ON THE SKEWNESS OF DISTRIBUTION OF THE GENERATORS OF A RULED SURFACE

By V. R. Chariar, F.A.Sc.

(*Science College, Patna*)

Received February 21, 1949

1. In a paper published in the *Bulletin of the Calcutta Mathematical Society*, the author has proved that the line of striction of the generators of a ruled surface is also the locus of the feet of transversals intersecting consecutive and ultimately coincident generators of the ruled surface at the same constant angle. A certain measure, called the skewness of distribution of the generators has been introduced, the vanishing of which means that the generators are all parallel to the same plane. It is also proved that this measure is independent of the directrix chosen and that it is connected with the curvature properties of the surface and in particular, the skewness of distribution is equal to the sum of the product of the parameter of distribution and the first curvature at the central point and the cotangent of the angle in which the line of striction cuts the generators.

2. The object of the present paper is to examine under what circumstances the skewness—μ—is constant. It is proved that if μ be constant, the generators make a constant angle with a fixed direction. It is also proved that if the osculating quadrics of the ruled surface are equilateral hyperboloids, μ is equal to half the cotangent of the angle in which the line of striction cuts the generators. Dini-Beltrami's Theorem about ruled W-surfaces, which appears to be incomplete in its statement, is completed and it is proved that ruled W-surfaces are of two types but in both the types the relation between J and $\sqrt{-\mathrm{K}}\,(=\tau)$ is of the form

$$a\mathrm{J} = \tau^{1/2} + b\tau^{3/2}.$$

The relation between the Laguerre function along a curve and the skewness of distribution of the normals to the surface along the curve is obtained.

3. Lines drawn through a point, parallel to the generators of a scroll, cut the unit sphere with the point as centre, in points which trace out a curve called the spherical indicatrix of the generators. For the spherical indicatrix, in the notation of Weatherburn's Differential Geometry, we have

$$r = d, \qquad ts' = d'$$

$$kns'^2 + ts'' = d''$$

$$\therefore\ [r, t, n]\, ks'^3 = [d, d', d'']$$

where k is the curvature, n the principal normal of the spherical indicatrix and dashes denote differentiation w.r.t. the arc of the directrix curve. Now because, $|d'| = s'$,

$$[d, d', d'']/|d'|^3 = k\,[r, t, n].$$

Again $n \times r = t \cos \omega$, ω being the angle that the binormal makes with the radius vector and hence

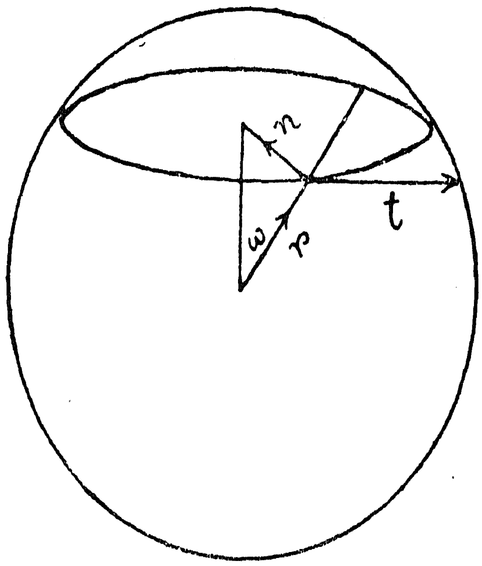
$$[n, r, t] = \cos \omega$$

and therefore, $\mu = k \cos \omega$

$$= \cot \omega \qquad (\because \; k = 1/\sin \omega)$$

$$= \rho'\sigma/\rho,$$

where ρ and σ are the radii of curvature and torsion of the spherical indicatrix and dashes denote differentiation w.r.t. the arc. Again, because the curve is spherical,



$$\rho/\sigma + \frac{d}{ds}(\rho'\sigma) = 0,$$

i.e.,

$$\rho'/\mu + \frac{d}{ds}(\mu\rho) = 0,$$

or

$$\rho'(\mu + 1/\mu) + \mu'\rho = 0.$$

If now, μ be constant, $\rho' = 0$ and therefore ρ is constant and hence $\tau = 0$ or the spherical indicatrix is a circle and the generators are inclined at a constant angle to a fixed line—the normal to the plane of the circle. Hence we have the theorem—*The skewness of distribution is equal to the cotangent of the angle which the binormal of the spherical indicatrix makes with the radius vector and if the skewness be constant, the generators are inclined at a constant angle to a fixed line.*

