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Families of Bose Rays in Quantum Optics

N. Mukunda,[1] **E. C. G. Sudarshan,**[2,3] **and R. Simon**[4]

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Having known classical wave optics and wave mechanics, can we reverse Schrödinger's path and extend the concept of families of rays of light to provide a new exact rendering of quantum optics including the Bose nature of photons? This question is answered in the affirmative, and the implications of the Bose symmetry for certain nonlocal correlations of the many-ray distribution functions are worked out. The similarities and the differences between classical and quantum wave optics are brought out. The ray-ray Bose correlation is analyzed. The generating functional for the many-ray distribution functions is formulated; and the notion of paraxial illumination for quantum optics is made precise.

1. PREAMBLE

Schrödinger,[1] in his discovery of wave mechanics, was much inspired by Hamilton's work[2] on the analogy between optics and mechanics. Hamilton first worked on and discovered the usefulness of the "characteristic function" in geometrical optics, based on Fermat's least-time principle in optics. Later Hamilton arrived at his "principal function" in mechanics.

In optics Hamilton found, as an infinitesimal expression of Huygens' principle, the simplest instance of the eikonal equation:

$$|\nabla S(\mathbf{x})| = n(\mathbf{x}) = \text{refractive index} \tag{1}$$

[1] Center for Theoretical Studies and Department of Physics, Indian Institute of Science, Bangalore, India 560012.

[2] Center for Particle Theory, University of Texas, Austin, Texas 78712.

[3] Center for Theoretical Studies, Indian Institute of Science, Bangalore, India 560012.

[4] Institute of Mathematical Sciences, Madras, India 600113.

Here, $S(\mathbf{x})$ has the dimensions of length, so both sides are dimensionless. (To keep track of dimensions of various things, $\hbar$ and c will appear explicitly.) In mechanics he found the Hamilton–Jacobi partial differential equation (P.D.E.), which, for the simplest case of a particle of mass m and energy E moving in a potential $V(\mathbf{x})$ in three dimensions, is

$$|\nabla S(\mathbf{x})| = \sqrt{2m(E - V(\mathbf{x}))} \tag{2}$$

Here $S(\mathbf{x})$ has the dimension of action, so both sides carry dimensions mlt^{-1}. Hamilton's discovery was the similarity of the above two P.D.E.'s in different contexts.

The P.D.E. (2) describes special families of classical particle trajectories in phase space suitable for mechanics. The most elementary notion in mechanics is that of an individual trajectory traced out by a mass point following Hamilton's ordinary differential equations of motion. In optics, though, while (1) describes a bundle of rays in the geometrical optics limit of wave optics,[(3)] this bundle or family is not in any physical sense built up from individual trajectories of some localizable physical object or entity. In optics in this limit, the *family* of rays *is* the really primitive concept, even though formally one can imagine a hypothetical "point" tracing a path in space according to Fermat's principle, formally "quantize"[(4)] such a system with the wavelength playing the role of $\hbar$, and thus arrive at classical wave optics.

In any case, it is seen that in the geometrical optics limit of wave optics, the basic notion is that of a *bundle of rays*, the bundle described by the single function $S(\mathbf{x})$ of (1). Schrödinger asked whether the bundle of classical particle trajectories described by (2) could similarly be the "geometrical" limit of an underlying wave theory, and was led to this wave mechanics. The "geometrical" limit now corresponds to $\hbar \to 0$ rather than wavelength $\lambda \to 0$; and in this limit of wave mechanics again the primitive notion is that of a bundle of trajectories tied together by one principal function $S(\mathbf{x})$, *not individual phase space trajectories.*

