\cdot\int_a^x \psi_{x_0}\left(\frac{1}{t}\right)\cdot\frac{B_t}{t}\,dt \to 0 \quad (a > 0)$, $x \to \infty$ (3·1)

and $\frac{1}{B_N'}\sum_{n_0}^{N}\frac{b_n}{B_n}\cdot B_n'$ is bounded for all N (3·2)

then $\frac{1}{B_n'}\cdot\sum_{0}^{n} b_r' S_{n-r}(x_0) \to f(x_0)$, as $n \to \infty$. (3·3)

Particular case of A of interest is as follows :

Let $\theta(t). = \psi_{x_0}\left(\frac{1}{t}\right)\cdot\frac{B_t}{t\,b_t}$.

THEOREM A_1:

If (*a*) $\theta(t) \to 0$ or (*b*) $\frac{1}{t}\int^t \theta(u)\,du \to 0$ as $t \to \infty$ then (3·3) will be true.

[It is to be noted that (*b*) of (3·4) could be given in another form completely equivalent to it namely $\frac{1}{t}\int_0^t \theta\left(\frac{1}{u}\right) du \to 0$ as $t \to 0$; also it can be shewn that (3·4) (*a*) implies (3·4) *b*, and (3·4) *b* implies (3·1)].

Subsidiary theorem B.

If $\frac{1}{B_N'} \sum_{n_0}^{N} \frac{b_r}{B_r} B_r'$ is bounded for all N ($\geqslant n_0$) (*i.e.* $\leqslant$ some positive *k*) (3·5)

then $\frac{1}{B_x} \int_a^x \psi_{x_0}\left(\frac{1}{t}\right) \frac{B_t}{t}\, dt \to 0$ ($a > 0$) implies
$x \to \infty$

$$\frac{1}{B_x'} \int_a^x \psi_{x_0}\left(\frac{1}{t}\right) \cdot \frac{B_t'}{t}\, dt \to 0; \tag{3·6}$$

If (3·5) is not true (3·6) is not necessarily true. (3·7)

[*Note* 1.—It is to be noticed that (F. L. H.) hypothesis can be put in the equivalent form $\frac{1}{x}\int_a^x \psi_{x_0}\left(\frac{1}{t}\right) dt \to 0$ $x \to \infty$; so that from conditions in 2 about b_n, it follows from theorem B that hypothesis $\frac{1}{B_x}\int_a^x \psi_{x_0}\left(\frac{1}{t}\right) \cdot \frac{B_t}{t}\, dt \to 0$ implies the (F. L. H.) hypothesis and if $\frac{1}{B_N} \sum_{n_0}^{N} \frac{B_r}{r}$ is bounded for all $N \geqslant n_0$ the converse is also true; so that in this case (F. L. H.) hypothesis is equivalent to hypothesis (3·1). But the form of theorem A allows us to enlarge the field of (C, *r*) summability for the Fourier series of $f(x)$ at x_0 as the following example shews:—

Example 1.—Let $0 < \rho_1 < \rho_2$ and $\rho_1 + \rho_2 < 1$ and $\psi(x) =$ logloglog x. Let $F_n = n^{\rho_2 + \rho_1 \cos \psi(n)}$ and $F_n - F_{n-1} = b_n$ for sufficiently large n, say $n \geqslant n_0$. b_0, b_1, b_{n_0-1} being defined suitably to suit (1) and (2) of 2. then it can be proved that (*a*) b_n and $B_n = \sum_0^n b_r$ satisfy (1), (2), (3) of 2, and (*b*) $\frac{1}{B_N} \sum_{n_0}^{N} \frac{B_r}{r}$ is bounded for all $N \geqslant n_0$. so that from theorem A it follows $\frac{1}{B_n} \sum_0^n b_r S_{n-r}(x_0) \to f(x_0)$ if (F. L. H.) hypothesis is true. It is to be noted $n \to 0$
that $\frac{\text{Log } B_n}{\text{Log } n}$ in this case oscillates between $\rho_1 + \rho_2$ and $\rho_2 - \rho_1$.] (3·8)

[*Note* 2.—Hypothesis (3·2) namely boundedness of $\frac{1}{B_N'} \sum_{m_0}^{N} \frac{b_r}{B_r} B_r'$ implies $\frac{\text{Log } B_n}{\text{Log } B_n'}$ is bounded above, in particular $\frac{1}{B_N} \sum^{N} \frac{B_r}{r}$ implies $\frac{\text{Log } B_n}{\text{Log } n}$ is bounded above for $n \geqslant 2$.] (3·9)

[*Note* 3.—The following example reveals the importance of condition (3·2) in theorem A.

