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Sensitivity of launch systems to gravity field

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Abstract. This paper considers the question of how sensitive inertially guided systems are to variations in the gravity field. There are systems which use other kinds of information for guidance, such as terrain. But, they will not be considered here.

Keywords. Launch systems; gravity field; inertially guided systems.

Inertially guided systems (IGS) are intended to keep an object moving along a predetermined path so as to reach a specified target location (Wheelon 1959). The intended path can be defined to a desired degree of accuracy, and actions for correcting any deviation from the desired path may be taken if the deviation is known. A guidance system provides information about such deviations. An IGS uses the acceleration of the vehicle, all three components of which are measured by accelerometers, to compute the position of the vehicle at any time. However, if a vehicle is moving freely under the action of a gravity field, the accelerometers mounted on the vehicle will indicate no accelerations. In other words, the accelerometers only provide information about forces other than the force of gravity (e.g. air drag), that may be acting on the vehicle. In order to correctly compute the trajectory of a vehicle, it is necessary to know the gravity field as a function of position. Any inaccuracy in the knowledge of the gravity field therefore results in a corresponding inaccuracy in the computed position of the vehicle.

It is therefore instructive to examine what should be the order of accuracy with which the gravity field along the path of the vehicle must be known in order that the error in targetting is less than some prescribed value.

It may be recalled that the actual gravity value $g(x, y, z)$ at a point in space, on the path of a vehicle controlled by an IGS, for example, can be calculated by adding the following components:

(i) g_0—the theoretical gravity for a standard earth spheroid at sea level at a given longitude and latitude;
(ii) g_1—the variation in gravity expected to arise from the height of the point above sea level;
(iii) g_2—the effect of solar and lunar tides on the value of gravity;
(iv) g_3—the effect of known anomalous masses; and
(v) g_4—gravity anomaly arising from any unknown anomalous masses in the earth.

We can thus write

$$g(x, y, z) = g_0 + g_1 + g_2 + g_3 + g_4. \qquad (1)$$

The first four terms on the right hand side can be calculated for any specified point. The last term, g_4, is an unknown term which is the source of uncertainty in the knowledge of the gravity field.

Geophysicists and geodesists have measured the gravity field in various parts of the earth and maps of the gravity field, at different accuracies, are available for most parts of the world. The unit of measurement is a gal, which is equal to 1 cm/s^2. It is possible to measure differences between the gravity fields at two points with an accuracy of about 10^{-2} milligal (mgal), or better, using a gravimeter, which is basically an extremely sensitive spring balance. The absolute value of gravity can be measured by using a pendulum and measuring its period of free oscillation. Such measurements can be made with an accuracy better than 0·1 milligal (10^{-4} gal).

Measurements of the earth's gravity field made on the ground, or at sea, have been supplemented by information gathered by tracking orbiting satellites to determine the perturbations in the orbits caused by anomalous changes in the gravity field (Marsh *et al* 1988). Expressed in spherical harmonics, the gravity field at any point can be computed by using a set of coefficients, which are now available up to 360th degree and order as coefficients of associated legendre functions of the first kind. This makes it possible to find the gravity field at any height where higher harmonics have negligible amplitude to an accuracy of about 5 mgal. The gravity field near the ground surface, however, cannot be accurately computed by using these harmonic coefficients. This is because the components of the gravity field at short spatial wavelengths less than about 100 km cannot be determined from data gathered from orbit perturbations of artificial satellites, although all gravity anomalies having a spatial wavelength of 200 km, or more, are known to the above mentioned accuracy.

The overall position is that the gravity field at the ground surface is known from published information to an accuracy of the order of $\pm$ 10 mgal in most parts of the world, except in some inaccessible, mountaneous areas like the Himalayan region, where the uncertainty in the knowledge of the total gravity field is no more than about 150 mgal.

Thus the inaccuracy in the knowledge of the gravity field is different for different heights above the ground surface. If the gravity field on the ground surface is known over a large area, the field at a higher level can be computed with reasonable accuracy. The reverse process of calculating the gravity field near the ground surface from the fields measured at a higher level, say by satellites, is theoretically possible to do, but suffers from the disadvantage that errors in the estimate of the fields at a higher level get magnified in the computation of the fields at a lower level.