4. If t be the unit tangent to the line of striction we have $t \cdot d' = 0$ and $d \cdot d' = 0$. Also if μ be const., $a \cdot d =$ const. and therefore $a \cdot d' = 0$ and hence $[a, t, d] = 0$ or the vector a lies in the plane of t and d. Hence to draw a surface with a given line of striction and skewness we draw a set of parallel straight lines at points of the given curve and in the tangent plane to the cylinder so drawn, we draw a set of lines making a constant angle with the generators. Thus there are ∞^2 of ruled surfaces with a given line of striction and skewness of distribution.

In terms of t, n, b of the line of striction, let

$$a = t \cos \theta - n \sin \theta \sin \omega + b \sin \theta \cos \omega.$$

Now because a is a constant vector, $a' = 0$ and therefore

$$\frac{d\theta}{ds} = k \sin \omega \qquad \text{(unless } \theta = 0) \tag{4. 1}$$

and
$$\frac{d\omega}{ds} + \tau = k \cos \omega \cot \theta \tag{4. 2}$$

unless $\omega = \pi/2$ when the line of striction is an asymptotic line. The first equation gives the geodesic curvature and the second, the geodesic torsion of the line of striction which will be a geodesic if θ be constant and then $\omega = 0$ (or $k = 0$) and the line of striction is a helix. Hence we have the

THEOREM.—*If the line of striction of the generators of a ruled surface whose skewness is constant be a geodesic, it must be a helix* (*or a straight line*).

If $\theta = 0$, the line of striction is a straight line and the generators make a constant angle with the given line as in oblique or right helicoid. If $\omega = \pi/2$, we have from the constancy of a, $\tau \sin \theta = 0$. Therefore either the curve is a plane curve when the surface is a plane or $\theta = 0$ when the line of striction is a straight line. Eliminating ds between the two equations and integrating we have

$$\sin \theta \cos \omega = c - \int \frac{\tau}{k} \sin \theta \, d\theta.$$

5. Taking the generators and their orthogonal trajectories as v = const. and u = const. and in the notation of Weatherburn's *Differential Geometry*, Vol. II, we have

$$\psi^2 = (u - \alpha)^2 + \beta^2,$$

$$\mathrm{J} = \frac{1}{\psi}\left[\nu - \frac{\beta' (u - \alpha) + \alpha'\beta}{\psi^2}\right],$$

and
$$\mathrm{K} = -\beta^2/\psi^4.$$

The line of striction is given by $u = \alpha$ which cuts the generators at an angle ϕ given by $\cot \phi = \pm \alpha'/\beta$. Now the value of J at the central point is given by

$$\mathrm{J}_0 = \frac{1}{\beta}[\nu - \alpha'\beta/\beta^2] = \frac{1}{\beta}[\nu - \cot \phi]$$

and therefore $\nu = \mu$ = the skewness of distribution. This gives the geometrical interpretation of ν involved in J. From the equation $\mathrm{J}_1\mathrm{K}_2 - \mathrm{J}_2\mathrm{K}_1 = 0$ which must reduce to an identity we have

$$\mu = \text{const.}, \ \beta = \text{const. and } \alpha' = \text{const.}$$

From the constancy of α' and β we have Dini-Beltrami's Theorem for a ruled W-surface.

Now because $\mu =$ const., the generators make a constant angle with a fixed vector a and therefore

$$\frac{d\theta}{ds} = k \sin \omega = 0. \qquad (\because \quad \phi = \cot^{-1} \alpha'/\beta = \text{const.}).$$

Hence either $k = 0$, *i.e.*, the line of striction is a straight line or $\omega = 0$. From the equation (4. 2) we have because θ is const.

$$\tau/k = \cot \theta,$$

and therefore the line of striction is a helix. Again because in the notation of the previous section, the value of β is given by

$$\beta = \frac{\sin (\theta - \alpha) \sin \theta}{k \sin \alpha \cos \omega} = \frac{\sin (\theta - \alpha) \sin \theta}{k \sin \alpha},$$

we have $k =$ const. because β is so and therefore τ is also constant and the line of striction is a right circular helix, the generators being drawn in the tangent planes of the cylinder making a constant angle with the generators of the cylinder.