Example 2.—Let $f(t)$ be an even periodic bounded summable function and let its Fourier coefficients be such that $\text{Lim sup } \frac{\bar{S}_n(0)}{\text{Log } n} > 0$

where $S_n(0) = \frac{1}{2} a_0 + \sum_1^n a_r$.

$$\text{Let } \psi(t) = \frac{f(t)}{|\text{Log } |t||} \text{ in } 0 \leqslant |t| \leqslant \delta < 1$$

$$= 0 \text{ otherwise.}$$

In this case the Fe Jer theorem at $t = 0$ about $\psi(t)$ is true.

Taking $b_n = \frac{1}{n+1}$ it is easy to see that $\frac{1}{B_N} \sum^{N} \frac{B_r}{r}$ in this case $\to \infty$ as $N \to \infty$.

It can be established that $\dfrac{\sum_0^n \frac{S_{n-r}(0)}{1+r}}{\sum_0^n \frac{1}{1+r}}$ does not converge to $\psi(0) = 0$,

$S_n(0) =$ the sum of the 1st $\overline{n+1}$ Fourier coefficients of $\psi(t)$. [Incidentally this example establishes (3·7) of B, *via* A.] (3·9)

[*Note* 4.—In view of theorem B, it is sufficient to establish theorem A in the following form:—

$\bar{A}$. Hypothesis (3·1) implies $\frac{1}{B_n} \sum_0^n b_r S_{n-r}(x_0) \to f(x_0)$.] (3·10)

4. *Proof of theorem B.*—

$$\text{Let } a_n = \int_n^{n+1} \psi_{x_0}\left(\frac{1}{t}\right) dt. \tag{4·1}$$

It can be easily established that

$\frac{1}{B_n} \cdot \int_a^x \psi_{x_0}\left(\frac{1}{t}\right) \frac{B_t}{t}\, dt \to 0$ as $x \to \infty$ is completely

equivalent to $\frac{1}{B_n} \sum_{n_0}^{n} \frac{B_r a_r}{r} \to 0$ as $n \to \infty$. (4·2) and similarly with regard to $\frac{1}{B_x'} \int_a^x \psi_{x_0}\left(\frac{1}{t}\right) \frac{B_t'}{t}\, dt \rightarrow 0$ (4·2)

so that we will have established theorem B if we prove the following:

$\bar{B}$. If $x_n = \frac{1}{B_n} \sum_{n_0}^{n} \frac{B_r a_r}{r}$, and $y_n = \frac{1}{B_n'} \sum_{n_0}^{n} \frac{B_r' a_r}{r}$, and

$\frac{1}{B_n'} \sum_{n_0}^{n} \frac{b_r}{B_r} B_r'$ is bounded, then $x_n \rightarrow 0$ implies $y_n \rightarrow 0$. (4·3)

Without the boundedness condition it (4·3) is not necessarily true (4·4)

Proof of $\bar{B}$.—From definition of x_n in $\bar{B}$, we have

$$a_n = n\,(x_n - x_{n-1}) + \frac{n\, b_n}{B_n} x_{n-1}$$

and $y_n = \frac{1}{B_n'} \cdot \sum_{n_3}^{n} B_r'\,(x_r - x_{r-1}) + \frac{1}{B_n'} \cdot \sum_{n_0}^{n} \frac{b_r}{B_r} \cdot B_r' \cdot x_{r-1}$

$= y_{n,1} + y_{n,2}.$ (4·5)

From the boundedness of $\frac{1}{B_n'} \cdot \sum_{0}^{n} \frac{b_r}{B_r} B_r'$, and $x_n \rightarrow 0$ and hypothesis in, 2, about B_n', it follows that

$$y_{n,2} \rightarrow 0 \text{ as } n \rightarrow \infty. \qquad (4·6)$$

From the fact that B_n' increases monotonically to ∞ and $x_n \rightarrow 0$ it follows from Kroneker's theorem that

$$y_{n,1} \rightarrow 0 \text{ as } n \rightarrow \infty \qquad (4·7)$$

thus (4·5), (4·6) and (4·7) establish (4·3) of $\bar{B}$.