One can now "reverse" Schrödinger's point of view in a certain sense and ask: having known classical wave optics and having now learned wave mechanics, can the concept of rays of light be extended so as to provide an exact *new language* in which to describe all of wave optics, both classical and quantum, and not just its geometrical limit? It is appropriate that methods learned in wave mechanics guide us in this task. The proper framework for considering these questions is statistical optics, and we here restrict ourselves to time stationary states. For a review, see the classic work of Mandel and Wolf.[(5)] We have already done quite some work at the level of the two-point function,[(6–11)] and shown the usefulness of the idea of *generalized* *rays* at that level. A statistical state in classical (quantum) optics is described by a hierarchy of correlation functions $\Gamma^{(n,m)}(\cdots)(G^{(n,m)}(\cdots))$ which will be defined explicitly later. Throughout, for simplicity, we deal with scalar waves and ignore polarization. [A systematic procedure for passing from scalar optics to vector optics was developed in Ref. 9 and has been applied to several interesting problems.[12-14]] At the level of $\Gamma^{(1,1)}$ and $G^{(1,1)}$, there being no distinction between classical and quantum cases, a partial generalization of rays of light, allowing for *light and dark rays*,[6] has sufficed. Their usefulness comes from their simple behavior under various conditions.[7-11] Now one can raise several questions:

(1) Is it possible to totally transcribe all the complete information of a classical statistical state contained in the entire collection $\{\Gamma^{(n,m)}\}$ for $n, m = 0, 1, \ldots, \infty$, into a generalized ray language?

(2) Similarly, for a quantum state and the collection $\{G^{(n,m)}\}$ for $n, m = 0, 1, \ldots, \infty$?

(3) What are the differences between classical generalized rays and quantum generalized rays, which must exist and which could not be seen at the $n = m = 1$ level?

(4) What are the consequences of "Bose statistics" for generalized rays, classical or quantum?

(5) While the collections $\{\Gamma^{(n,m)}\}$, $\{G^{(n,m)}\}$ can be neatly handled via generating functionals, are there similar characteristic functional methods for handling collections of joint distribution functions for generalized rays by working "up in the exponent"?

(6) What is the ray distribution function for a black-body cavity in thermal equilibrium?

(7) In the quantum case, what is the most convenient definition of the practically important paraxial situation?

In order to investigate these and other questions, we must establish our notation and introduce the primary correlation functions in both classical and quantum wave optics. Since much of the development is parallel, we shall use similar notations. Without any essential loss of generality we will work with a complex scalar field $\phi(\mathbf{x}, t) \equiv \phi(x)$ and distinguish operators by putting a caret on the field symbol. We shall also restrict our attention to free space so that $\phi(x)$ satisfies the free-wave equation. Its positive frequency "analytic signal" part will be denoted by $\psi(x)$. Then the classical correlation functions are given by[15,16]

$$\begin{aligned}\Gamma^{(n,m)}&(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m)\\ &= \langle \psi^*(y_1) \cdots \psi^*(y_m)\, \psi(x_1) \cdots \psi(x_n) \rangle \end{aligned} \tag{3}$$

Of the several choices available in quantum theory, we will stick to the normal ordered correlation functions

$$\begin{aligned} G^{(n,m)}&(x_1, x_2,..., x_n; y_1, y_2,..., y_m) \\ &= \langle \hat{\psi}^\dagger(y_1)\cdots\hat{\psi}^\dagger(y_m)\,\hat{\psi}(x_1)\cdots\hat{\psi}(x_n)\rangle \\ &= \mathrm{Tr}\{\hat{\psi}(x_1)\cdots\hat{\psi}(x_n)\,\hat{\rho}\hat{\psi}^\dagger(y_1)\cdots\hat{\psi}^\dagger(y_m)\} \end{aligned} \tag{4}$$

For the (quasi) monochromatic situation, each of the x's and y's which stands for both a space vector $\mathbf{x}$, $\mathbf{y}$ and a time t, t' can just be replaced by the space part alone. For the full three-dimensional case, every $\mathbf{x}_1,..., \mathbf{x}_n$, $\mathbf{y}_1,..., \mathbf{y}_m$ is a 3-vector, but if one wants to study paraxial propagation problems, one simplifies further to $\mathbf{x}$'s and $\mathbf{y}$'s which are 2-vectors on a transverse plane. Detailed definitions of ψ, $\hat{\psi}$ are given in the sequel. Note here that $\Gamma^{(n,m)}$ and $G^{(n,m)}$ have the same physical dimensions.