If $k = 0$, we may take

$$r = [0, 0, f(\theta)] \text{ and } d = (\sin \alpha \cos \theta, \sin \alpha \sin \theta, \cos \alpha)$$

and because β is constant, $f(\theta) = c\theta$ and in that case the surface is a helicoid of angle α given by

$$r = (u \cos \alpha \cos \theta, \; u \sin \alpha \sin \theta, \; c\theta + u \cos \alpha).$$

If the line of striction is a right circular helix, the equation of the surface can be written as

$$\mathrm{R} = r + ud$$

where $$r = (a \cos \theta, \; a \sin \theta, \; a\theta \cot \alpha)$$

and $$d = (- \sin \theta \sin \beta, \; \cos \theta \sin \beta, \; \cos \beta).$$

When $a \to 0$, *i.e.*, the right circular cylinder shrinks to a straight line, and $\alpha \to 0$ such that $a \cot \alpha \to c$, the surface tends to

$$\mathrm{R} = (- u \sin \theta \sin \beta, \; u \cos \theta \sin \beta, \; c\theta + u \cos \beta)$$

which is a helicoid of angle β. Hence we have the

THEOREM.—*If a ruled surface is also a W-surface, the line of striction is a right circular helix (or a straight line) and the generators are lines in the tangent planes, making a constant angle with the generators of the cylinder (or a helicoid of angle* α*).*

6. The fundamental magnitudes of the first and second orders for the oblique helicoid

$$r = (u \sin \alpha \cos \theta,\ u \sin \alpha \sin \theta,\ c\theta + u \cos \alpha)$$

are given by $E = 1,\ F = c \cos \alpha,\ G = c^2 + u^2 \sin^2 \alpha$

and $HL = 0,\ HM = -c \sin^2 \alpha,\ HN = u^2 \sin^2 \alpha \cos \alpha.$

Hence $J = (u^2 - 2c^2) \cot \alpha/(u^2 + c^2)^{3/2}$

and $K = -c^2/(u^2 + c^2)^2$

and therefore $\tau = \sqrt{-K} = c/(u^2 + c^2).$

Hence for the oblique helicoid, the functional relation is given by

$$Jc^{1/2} \tan \alpha = \tau^{1/2} - 3c\tau^{3/2}.$$

For the surface $R = r + ud$ when the line of striction is the right circular helix $r = (a \cos \theta,\ a \sin \theta,\ a\theta \cot \alpha)$

and $d = (-\sin \theta \sin \beta,\ \cos \theta \sin \beta,\ \cos \beta)$ we have

$$E = 1,\ F = a \cos (\alpha - \beta)/\sin \alpha,\ G = a^2 \cos e/c^2\alpha + u^2 \sin^2\beta$$

$$HL = 0,\ HM = a \sin (\alpha - \beta) \sin \beta/\sin^2 \alpha,\ HN = a^2 \sin^2 (\alpha - \beta)/\sin \alpha + u^2 \sin^2 \alpha \cos \beta$$

whence $K = \dfrac{c^2}{(u^2 + c^2)^2}$ where $c = a \sin (\alpha - \beta)/\sin \alpha \sin \beta$

and $J = \cot \beta\, (u^2 - bc)/(u^2 + c^2)^{3/2}$

and therefore

$$Jc^{1/2} \tan \beta = \tau^{1/2} - (b + c)\, \tau^{3/2}.$$

We have thus the

THEOREM—*The relation between the first and second curvatures of a ruled W-surface is of the form*

$$aJ = \tau^{1/2} + b\tau^{3/2}.$$

7. The osculating quadric of a ruled surface is generated by the tangents to the curved asymptotic lines, at points of a given generator. These are perpendicular to the generator at points where $J = 0$ and if they are mutually perpendicular to one another, the values of u corresponding to $J = 0$ must belong to the involution formed by pairs of points on the generator the tangent planes at which are perpendicular to one another, *i.e.*, $\psi = 0$.