(4·4) is covered by example 2 in (3·9); thus establishing $\bar{B}$.

5. *Proof of $\bar{A}$.*—By the usual formula in Fourier series and Riemann-Lebesgue Lemma, we have

$$S_n(x_0) - f(x_0) = \frac{1}{2\pi} \int_0^{\delta} \{f(x_0 + t) + f(x_0 - t) - 2f(x_0)\} \frac{\sin (n + \frac{1}{2})\, t}{\sin t/2}\, dt + \epsilon\,(n, \delta) \qquad (5·1)$$

where $0 < \delta < \pi$, δ to be suitably chosen and fixed and $\epsilon\,(n, \delta) \rightarrow 0$ $n \rightarrow \infty$.

Hence $\left|\frac{1}{B_n} \sum_{0}^{n} b_r S_{n-r}(x_0) - f(x_0)\right| \leqslant \frac{1}{2\pi B_n} \cdot \int_0^{\delta} \psi_{x_0}(t) \cdot \frac{|k_n(t)|}{\sin \frac{t}{2}}\, dt$

$+ \epsilon'\,(n, \delta)$ (5·2)

where $k_n(t) = \sum_0^n b_r \sin\left(n+\frac{1}{2}-r\right)t$ and

$$\epsilon'(n, \delta) = \left|\sum_0^n \frac{b_r\,\epsilon(n-r, \delta)}{B_n}\right| \to 0 \text{ as } n \to \infty.$$

We shall here prove

$$\left.\begin{aligned} &(a) \quad \frac{|k_n(t)|}{\sin\frac{t}{2}} \leqslant (2n+1)\,B_n \\ &(b) \quad \frac{|k_n(t)|}{B_{\frac{1}{t}}} \leqslant k_1 \text{ (some positive constant) in } \frac{1}{n} \leqslant t \leqslant \delta \end{aligned}\right\} \quad (5\cdot3)$$

(*a*) follows from $\left|\frac{\sin mz}{\sin z}\right| \leqslant m$ for integral m.

Proof of 5·3 (*b*).

$$-\sum_0^n b_r \sin\left(n+\frac{1}{2}-r\right)t$$

$$= \text{Imaginary part of } \left\{e^{-i\left(n+\frac{3}{2}\right)t} \cdot \sum_0^n b_r\, e^{i(r+1)t}\right\}$$

and $\sum_{r=0}^n b_r\, e^{i(r+1)t} = \sum_{r=0}^{\left[\frac{1}{t}\right]} + \sum_{\left[\frac{1}{t}\right]+1}^{n} \quad \left(\frac{1}{n} \leqslant t\right)$.

$$= A_{n,1} + A_{n,2}.$$

By hypothesis in 2 about b_n and Abel's inequality we have

$$|A_{n,2}| \leqslant \frac{k}{t} \cdot b_{\frac{1}{t}} \leqslant k'\, B_{1/t};$$

and

$$|A_{n,1}| \leqslant \cdot\, B_{1/t}.$$

Hence $|k_n(t)| \leqslant k_2 \cdot B_{1/t}$ in $\frac{1}{n} \leqslant t \leqslant \delta$ thus establishing (5·3) *b*.

Now $$\frac{1}{2\pi B_n}\int_0^\delta \psi_{x_0}(t)\,\frac{|k_n(t)|}{\sin\frac{t}{2}}\,dt = \frac{1}{2\pi B_n}\int_0^{\frac{1}{n}} + \frac{1}{2\pi B_n}\int_{\frac{1}{n}}^{\delta}$$

$$= E_n + F_n \quad (5\cdot4)$$

By (5·3) *a*, $E_n \leqslant \frac{2n+1}{2\pi} \cdot \int_0^{\frac{1}{n}} \psi_{x_0}(t)\,dt.$

and by Note 1, under B, hypothesis of $\bar{A}$ or A, namely (3·1), implies

$$n \int_0^{\frac{1}{n}} \psi_{x_0}(t)\, dt \to 0 \; n \to \infty.$$

Hence $$E_n \to 0 \text{ as } n \to \infty \tag{5·5}$$

$$F_n \leqslant \frac{k_3}{2\pi B_n} \cdot \int_{\frac{1}{n}}^{\delta} \psi_{x_0}(t) \cdot \frac{B_{1/t}}{t}\, dt \text{ by } 5\cdot 3\,(b)$$

$$\leqslant \frac{k_3}{2\pi B_n} \cdot \int_{\frac{1}{\delta}}^{n} \psi_{x_0}\left(\frac{1}{t}\right) \frac{B_t}{t}\, dt \text{ which by hypothesis of A}$$

converges to 0 with $n \to \infty$. (5·6)