If we assume that corrections of the trajectory performed by an IGS are perfect in the sense that the effect of all forces other than that of gravity is completely taken care of, the error in guiding a vehicle to a target would depend only on the uncertainty in the knowledge of the gravity field along its trajectory. The extent to which the uncertainty in knowledge of the gravity field can contribute to errors in reaching the target location may thus be studied by considering a model anomaly in the gravity field and calculating its effect on the target error for a ballistic projectile. Such calculations have been made by assuming model gravity anomalies of specified shape and amplitude, representing the uncertainty in the gravity field. These calculations take into account a spherical, rotating earth with uniformly distributed mass except for the assumed anomaly. The scheme of these calculations is described in Appendix 1.

Although the uncertainties in the gravity field are of a random nature, there are no unknown anomalies larger than 5 mgal with spatial wavelengths of the order of 200 km or more. As such we have assumed anomalies having much larger extent and higher magnitudes to ensure that the computed errors represent outer bounds within which the actual errors must lie.

The assumption of a spherical earth with uniformly distributed mass is as good as the assumption of any other shape with any other known mass distribution because the effect of all known masses can always be taken into account.

As shown in Appendix 1, the calculations are made by taking the actual magnitude and direction of the net gravity field all along the path of the projectile. Therefore, it not only takes care of the deflection of the vertical, but also the changes in the magnitude of the gravity field along the path. This is how these calculations produce the resulting targetting error, taking into account changes in the osculating plane of the trajectory caused by the assumed gravity anomaly.

Calculations with anomalies of different size and location, and ballistic shots in different directions of azimuth showed the following general results:

(i) The effect of a perturbation in the gravity field is much more important when it is at the shot end, compared to the effect of a similar perturbation near the target end.
(ii) The target error is related to a quantity like the product of the amplitude and the areal extent of a gravity anomaly and increases with this product.
(iii) The target error depends on the direction of the azimuth of shot. Thus an eastward shot does not result in the same target error as a shot directed westward even if the gravity anomaly is located at corresponding points. This is due to the effect of the earth's rotation.
(iv) If the gravity anomaly on the ground surface is physically located away from the osculating plane of the projectile at the shot instant, the osculating plane does not remain confined to the plane of a great circle, but instead, changes constantly along the trajectory. This results in an error in a direction perpendicular to the line through the target point (across-track error) in addition to the error along the shortest path from shot point to the target (along-track error). The total error which may be taken as the direct distance between the intended target position and the actual point of landing of the projectile also changes with the azimuth of the shot.
(v) For a given initial direction, with a given anomaly at a given location with respect to the shot point the targetting error becomes largest for some specific value of the initial velocity of the shot.
(vi) If the gravity anomaly is circularly symmetric, the worst case of target error happens when the shot point is close to the centre of the anomaly, and the shot is eastward (figure 1).
(vii) With respect to the latitude of the shot point the worst case occurs when the shot point is on the equator.

On the basis of these findings, the anomaly chosen was that caused by a circular segment of an earth-concentric spherical shell at a depth of 20 km from the earth surface, the circular segment having a radius of 200 km. This shell is assigned a circularly symmetric distribution of mass/unit area, such that the vertical component of acceleration on this surface at the centre is 500 mgal reducing cosinusoidally to zero at the rim of the circular segment 200 km away. The gravity anomaly on the surface of the earth for this anomalous mass is like a circularly symmetric dome with about 406 mgal vertical component of gravity at the centre. A section of this gravity