Hence $\mu (u - \alpha)^2 - \beta' (u - \alpha) + (\mu\beta^2 - \alpha'\beta) = 0$

must be harmonic with

$$(u - \alpha)^2 \qquad + \beta^2 = 0$$

i.e., $$2\mu - \alpha'/\beta = 0$$

or $$2\mu \mp \cot\theta = 0 \qquad (\because \ \alpha'/\beta = \pm \cot\theta).$$

Hence we have the

THEOREM.—*If the osculating quadric of a ruled surface be equilateral, the skewness of distribution is half the cotangent of the angle in which the line of striction cuts the generator.*

8. Dr. Ram Behari has given the geometrical interpretation of the vanishing of the Laguerre function along a curve drawn on a surface, *viz.*, that the ruled surface generated by the normals to the surface has equilateral osculating quadrics. A new proof is given below in which it is also shown that the value of the Laguerre function (L) along a curve is given by

$$\mathrm{L} + \tau(2\mu + \cot\phi)/\sqrt{\tau^2 + k^2} = 0.$$

Taking the curve as $v =$ const. with unit tangent a, we have in the notation of Weatherburn's *Differeniial Geometry*

$$d = \bar{n},\ d' = a.\nabla\bar{n} = -k_n a - \tau b$$

$$d \times d' = -k_n b + \tau a$$

$$\therefore \quad \beta = [t\ d'd]/|\,d'|^2 = -\tau/(\tau^2 + k_n^2).$$

$$d'' = -k_n\, a.\nabla\ a - \tau a.\nabla b - ak_n' - b\tau'$$

$$= -k_n(k_n\bar{n} + \gamma b) - \tau(\tau\bar{n} - \gamma a) - ak_n' - b\tau'.$$

$$= -(k_n^2 + \tau^2)\,n + (\tau\gamma - k_n')\,a - (k_n\gamma + \tau')\,b.$$

$$[dd'd''] = \{\gamma(k_n^2 + \tau^2) + k_n\tau' - \tau k_n'\}$$

$$\therefore \quad \mu = \{\gamma(k_n^2 + \tau^2) + k_n\tau' - \tau k_n'\}/(\tau^2 + k_n^2)^{3/2}.$$

Again, the line of striction is given by

$$u(\tau^2 + k_n^2) + a.\bar{n}' = 0$$

i.e., $$u = k_n/(\tau^2 + k_n^2).$$

If T denotes the unit tangent to the line of striction

$$\mathrm{TS}' = a + ua.\nabla\bar{n} + \bar{n}\frac{d}{ds}(k_n/\overline{\tau^2 + k_n^2})$$

$$= a\frac{\tau^2}{\tau^2 + k_n^2} - \frac{k_n\tau}{\tau^2 + k_n^2}\,b + \bar{n}\,\frac{d}{ds}\left(\frac{k_n}{\tau^2 + k_n^2}\right),$$

$$\therefore \quad \mathrm{S}'\cos\phi = \frac{d}{ds}(k_n/\overline{\tau^2 + k_n^2})$$

$$\therefore \quad \cot\phi = \frac{(\tau^2 + k_n^2)^{1/2}}{\tau}\,\frac{d}{ds}\left(\frac{k_n}{\tau^2 + k_n^2}\right). \qquad (\tau \neq 0)$$

Hence because β is negative,

$$2\mu + \cot\phi = (2\gamma\tau - k_n')/\tau\sqrt{\tau^2 + k_n^2}$$
$$= -L/\tau\sqrt{\tau^2 + k_n^2}$$

or
$$\frac{\tau}{\sqrt{\tau^2 + k_n^2}}\{2\mu + \cot\phi\} + L = 0.$$

Hence if $L = 0$, μ and $\frac{\pi}{2} - \phi$ are both zero or constants. We have the

THEOREM.—*If a curve on a surface is a Laguerre line and if the normals along the curve make a constant angle with a given line, the line of striction of the normals cut them at a constant angle and conversely.*

REFERENCES

1. Chariar, V. R. .. *Bulletin of the Calcutta Mathematical Society*, 1944.
2. Weatherburn, C. E. .. *Differential Geometry*.
3. Behari, R. .. *Journal of the Indian Mathematical Society*, 1939, 3.