A PROPERTY OF INTEGRAL FUNCTIONS WITH REAL ROOTS AND OF ORDER LESS THAN TWO

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I

ERDOS AND GRUNWALD* have proved the following theorem:—

Let $f(x)$ be a real polynomial, with real roots and let (1) $x=a_1$ and $x=a_2$ be consecutive single roots of $f(x)$, (2) Y, the height of the point of intersection of the tangents to the curve $y=f(x)$ at $(a_1, 0)$, $(a_2, 0)$ above the X-axis, (3) M the maximum of $|f(x)|$ in $(a_1\ a_2)$, then taking $a_1 < a_2$

$$\tfrac{2}{3}\,\mathrm{M}\,(a_2-a_1) \geqslant \int_{a_1}^{a_2} |f(x)|\,dx \geqslant \tfrac{1}{3}\,|\,\mathrm{Y}\,|\cdot(a_2-a_1). \qquad (1\cdot1)$$

It is the object of this paper to give four very comprehensive theorems which yield the best possible inequality [given as (3·3)] of the type (1·1). It will be obvious in the course of the paper that the inequality (1·1) is the best possible only in cases of symmetry. It may be noted that the method adopted here is quite different from that of Erdos, and Grunwald.

II. Statement of Theorems

THEOREM (T_1): (1) Let $\phi(x)$, a real integral function, be defined by

$$\phi(x) = e^{\alpha x+\beta}\ \prod_{1}^{\infty}\left(1-\frac{x}{a_n}\right)\ e^{\frac{x}{a_n}},$$

where α, β, $a_1, \cdots a_n \cdots$ are all real and $\Sigma\,\frac{1}{a_n^{\,2}}$ is convergent.

(2) Let $x=a_1$, $x=a_2$ $(a_1 < a_2)$ be consecutive single roots of $\phi(x)=0$ and $\phi(x)>0$ in $a_1<x<a_2$.

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* "On polynomials with real roots", in *Annals of Mathematics*, July 1939 (3), 537, **40**.

(3) $\phi'(a_1) = m_1$ and $\phi'(a_2) = -m_2$.

Then, if $A = \frac{\sqrt{m_1 m_2}}{a_2 - a_1} \cdot e^{-B\left(\frac{a_1 + a_2}{2}\right)}$ and $B = \log \frac{m_2}{m_1} \Big/ (a_2 - a_1)$

all the curves $Y = \phi(x)$ [where $\phi(x)$ satisfies conditions (2) and (3) and is of the type defined in (1)] in $a_1 \leqslant x \leqslant a_2$, lie above the minimum curve

$$Y = m(x) = A \cdot e^{Bx} \cdot (x - a_1)(a_2 - x) \qquad (2\cdot 1)$$

and you can find real polynomials with real roots satisfying conditions (2) and (3) as near as you like to the minimum curve of (2·1).

THEOREM (T_2): Let $\phi(x)$ be a function satisfying conditions (1) and (2) of Theorem I and (3)′:—let $\phi(x)$ attain its maximum value in $a_1 \leqslant z \leqslant a_2$ at x_0 and $\phi(x_0) = M_0$.

Then all the curves $Y = \phi(x)$, [where $\phi(x)$ satisfies (1), (2) and (3)′], in $a_1 \leqslant x \leqslant a_2$ lie below the curve

$$Y = \bar{A}(x - a_1)(a_2 - x)\, e^{\bar{B} \cdot x} = M(x) \qquad (2\cdot 2)$$

where $\bar{B} = \frac{2x_0 - a_1 - a_2}{(x_0 - a_1)(a_2 - x_0)}$ and $\bar{A} = \frac{M_0\, e^{-\bar{B} x_0}}{(x_0 - a_1)(a_2 - x_0)}$

and you can find the real polynomials with real roots satisfying (1), (2) and (3)′ as near as you like to the maximum curve of (2·2).

THEOREM (T_3): The curve $Y = m(x)$ of (2·1) (of Theorem T_1) lies above the parabola which touches the lines $Y = m_1(x - a_1)$, $Y = -m_2(x - a_2)$, at $(a_1, 0)$, $(a_2, 0)$ respectively. It coincides with it in case of symmetry, namely, $m_1 = m_2$.

THEOREM (T_4): The curve $Y = M(x)$ of (2·2) (of Theorem T_2) lies below the parabola passing through $(a_1, 0)$, $(a_2, 0)$ and touching the line $Y = M_0$ at $x = x_0$. If $x_0 = \frac{a_1 + a_2}{2}$, then the parabola will coincide with $Y = M(x)$.