Now in dealing with $\Gamma^{(1,1)}$ (or $G^{(1,1)}$) the Wigner–Moyal transform[17] of quantum mechanics has played the key role. Its usefulness has been established beyond any doubt,[10,11] so we must exploit it. Even though $\Gamma^{(1,1)}$ is purely classical, we have treated it "as though" it were the (unnormalized) configuration space density matrix of a suitable single particle in quantum mechanics, and then exploited the Weyl–Wigner Moyal methods. Now we make some observations:

(1) It is *impossible* to have a classical looking description of quantum mechanics at the level of the wave function ψ, *linear* in the wave function; while in the WKB *limit* classical pictures can be used, in an exact sense any classical-like version of quantum mechanics *must* use *bilinears* $\sim\psi\psi^*$.

(2) Similarly, while in the eikonal *limit* a classical wave is describable in ray language, there is *no exact* ray-like descrition of the classical wave which is *linear* in the wave amplitude. What has been earlier demonstrated in this direction is at the $\Gamma^{(1,1)}$ level.

(3) Hence a generalized ray language is *not possible* for *all* $\Gamma^{(n,m)}$ and $G^{(n,m)}$; it can be achieved only for all *diagonal* $\Gamma^{(n,n)}$ and $G^{(n,n)}$.

(4) To handle, say, $\Gamma^{(N,N)}$ and $G^{(N,N)}$, we would compare them with unnormalized quantum-mechanical density matrices for N particles, moving in two or three dimensions as appropriate, and then borrow the technology of WWM (Weyl–Wigner–Moyal) to talk of N-fold joint quasi-probability distributions of generalized (classical or quantum) rays.

(5) Thus *all* the information about a statistical state in optics *cannot* be recast in ray language; only that part relevant for photon counting or intensity correlations, for example, can be given a generalized description.

(6) In this context, it is but right to view the state space of N identical quantum particles as a subset of the state space of N possibly distinguishable quantum particles. Therefore to deal with $\Gamma^{(N,N)}$ and $G^{(N,N)}$ the way we have handled $\Gamma^{(1,1)}$ previously, we must develop WWM for N possibly distinguishable particles, and then impose the "Bose statistics" requirement.

(7) We will see later how $\Gamma^{(N,N)}$ leads to an N-fold ray distribution function $\omega_N(\ldots)$; and similarly $G^{(N,N)}$ to a function $W_N(\ldots)$. It will turn out that the "Bose condition" on $\omega_N(\ldots)$ and $W_N(\ldots)$ is *essentially nonlocal* in terms of ray variables.

(8) We can obtain a generating functional for the ray density functions.

(9) It is clear that the collections $\{\omega_N\}$, $\{W_N\}$ cannot give a generalized ray description of *all* features of a statistical optical state. What then can be done for this purpose?

(10) We can fall back on the following general principle: for *any* quantum system based on canonical $\hat{q}$'s and $\hat{p}$'s, the Wigner distribution is *always* available for a classical-looking descrition of a general state. For the quantum optical field, then, we can define the Wigner distribution functional for the entire field, and this certainly contains all information in the full density operator $\hat{\rho}$. How are $W_N(\ldots)$ and $G^{(N,N)}$ and even $G^{(n,m)}$ obtained from the field's Wigner distribution?

This essay, undertaken as an offering in homage to the memory of Erwin Schrödinger, has thus exceeded our original limited purpose of seeing how reversing Schrödinger's path leads us to use the concept of generalized bundles of light rays to provide a new language of descrition. By the first fundamental theorem of quantum optics[18] the two point functions $G^{(1,1)}$ and $\Gamma^{(1,1)}$ are in one-to-one correspondence and hence (apart from the generalization to dark rays needed in wave optics!) no specific quantum features come into the ray distribution function. But the situation is radically different even for the four point functions $G^{(2,2)}$ and $\Gamma^{(2,2)}$; and the correlated distribution of light rays reveals a rich structure which we are only able to outline in this paper due to limitations beyond our control.