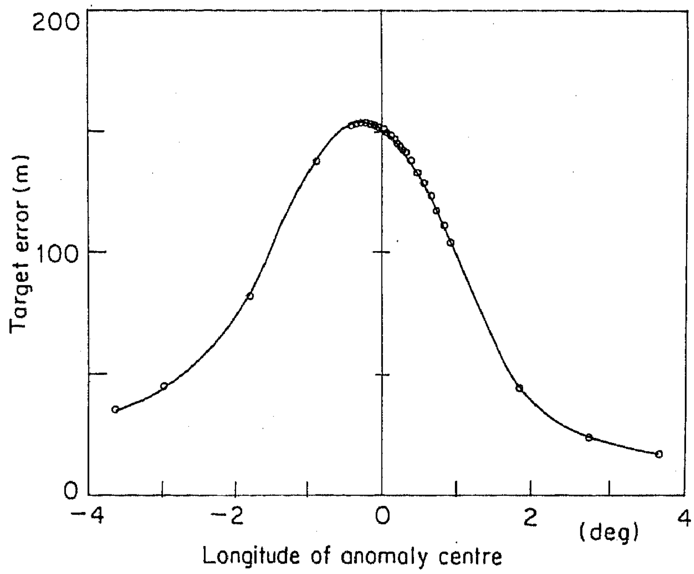

Figure 1. Variation of total target error with position of the anomaly (shown in figure 2) on the equator. Projectile shot eastward at 45° elevation at 5000 m/s from 0° longitude along the equator.

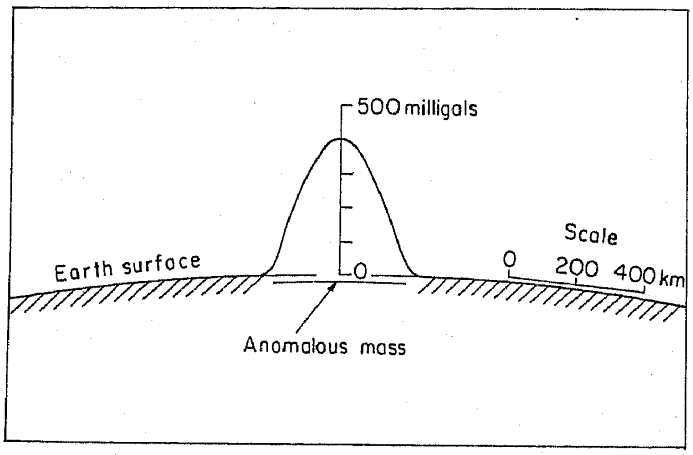

Figure 2. The vertical component of the model gravity anomaly on the ground surface (due to an anomalous mass of 200 km radius located 20 km below the ground surface).

anomaly is shown in figure 2. Such an anomaly is very much larger than any uncertainty in the calculated gravity field based on published information.

The target error for such an anomaly was computed for various positions of the anomaly, and for different values of the azimuth and velocity of the shot at an elevation of 45°.

The target error changes with changes in these parameters. Various values of error along-track and across-track are plotted in figure 3 to show the range of variations. But the largest possible error due to this gravity anomaly was less than 200 m (figure 4). This means that regardless of the positions of the shot point and the centre of the anomaly, or the orientation or velocity of the shot, the target error caused by this anomaly is < 200 m. Since the anomaly chosen is much larger than the possible uncertainty in the knowledge of gravity field anywhere in the world, and even with

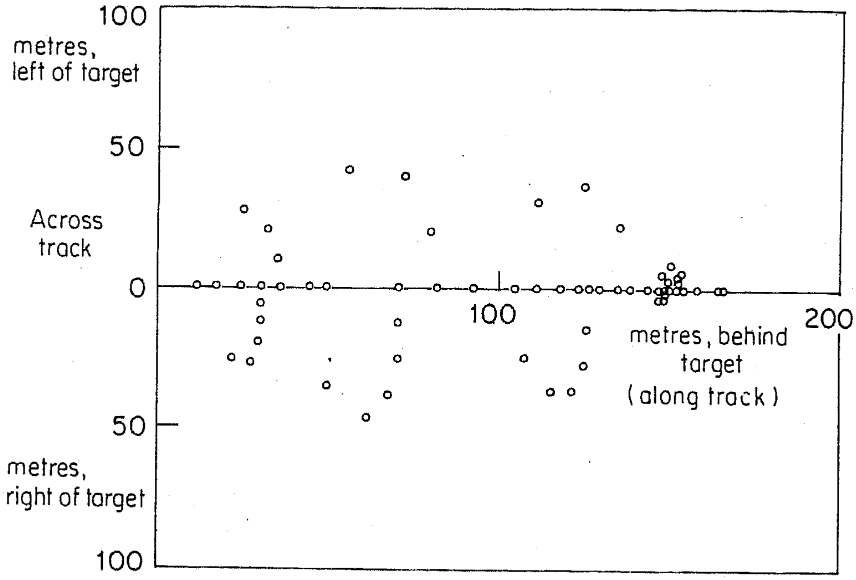

Figure 3. Scatter of along-track and across-track errors with anomaly located at various distances from the shot point and for different values of the velocity of the shot. The shot was taken to be fired at 45° elevation in different directions.