III

From Theorems T_1 and T_3 we obtain

$$\int_{a_1}^{a_2} \phi(x)\, dx \geqslant \int_{a_1}^{a_2} m(x)\, dx = (a_2 - a_1)^2 \sqrt{m_1 m_2} \left\{ \frac{1}{6} + \frac{1}{2} \int_0^{B_1} \frac{u(\sinh u - u)}{B_1^{\,3}} \cdot du \right\}$$

$\left[\text{where } B_1 = \frac{B(a_2 - a_1)}{2}, \text{ B being defined in } (2\cdot 1)\right]$

$\geq$ Area of the parabola of contact (of Theorem T_3)† $= \frac{1}{3} \cdot Y(P) \cdot (a_2 - a_1)$. (3·1)

where Y (P) is the Y co-ordinate of P, the point of intersection of tangents at $(a_1, 0)$ $(a_2, 0)$ to the curves $Y = \phi(x)$.

From Theorems T_2 and T_4 we get,

$$\int_{a_1}^{a_2} \phi(x)\,dx \leq \int_{a_1}^{a_2} M(x)\,dx = \frac{e^{\frac{2u_0^2}{1-u_0^2}}}{1-u_0^2} \cdot 2M_0(a_2 - a_1) \cdot \int_0^{B_2} \frac{u \sinh u}{B_2^3} \cdot du$$

$\left[\text{where} \qquad u_0 = \frac{2x_0 - a_1 - a_2}{a_2 - a_1} \text{ and } B_2 = \frac{2u_0}{1-u_0^2}\right]$

$\leq$ Area of the parabola of (Theorem T_4)‡ $= \frac{2M_0}{3}(a_2 - a_1)$ (3·2)

so that combining (3·1) and (3·2), if $\phi(x)$ satisfies (1), (2), (3) and (3)′ we obtain

$$\frac{2}{3}M_0(a_2 - a_1) \geq 2M_0(a_2 - a_1) \cdot \frac{e^{\frac{2u_0^2}{1-u_0^2}}}{1-u_0^2} \cdot \int_0^{B_2} \frac{u \cdot \sinh u}{B_2^3}\,du \geq \int_{a_1}^{a_2} \phi(x)\,dx \geq$$

$$\geq \sqrt{m_1 m_2}(a_2 - a_1)^2 \left\{\frac{1}{6} + \frac{1}{2}\int_0^{B_1} \frac{u(\sinh u - u)}{B_1^3} \cdot du\right\} \geq \frac{1}{3} Y(P) \; (a_2 - a_1). \quad (3\cdot 3)$$

It will be obvious from the proofs of Theorems T_3 and T_4 that equality sign can occur in the first and last inequality signs in (3·3), only in case of symmetry, *i.e.*, $m_1 = m_2$ and $x_0 = \frac{a_1 + a_2}{2}$.

Note.—Generalization of Theorem T_1 is possible to the case where conditions (2) and (3) may be replaced by

$(2)_1$ $\phi(a_1) = 0$ a zero of q_1th degree and $\phi(a_2) = 0$ of q_2th degree and $\phi(x) > 0$ in $a_1 < x < a_2$.

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†, ‡ The following two elementary propositions are implied :

(1) If $Y = P(x)$ be a parabola touching the lines $Y = m_1(x - a_1)$ and $Y = -m_2(x - a_2)$ at $(a_1, 0)$, $(a_2, 0)$ respectively, then $\int_{a_1}^{a_2} P(x)\,dx = \frac{1}{3} Y(P)(a_2 - a_1) = \frac{m_1 m_2}{m_1 + m_2} \cdot \frac{(a_2 - a_1)^2}{3}$.

(2) If $Y = P(x)$ a parabola passes through $(a_1, 0)$ and $(a_2, 0)$ and touches $Y = M_0$ at $x = x_0$ then $\int_{a_1}^{a_2} P(x)\,dx = \frac{2}{3} M_0 (a_2 - a_1)$.

$(3)_1$ $\phi^{(q_1)}(a_1) = m_1$ $\phi^{(q_2)}(a_2) = \{(-1)^{q_2}\} m_2$. Then the minimum curve will be of the type $Y = A(x-a_1)^{q_1} (a_2 - x_1)^{q_2} e^{B \cdot x}$ in $a_1 \leqslant x \leqslant a_2$.