2. QUANTUM MECHANICS FOR ONE DEGREE OF FREEDOM: THE WEYL–WIGNER–MOYAL METHOD

Let $\hat{q}$ and $\hat{p}$ have dimensions of length and momentum, respectively, and obey

$$[\hat{q}, \hat{p}] = i\hbar \tag{5}$$

q and p are corresponding classical real variables with the same dimensions as $\hat{q}$ and $\hat{p}$, respectively. For any classical function $f(q, p)$ with the representation

$$f(q, p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\sigma \, d\tau \, \tilde{f}(\sigma, \tau) \, e^{i(\sigma q - \tau p)} \tag{6}$$

the Weyl–Wigner–Moyal method (WWM)[17] associates the quantum operator F:

$$F = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\sigma \, d\tau \, \tilde{f}(\sigma, \tau) \, e^{i(\sigma \hat{q} - \tau \hat{p})} \tag{7}$$

Therefore dimension of f = dimension of F. To relate f and F directly,[19] let us introduce the family of operators:

$$\hat{W}(q; p) = \frac{\hbar}{2\pi} \int d\sigma \int d\tau \, e^{i\sigma(\hat{q} - q) - i\tau(\hat{p} - p)} \tag{8}$$

Here $-\infty < q, p < \infty$. The important properties are:

$$\begin{gathered} \hat{W}(q; p)^\dagger = \hat{W}(q; p) \\ \mathrm{Tr}\, \hat{W}(q; p) = 1 \\ \mathrm{Tr}\, \hat{W}(q; p) \, \hat{W}(q'; p') = 2\pi\hbar \, \delta(q' - q) \, \delta(p' - p) \\ \hat{W}(q; p) \text{ is dimensionless} \\ \tfrac{1}{2}\{\hat{q} \text{ or } \hat{p}, \hat{W}(q; p)\} = (q \text{ or } p) \, \hat{W}(q; p) \\ [\hat{q} \text{ or } \hat{p}, \hat{W}(q; p)] = -i\hbar \left(\frac{\partial}{\partial p} \text{ or } -\frac{\partial}{\partial q} \right) \hat{W}(q; p) \end{gathered} \tag{9}$$

We also recall that the trace operation in quantum mechanics (QM), and also the density operator $\hat{\rho}$, are both dimensionless. Using the above we now have:

WWM rule

$$\begin{aligned} F &= \iint \frac{dq \, dp}{2\pi\hbar} f(q, p) \, \hat{W}(q; p) \\ \leftrightarrow f(q, p) &= \mathrm{Tr}\, F\hat{W}(q; p) \\ &= \int dq' \, e^{ipq'/\hbar} \langle q - \tfrac{1}{2}q' | F | q + \tfrac{1}{2}q' \rangle \\ &\qquad \text{in the position representation} \end{aligned} \tag{10}$$

Then for two operator F and G

$$\mathrm{Tr}\, FG = \iint \frac{dq\, dp}{2\pi\hbar} f(q, p)\, g(q, p) \tag{11}$$

and if $FG = H$ corresponds to $h(q, p)$, we have

$$\begin{aligned} h(q, p) &= (f * g)(q, p) \\ &= \exp\left[\frac{i\hbar}{2}\left(\frac{\partial}{dq}\frac{\partial}{\partial p'} - \frac{\partial}{\partial p}\frac{\partial}{\partial q'}\right)\right] f(q, p)\, g(q', p')|_{q'=q, p'=p} \end{aligned} \tag{12}$$

In all the above, $\hbar$ is explicit. But if we want to use WWM in a purely classical context, we retain q as position with dimension of length, and replace momentum p by $\hbar k$, the wave number k being an inverse length. Then $\hbar$ disappears completely. If we wish, in that context we can work with operators $\hat{W}(q; k)$, which are really the same as $\hat{W}(q; p)$:

$$\begin{aligned} \hat{W}(q; k) &= \frac{1}{2\pi} \int d\sigma \int d\tau\, e^{i\sigma(\hat{q}-q) - i\tau(\hat{k}-k)} \\ [\hat{q}, \hat{k}] &= i, \qquad \hat{k} = \frac{1}{i}\frac{\partial}{\partial q}, \text{ etc.} \end{aligned} \tag{13}$$