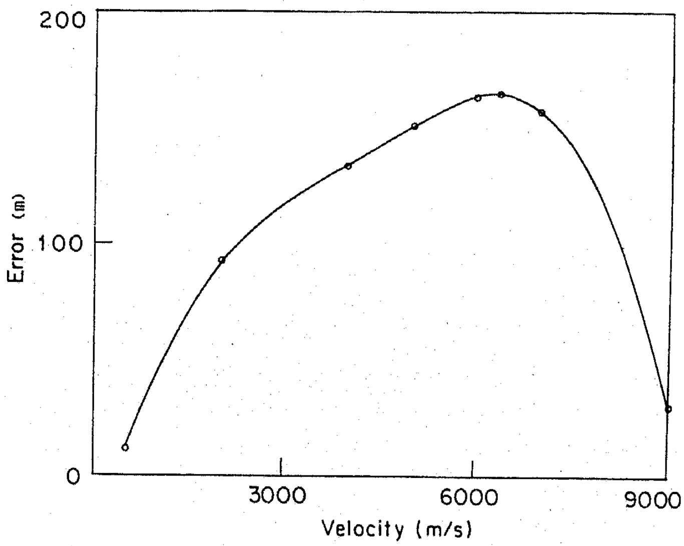

Figure 4. Along-track error for different shot velocities for an eastward shot at 45° elevation from a point on the equator. The model anomaly was centered at the shot point.

such an anomaly the target error is less than what seems tolerable for a missile weapon anyway, it may be concluded that surface gravity data at any finer level of accuracy than what is already available is of no interest for the design or implementation of a ballistic, inertially guided delivery system.

Since it is always possible in principle to deliver a projectile launched at a shot elevation of 45°, to any point on the earth, simply by choosing an appropriate velocity, the target error, caused by the above mentioned anomaly, can be kept within this limit.

In practice, however, shot elevations other than 45° are used. For very long ranges covering intercontinental distances, the launch angle is more near vertical. Since the burn-out process is not instantaneous, this means that the portion of the trajectory where the missile is entirely under the influence of gravity is usually at a very large altitude. In such a case perturbations due to unknown anomalous gravity fields are negligible. The powered and re-entry phases are subjects in themselves which do not concern the question we are discussing here because these can be adjusted to obtain a high accuracy if the intended free flight trajectory is correctly achieved.

For shorter ranges within 1000 km, target errors due to uncertainties in gravity computed on the same basis are also within 200 m.

We now consider some cases other than that of a ballistic projectile. In the case of an orbiting vehicle the extent of possible discrepancy between the expected orbit and the actual orbit as a consequence of unknown gravity anomalies accumulate with passing time. The errors can grow indefinitely in this case if the elapsed time is sufficiently large. Therefore, it becomes essential to monitor the actual orbit by tracking (e.g., from ground stations), and update the position of the vehicle at suitable intervals. Such an operation does not need gravity field information. On the other hand such tracking data provide additional information about the gravity field.

We next consider the case of a vehicle which is propelled during most of its flight to cruise at a low height above the ground, instead of undergoing a 'free fall' under gravity for most of its path. The total deviation in such a case due to possible anomalous masses may be seen in the following manner.

We consider a path that is fixed a priori and is the intended trajectory to be followed by the vehicle. In order to keep to this path, all accelerations due to forces of gravity, air drag, etc., which tend to deflect the vehicle from the intended path, must be exactly compensated by the thrust of the propellant mechanism. This can be done in principle for all known variations of g. As we have seen, the error along track due to uncertainties in gravity will not exceed 200 m. Since we are interested in deviations from the intended path, only the component of the unknown horizontal gravity field across the path is important. Since the uncertainty in the total field is no more than 150 mgal, and the longest wavelength of unknown anomalies is about 100 km, a situation more severe than the actual worst case can be constructed in the following way.

We compute the maximum horizontal gravity field on the slope of a flat-topped mountain which is 4 km high and covers an area of 200 km × 200 km. The slope is taken to have an inclination of 8°, and the density of the rock is taken to be 2·7 g/cc.