Hence (5·2), (5·5) and (5·6) establish $\bar{A}$ (and with B, A also).

6. THEOREM C: If $f(x)$ be a periodic summable function and

$$(a)\ \frac{1}{B_x} \int_a^x \theta(t) \cdot b_t\, dt \to 0 \text{ or } (a')\ \frac{1}{x} \int_a^x \theta(t)\, dt \to 0\ (a'')\ \theta(x) \to 0 \text{ as } x \to \infty$$

where $$\theta(t) = \psi_{x_0}\left(\frac{1}{t}\right) \cdot \frac{B_t}{t \cdot b_t}, \tag{6·1}$$

and (b) the Fourier coefficients of $f(x) \leqslant \frac{A}{p_n}$, $(A > 0)$. (6·2)

then the Fourier series of $f(x)$ converges at x_0 to $f(x_0)$;

the terms b_t, B_t, p_n, $\psi_{x_0}(t)$ being as follows:—

[(1) $b_0 > 0$;

(2) $\frac{b_0}{b_1} \geqslant \frac{b_1}{b_2} \geqslant \cdots \geqslant \frac{b_n}{b_{n+1}} \to 1$ as $n \to \infty$;

(3) $\sum_0^n b_r = B_n \to \infty\ n \to \infty$;

(4) $b_t = b_{[t]}$, $B_t = B_{[t]}$;

(5) p_n is defined by $\frac{B_{p_n}}{B_n} \leqslant \delta < \frac{B_{(p_n+1)}}{B_n}$, $0 < \delta < 1$, (δ being fixed);

(6) $\psi_{x_0}(t)$ as in 1·1]. (6·3)

Conditions (1), (2), (3) and (4) of (6·3) imply that b_n's and B_n's satisfy (1) and (2), (3) and (4) of 2.

Hence by theorems A and A_1, (6·1) implies that

$$\frac{1}{B_n}\sum_0^n b_r S_{n-r}(x_0) \to f(x_0) \text{ as } n \to \infty \tag{6·4}$$

In a paper[8] to be shortly published, I have established the following Tauberian Theorem:—"Under conditions (1) to (5) about b_n, if a sequence S_n is such that

$$(a)\ \frac{1}{B_n}\sum^n b_r S_{n-r} \to S \text{ as } n \to \infty \text{ and } (b)\ S_n - S_{n-1} \leqslant \frac{A}{p_n}\ (A>0)$$

then S_n converges to S in the ordinary sense. (6·5)

In view of this, conditions (6·2) and (6·4) establish theorem C.

7. *Special Cases of A and C.*—In these special cases we shall consider, the coefficients satisfy (6·3) thence the conditions of 2, so that we can consider the special cases of A and C, together.

(1) $b_n \sim \frac{1}{n^\sigma}$ $0 \leqslant \sigma < 1$. Theorem $\bar{A} \equiv$ (F. L. H.) theorem, and the Tauberian part of C = Cesaro-Tauber theorem for (C, r) $0 < r \leqslant 1$.

(2) $b_n = \frac{1}{n+1}$. Theorem A for this case is discussed in my paper[7] Theorem C in this case is the Hardy-Littlewood[5] (1932) test with slightly broader conditions.

(3) Corresponding to $b_n \sim \frac{1}{n \log n}, \frac{1}{n \log n \log \log n}$, etc., the point condition of A or A_1 gets stiffer and stiffer, and the Tauberian condition in C gets broader and broader, for example in the case $b_n \sim \frac{1}{n \log n}$, the point condition is in its simplest form is $\psi_{x_0}(t).\ \log t \cdot \log|\log t| \to 0\ t \to 0$.

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