WWM in complex representation

For ray distribution functions, we would generalize the above to many degrees of freedom. But for handling the scalar field analytic signal, it may be useful to develop WWM in a different notation, but with no essential change. To make the necessary changes, let a mass m and a frequency ω with appropriate dimensions be given. Then define:

$$\begin{aligned} \hat{a} &= \frac{m\omega\hat{q} + i\hat{p}}{\sqrt{2m\omega}}, \qquad \hat{a}^\dagger = \frac{m\omega\hat{q} - i\hat{p}}{\sqrt{2m\omega}} \\ &[\hat{a}, \hat{a}^\dagger] = \hbar \\ \hat{q} &= \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2m\omega}}, \qquad \hat{p} = -i\sqrt{\frac{m\omega}{2}}(\hat{a} - \hat{a}^\dagger) \end{aligned} \tag{14}$$

So $\hat{a}$ and $\hat{a}^\dagger$ both have dimensions of $(\text{action})^{1/2}$, but no $\hbar$ is included in the definition of $\hat{a}$ and $\hat{a}^\dagger$ in terms of $\hat{q}$ and $\hat{p}$, so that these equations make sense classically too.

To accompany (14) we also define at the c-number level

$$a = \frac{m\omega q + ip}{\sqrt{2m\omega}}, \qquad a^* = \frac{m\omega q - ip}{\sqrt{2m\omega}} \tag{15}$$

and replace the integration variables σ and τ by

$$\xi = \frac{m\omega\tau + i\sigma}{\sqrt{2m\omega}}, \qquad \xi^* = \frac{m\omega\tau - i\sigma}{\sqrt{2m\omega}} \tag{16}$$

Hence,

dimension of a and $a^* = (\text{action})^{1/2}$, and dimension of ξ and $\xi^* = (\text{action})^{-1/2}$ (17)

Then

$$\begin{aligned} d^2 a &\equiv d\,\mathrm{Re}\,a\; d\,\mathrm{Im}\,a = \frac{1}{2}\,dq\,dp = \frac{i}{2}\,da^*\,da \\ d^2\xi &\equiv d\,\mathrm{Re}\,\xi\; d\,\mathrm{Im}\,\xi = \frac{1}{2}\,d\sigma\,d\tau = \frac{i}{2}\,d\xi^*\,d\xi \end{aligned} \tag{18}$$

Hence

$$\begin{aligned} &\hat{W}(q; p) \text{ as defined in Eq. (8)} \\ &\quad \equiv \hat{W}(a; a^*) \text{ (abuse of notation!)} \\ &\quad = \frac{\hbar}{\pi} \int d^2\xi\, e^{\xi \hat{a}^\dagger - \xi^* \hat{a} + \xi^* a - \xi a^*} \\ &\quad = \frac{\hbar}{\pi} \int d^2\xi\, e^{\xi \hat{a}^\dagger}\, e^{-\xi^* \hat{a}}\, e^{-(\hbar/2)\xi^*\xi + \xi^* a - \xi a^*} \end{aligned} \tag{19}$$

If $f(q, p) \equiv f(a, a^*) \overset{\text{WWM}}{\longleftrightarrow} F$, then

$$\begin{aligned} f(a, a^*) &= \mathrm{Tr}\, F\hat{W}(a; a^*), \qquad F = \frac{1}{\pi\hbar} \int d^2 a\, f(a, a^*)\, \hat{W}(a; a^*) \\ \mathrm{Tr}\, FG &= \frac{1}{\pi\hbar} \int d^2 a\, f(a, a^*)\, g(a, a^*) \end{aligned} \tag{20}$$

After generalization to many, in fact infinite, dimensions, this will be useful in handling the field after quantization.