A and B being expressible in terms of m_1 m_2, a_1, a_2 q_1 q_2.
similar generalization for Theorem T_2 is possible.

Proof of Theorem (T_1): Let $m(x)$ be the function defined in (2·1) and $\phi(x)$ the function satisfying (1), (2) and (3).

Let $F(x) = \frac{\phi(x)}{m(x)}$. Then from condition (3) of Theorem T_1

$$F(a_1) = F(a_2) = 1 \tag{4·1}$$

and from condition (1) of Theorem T_1 we obtain

$$\frac{d^2}{dx^2} \log F(x) = -\sum_{n=3}^{\infty} \frac{1}{(x-a_n)^2} = -\mu(x), \tag{4·2}$$

where all the a_n's for $n \geqslant 3$ lie outside the interval $a_1 \leqslant x \leqslant a_2$ by condition (2). Since $\log F(a_1) = \log F(a_2) = 0$, we can easily prove

$$\text{Log}\, F(x) = \frac{(x-a)_1 \, (a_2 - x)}{\underline{|2}} \mu(\xi), \text{ where } a_1 < \xi < a_2 \tag{4·3}$$

so that $\log F(x) \geqslant 0$, equality occurring only when $F(x) \equiv 1$ so that

$$\phi(x) \geqslant m(x). \tag{4·4}$$

To prove the latter half of the theorem, we assume without any loss of generality $a_1 = -1$, $a_2 = +1$ and let $\acute{m}(x)$ stand for the corresponding minimum function of (2·1).

Let $P(x)$ be defined by $P(x) = D(1-x^2)\left(1-\frac{x}{a}\right)^n$ (4·5)

D, and a, to be suitably chosen such that $|a| > 1$ and $P'(1) = -m_2$, $P'(-1) = m_1$.

Since $P'(x) = -2Dx\left(1-\frac{x}{a}\right)^n + (1-x^2) \cdot D \cdot \{\cdot \ \cdot\} \cdot$ and $P'(1) = -m_2$,

$$P'(-1) = m.$$

We obtain $D\left(1-\frac{1}{a}\right)^n = \frac{m_2}{2}$, $D\left(1+\frac{1}{a}\right)^n = \frac{m_1}{2}$

so that
$$a = \frac{\left(\frac{m_1}{m_2}\right)^{\frac{1}{n}} + 1}{\ldots} \text{ if } m_1 \neq m_2. \tag{4·6}$$

Let $m_1 > m_2$ (argument being of the same type when $m_2 > m_1$)

and let $$\theta(x) = \log \frac{P(x)}{m(x)}.$$

Then $$\theta(\pm 1) = 0 \text{ and } \theta''(x) = -\frac{n}{(x-a)^2}$$

and if $-1 \leqslant x \leqslant 1$ $0 \leqslant \theta(x) \leqslant \frac{1}{2}(1-x^2)\cdot$ [Max. of $-\theta''(x)$ in $(-1, 1)$]
since $m_1 > m_2$, $a > 1$

and Max. of $-\theta''(x)$ will be $$\frac{n}{(1-a)^2} = \frac{n}{4}\left\{\left(\frac{m_1}{m_2}\right)^{\frac{1}{n}} - 1\right\}^2 \qquad (4\cdot 7)$$

and will be of order $O\left(\frac{1}{n}\right)$ for large n

so that for large n, $P(x) = m(x)\cdot e^{O\left(\frac{1}{n}\right)}$ or $|P(x) - m(x)| = O\left(\frac{1}{n}\right)$ in $|x| \leqslant 1$.
(4·8)

In case $m_1 = m_2$, $m(x)$ itself will be a quadratic, thus proving the latter half of Theorem T_1.

Proof of Theorem T_2: Let $M(x)$ be the function defined in (2·2) and $\phi(x)$ the function satisfying conditions (1), (2) and (3)′, and let

$$F(x) = \log \frac{\phi(x)}{M(x)}.$$

Then from condition (3)′ we obtain $F(x_0) = F'(x_0) = 0$ (4·9)

and from condition (1) $$\frac{d^2}{dx^2}F(x) = -\sum_{3}^{\infty} \frac{1}{(x-a_n)^2} = -\mu(x), \qquad (4\cdot 10)$$

where all a_n's for $n \geqslant 3$ lie outside $a_1 \leqslant x \leqslant a_2$

Now $$F(x) = F(x_0) + (x-x_0)F'(x_0) - \frac{(x-x_0)^2}{\lfloor 2}\cdot \mu(\xi)$$

$$= -\frac{(x-x_0)^2}{\lfloor 2}\cdot \mu(\xi) \quad \text{for } a_1 \leqslant x \leqslant a_2 \qquad (4\cdot 11)$$

ξ being in the interval (x_0, x).