The maximum horizontal gravity field due to such a body at the surface of the ground is < 500 mgal. At a height 50 km above the ground it is < 150 mgal. We take such a horizontal anomaly to be present over as much as 200 km along the path of the propelled vehicle. The horizontal velocity along the path is taken to be v m/s. The anomalous horizontal gravity field is taken to be constant, in a direction perpendicular to the path.

The across track velocity is zero to start with. At any later time it is equal to the time integral of the acceleration across track. The deviation is equal to the integral of the across track velocity, and is given by $ad^2/2v^2$ metres where a is the horizontal acceleration due to the gravity field in m/s^2, and d is the distance traversed along the intended path in metres.

The deviation from the intended path increases with time-of-flight, and is inversely proportional to the square of the velocity v along the path. For a projectile flying at a height as low as 50 km and with a velocity of 1 km/s, the horizontal deviation is only 30 cm over 200 km. Even if the target is 1000 km away, the error is < 240 m.

Appendix 1

Scheme used for the approximate computation of the trajectory of a projectile and the perturbation caused by a gravity anomaly.

1. *Coordinate system used*

A cartesian coordinate system with its origin at the earth centre is used. The earth is assumed to be spherically symmetric, except for an anomalous mass as described in the text. The z axis coincides with the north along the earth's axis of rotation. The earth's radius is taken to be 6,371 km.

The earth rotates in this inertial frame, changing the cartesian coordinates of all points on the earth's surface accordingly with passing time. Moreover, the projectile has a velocity, tangential to the earth's surface, added to the relative velocity with which it leaves the surface.

2. *Computation of trajectory*

The position of the projectile and its velocity are known at time $t = 0$ at the starting point as vectors $\mathbf{p}_0$ and $\mathbf{v}_0$. The accelaration $\mathbf{a}$ due to gravity is computed for the point $\mathbf{p}_0$. The time rate of change of acceleration $\mathbf{a}'$ is approximated by finding the position $\mathbf{p}_{-1}$ of the projectile at a very short interval t_s prior to the starting time by using $\mathbf{v}_0$ and $\mathbf{a}$. Its position $\mathbf{p}_{+1}$ after the same interval t_s after start is similarly calculated. The acceleration $\mathbf{a}_{-1}$ at $\mathbf{p}_{-1}$ and $\mathbf{a}_{+1}$ at $\mathbf{p}_{+1}$ are found. The time derivative of acceleration $\mathbf{a}'$ is then

$$\mathbf{a}' = (\mathbf{a}_{+1} - \mathbf{a}_{-1})/2\,t_s.$$

Using these, the position $\mathbf{p}_T$ after a short-time interval T is calculated as

$$\mathbf{P}_T = \mathbf{P}_0 + \mathbf{v}_0\,T + 1/2\mathbf{a}T^2 + 1/6\mathbf{a}'\,T^3,$$

and the new velocity is approximated as

$$\mathbf{v}_T = \mathbf{v}_0 + \mathbf{a}T + 1/2\mathbf{a}'\,T^2.$$

These values of position and velocity then become the values of $\mathbf{P}_0$ and $\mathbf{v}_0$ for the next time step of calculation.

Landing of the projectile is indicated when after a time step of computation the size of the position vectors is found to be less than or equal to the earth radius. The time and position of crossing the earth's surface is computed by appropriate nonlinear interpolation over the last computation time-step, and applying the correction due to earth rotation.

3. *Computation of acceleration*

Acceleration is a function of position with respect to the rotating earth, because the anomalous mass also rotates with it. It has two components: one directed towards the earth centre, given by GM/R^2 where G is the universal gravitation constant, M the earth mass, and R the size of the position vector. The other is due to the anomalous mass causing the gravity anomaly. The vector attraction of this anomalous mass is computed for the point in question by Gaussian integration over the circular segment of the spherical shell constituting the mass. The number of points used for the quadrature is adjusted, according to the requirements of accuracy from 16 to 4096.

This approach makes it easy to take a spherical, rotating earth and compute the acceleration vector at any specified point.

4. *Calculation of target error*

The point of landing is computed first without the gravity anomaly, and then again by taking the anomalous mass. The target error is computed by taking the difference between these two positions. It is then decomposed into two parts: one along the great circle path from the shot point to the intended target, the 'along-track' error, and the other in a direction perpendicular to the first, the 'across-track' error.

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