3. WWM TECHNIQUES FOR MANY DEGREES OF FREEDOM: BOSE SYMMETRY

Now, going back to the real $q - p$ formalism, we *generalize WWM for N particles each in n-dimensional space*. The basic operators are

$\hat{q}_{\alpha j}, \hat{p}_{\alpha j}: \alpha = 1, 2, \ldots, N =$ particle label

$j = 1, 2, \ldots, n =$ Cartesian component in n-dimensional space

$$[\hat{q}_{\alpha j}, \hat{p}_{\beta k}] = i\hbar\, \delta_{\alpha\beta}\, \delta_{jk} \tag{21}$$

The compressed notation is

$$\begin{aligned}\{\hat{\mathbf{q}}\} &\equiv \{\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2, \ldots, \hat{\mathbf{q}}_N\} \\ \hat{\mathbf{q}}_\alpha &\equiv \{\hat{q}_{\alpha 1}, \hat{q}_{\alpha 2}, \ldots, \hat{q}_{\alpha n}\}, \text{ similarly for } p\end{aligned} \tag{22}$$

Then

$$\begin{aligned}\hat{W}(\{\mathbf{q}\};\{\mathbf{p}\}) &= \left(\frac{\hbar}{2\pi}\right)^{nN} \int d^{nN}\sigma\, d^{nN}\tau \\ &\quad \times \exp\left[i \sum_{\alpha=1}^{N} (\boldsymbol{\sigma}_\alpha \cdot (\hat{\mathbf{q}}_\alpha - \mathbf{q}_\alpha) - \boldsymbol{\tau}_\alpha \cdot (\hat{p}_\alpha - \mathbf{p}_\alpha))\right] \\ &= \hat{W}(\{\mathbf{q}\};\{\mathbf{p}\})^\dagger \\ \tfrac{1}{2}\{\hat{q}_{\alpha j} \text{ or } \hat{p}_{\alpha j}, \hat{W}(\{\mathbf{q}\};\{\mathbf{p}\})\} &= (q_{\alpha j} \text{ or } p_{\alpha j})\, \hat{W}(\{\mathbf{q}\};\{\mathbf{p}\}) \\ [\hat{q}_{\alpha j} \text{ or } \hat{p}_{\alpha j}, \hat{W}(\{\mathbf{q}\};\{\mathbf{p}\})] &= i\hbar \left(\frac{\partial}{\partial p_{\alpha j}} \text{ or } -\frac{\partial}{\partial q_{\alpha j}}\right) \hat{W}(\{\mathbf{q}\};\{\mathbf{p}\}) \\ \operatorname{Tr} \hat{W}(\{\mathbf{q}\};\{\mathbf{p}\}) &= 1 \\ \operatorname{Tr} \hat{W}(\{\mathbf{q}\};\{\mathbf{p}\})\, \hat{W}(\{\mathbf{q}'\};\{\mathbf{p}'\}) & \\ = (2\pi\hbar)^{nN} \prod_{\alpha=1}^{N} \delta^{(n)}(\mathbf{q}'_\alpha - \mathbf{q}_\alpha)\, \delta^{(n)}(\mathbf{p}'_\alpha - \mathbf{p}_\alpha) &\end{aligned} \tag{23}$$

Again, $\hat{W}$ is dimensionless; and the WWM correspondence is

$$\begin{aligned}F &= \int \prod_{\alpha=1}^{N} \left(\frac{d^n q_\alpha\, d^n p_\alpha}{(2\pi\hbar)^n}\right) f(\{\mathbf{q}\}, \{\mathbf{p}\})\, \hat{W}(\{\mathbf{q}\};\{\mathbf{p}\}) \\ f(\{\mathbf{q}\}, \{\mathbf{p}\}) &= \operatorname{Tr} F\hat{W}(\{\mathbf{q}\};\{\mathbf{p}\}) \\ &= \int \prod_{\alpha=1}^{N} d^n q'_\alpha \exp\left(i \sum_{\alpha=1}^{N} \mathbf{p}_\alpha \cdot \mathbf{q}'_\alpha/\hbar\right) \\ &\quad \times \langle \{\mathbf{q} - \tfrac{1}{2}\mathbf{q}'\} | F | \{\mathbf{q} + \tfrac{1}{2}\mathbf{q}'\} \rangle\end{aligned} \tag{24}$$