Now $\mu(x) > 0$ in $a_1 \leqslant x \leqslant a_2$

so that $F(x) \leqslant 0$ in $a_1 \leqslant x \leqslant a_2$

equality occurring only when $F(x) = 0$.

i.e., in $a_1 \leqslant x \leqslant a_2$ $M(x) \geqslant \phi(x)$. (4·12)

The latter half of the Theorem T_2 can be proved in the same manner as the corresponding part of Theorem T_1.

Proof of Theorem T_3: Without loss of generality we take $a_1 = -1$, $a_2 = +1$, given $m_2 \neq m_1$, the parabola of contact will be given by

$$Y = \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta x} - \alpha - \beta x = P(x) \qquad \text{in } -1 \leqslant x \leqslant 1$$

where $$\alpha = \frac{2 m_1 m_2 (m_1 + m_2)}{(m_1 - m_2)^2} \qquad \beta = \frac{2 m_1 m_2}{m_1 - m_2}. \tag{4·13}$$

Let $F(x) = \log \frac{m(x)}{P(x)}$. Then since $\frac{1}{P(x)} = \frac{\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta x} + \alpha + \beta x}{\beta^2 (1 - x^2)}$

$$F(x) = \log \frac{m(x)}{\beta^2 (1 - x^2)} + \log \{\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta x} + \alpha + \beta x\}$$

$$= \quad ,, \quad + \log \theta(x) \tag{4·14}$$

Now $$F(\pm 1) = 0 \text{ and } \frac{d^2}{dx^2} F(x) = \frac{d^2}{dx^2} \log \theta(x) = \frac{\theta''}{\theta} - \frac{(\theta')^2}{\theta^2}$$

$$\text{and} \quad \theta'' = - \frac{(\alpha\beta)^2}{(\alpha^2 + \beta^2 + 2\alpha\beta x)^{\frac{3}{2}}}. \tag{4·15}$$

Since $\theta > 0$ and $\theta'' < 0$ in $|x| \leqslant 1$

$$\frac{d^2}{dx^2} F(x) < 0 \quad \text{in } |x| < 1. \quad \text{Let} - \mu(x) = \frac{d^2 F}{dx^2}.$$

Then if $|x| \leqslant 1$ $$F(x) = \left(\frac{1 - x^2}{2}\right) \cdot \mu(\xi) \quad -1 < \xi < 1$$

Hence $$F(x) > 0 \qquad \text{in } |x| < 1$$

or $$m(x) > P(x) \qquad \text{in } |x| < 1. \tag{4·16}$$

In case $m_1 = m_2$ the parabola of contact will be defined

$$\text{by} \quad Y = \frac{m_1}{2}(1 - x^2) \tag{4·17}$$

and coincides with $Y = m(x)$ the minimum curve of (2·1).

Proof of Theorem T_4: As in Theorem T_3 we take $a_1 = -1$, $a_2 = +1$. Let $x_0 \neq 0$. Then the parabola passing through $(\pm 1, 0)$ and touching $Y = M_0$ at x_0 will be

$$Y = \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta x} - \alpha - \beta x = P(x)$$

where $$\alpha = \frac{M_0}{2 x_0^2} \qquad \beta = -\frac{M_0}{x_0} \tag{4·18}$$

and let $$F(x) = \log \frac{M(x)}{P(x)}.$$

Then $$F(x_0) = F'(x_0) = 0$$

and arguing as in Theorem T_3 we prove

$$F''(x) < 0 \quad \text{in} \quad -1 \leq x \leq 1 \tag{4·19}$$

and as in Theorem T_2 for x in $-1 \leq x \leq 1$, $F(x) < 0$ when $x \neq x_0$,

i.e., $$M(x) < P(x) \quad \text{when } x \neq x_0$$

$$\leq P(x) \quad \text{for} \quad x = x_0, \quad -1 \leq x \leq 1. \tag{4·20}$$

In case $x_0 = 0$ the parabola will be $Y = M_0(1 - x^2)$ and coincides with the maximum curve of (2·2). (4·21)