For products of operators we get

$$\begin{aligned}FG &= H : h = f * g \\ h(\{\mathbf{q}\}, \{\mathbf{p}\}) &= \exp\left[\frac{i\hbar}{2} \sum_{\alpha=1}^{N} \sum_{j=1}^{n} \left(\frac{\partial}{\partial q_{\alpha j}} \frac{\partial}{\partial p'_{\alpha j}} - \frac{\partial}{\partial p_{\alpha j}} \frac{\partial}{\partial q'_{\alpha j}}\right)\right] \\ &\quad \cdot f(\{\mathbf{q}\}, \{\mathbf{p}\})\, g(\{\mathbf{q}'\}, \{\mathbf{p}'\})|_{\mathbf{q}'=\mathbf{q}, \mathbf{p}'=\mathbf{p}} \\ \operatorname{Tr} FG &= \int \prod_{\alpha=1}^{N} \left(\frac{d^n q_\alpha\, d^n p_\alpha}{(2\pi\hbar)^n}\right) f(\{\mathbf{q}\}, \{\mathbf{p}\})\, g(\{\mathbf{q}\}, \{\mathbf{p}\})\end{aligned} \tag{25}$$

So far the N particles have been treated as *distinguishable*. How are particle permutation operators to be described in WWM? It is quite straightforward.

Consider the operation $\hat{P}_{12}$ which interchanges particles 1 and 2 and leaves the rest alone. Its definition and properties are

$$\begin{aligned} \hat{P}_{12}|\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, ..., \mathbf{q}_N\}\rangle &= |\{\mathbf{q}_2, \mathbf{q}_1, \mathbf{q}_3, ..., \mathbf{q}_N\}\rangle \\ \hat{P}_{12}(\hat{q}_{1j} \text{ or } \hat{p}_{1j})\,\hat{P}_{12}^{-1} &= \hat{q}_{2j} \text{ or } \hat{p}_{2j} \\ \hat{P}_{12}(\hat{q}_{2j} \text{ or } \hat{p}_{2j})\,\hat{P}_{12}^{-1} &= \hat{q}_{1j} \text{ or } \hat{p}_{1j} \\ \hat{P}_{12} = \hat{P}_{12}^{\dagger} &= \hat{P}_{12}^{-1} \end{aligned} \tag{26}$$

To find the WWM representative of $\hat{P}_{12}$, we just use the last formula of (24): calling the representative function P_{12} again, we find

$$\begin{aligned} P_{12}(\{\mathbf{q}\}, \{\mathbf{p}\}) &= \int \prod_{\alpha=1}^{N} d^n q'_\alpha \exp\left(\frac{i}{\hbar} \sum_{\alpha=1}^{N} \mathbf{p}_\alpha \cdot \mathbf{q}'_\alpha\right) \\ &\quad \times \langle\{\mathbf{q} - \tfrac{1}{2}\mathbf{q}'\}|\,\hat{P}_{12}\,|\{\mathbf{q} + \tfrac{1}{2}\mathbf{q}'\}\rangle \\ &= \int \prod_{\alpha=1}^{N} d^n q'_\alpha \exp\left(\frac{i}{\hbar} \sum_{\alpha=1}^{N} \mathbf{p}_\alpha \cdot \mathbf{q}'_\alpha\right) \\ &\quad \cdot \delta^{(n)}(\mathbf{q}_1 - \mathbf{q}_2 - \tfrac{1}{2}(\mathbf{q}'_1 + \mathbf{q}'_2))\, \delta^n(\mathbf{q}_2 - \mathbf{q}_1 - \tfrac{1}{2}(\mathbf{q}'_2 + \mathbf{q}'_1)) \\ &\quad \times \delta^n(\mathbf{q}'_3) \cdots \delta^n(\mathbf{q}'_N) \\ &= \delta^n(\mathbf{q}_1 - \mathbf{q}_2) \int d^n q'_1\, d^n q'_2 \\ &\quad \times e^{i/\hbar(\mathbf{p}_1 \cdot \mathbf{q}'_1 + \mathbf{p}_2 \cdot \mathbf{q}'_2)}\, \delta^{(n)}(\mathbf{q}'_1 + \mathbf{q}'_2) \\ &= (2\pi\hbar)^n\, \delta^{(n)}(\mathbf{q}_1 - \mathbf{q}_2)\, \delta^{(n)}(\mathbf{p}_1 - \mathbf{p}_2) \end{aligned} \tag{27}$$

For any other pair of particles α, β with $\alpha \neq \beta$, the WWM representative of $\hat{P}_{\alpha\beta}$ is

$$P_{\alpha\beta}(\{\mathbf{q}\}, \{\mathbf{p}\}) = (2\pi\hbar)^n\, \delta^{(n)}(\mathbf{q}_\alpha - \mathbf{q}_\beta)\, \delta^{(n)}(\mathbf{p}_\alpha - \mathbf{p}_\beta) \tag{28}$$

$\hat{P}_{\alpha\beta}$ is an element of the group $\mathrm{Sp}(2nN, R)$ acting on the nN canonical pairs, since it just interchanges the $\hat{q}$'s and $\hat{p}$'s of particle α with those of particle β. Thus, if an operator F is transformed to F' by

$$F' = \hat{P}_{\alpha\beta} F \hat{P}_{\alpha\beta}^{-1} \tag{29}$$

then the corresponding WWM representatives $f'(\{\mathbf{q}\}, \{\mathbf{p}\})$, and

$f(\{\mathbf{q}\}, \{\mathbf{p}\})$ are related by a simple exchange of arguments $\mathbf{q}_\alpha \leftrightarrow \mathbf{q}_\beta$, $\mathbf{p}_\alpha \leftrightarrow \mathbf{p}_\beta$. In fact, every element of the permutation group S_N acts in this simple way: conjugation of F by any element of S_N amounts to simple permutation of arguments in the WWM representative f of F.

However, if $\hat{\rho}^{(N)}$ is a density operator for N *identical* Bose particles in n dimensions, while it is true that

$$\hat{P}_{\alpha\beta}\hat{\rho}^{(N)}\hat{P}_{\alpha\beta}^{-1} = \hat{\rho}^{(N)} \qquad \text{for each pair } \alpha\beta \tag{30}$$

this is not the most primitive relation. Rather the primitive relation is

$$\hat{P}_{\alpha\beta}\hat{\rho}^{(N)} = \hat{\rho}^{(N)} \qquad \text{for each pair } \alpha, \beta \tag{31}$$

from which (30) follows by hermitian conjugation and the hermiticity of $\hat{\rho}^{(N)}$ and $\hat{P}_{\alpha\beta}$. That (31) must hold is clear since $\hat{\rho}^{(N)}$ has to be built up from vectors $|\psi\rangle$ which are themselves invariant under $\hat{P}_{\alpha\beta}$:

$$\begin{aligned}
&\hat{\rho}^{(N)} \sim |\psi\rangle\langle\psi|, \qquad \hat{P}_{\alpha\beta}|\psi\rangle = |\psi\rangle \text{ (Bose symmetry)}\\
&\quad\Rightarrow \hat{P}_{\alpha\beta}\hat{\rho}^{(N)} = \hat{\rho}^{(N)}\\
&\quad\Rightarrow \hat{\rho}^{(N)}\hat{P}_{\alpha\beta} = \hat{\rho}^{(N)}, \quad \text{and} \quad \hat{P}_{\alpha\beta}\hat{\rho}^{(N)}\hat{P}_{\alpha\beta}^{-1} = \hat{\rho}^{(N)}
\end{aligned} \tag{32}$$

In fact, if F is any hermitian N-particle operator obeying (31), what is the expression of this property in terms of the WWM representative f of F? We must use (28) and the first part of (25)! For $\alpha = 1$, $\beta = 2$ for simplicity:

$$\begin{aligned}
&\hat{P}_{12}F = F \Leftrightarrow F\hat{P}_{12} = F\\
&\quad\Leftrightarrow \langle \mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_N| F |\mathbf{q}'_1, \mathbf{q}'_2, ..., \mathbf{q}'_N\rangle\\
&\quad= \langle \mathbf{q}_2, \mathbf{q}_1, ..., \mathbf{q}_N| F |\mathbf{q}'_1, \mathbf{q}'_2, ..., \mathbf{q}'_N\rangle\\
&\quad= \langle \mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_N| F |\mathbf{q}'_2, \mathbf{q}'_1, ..., \mathbf{q}'_N\rangle\\
&\quad\Leftrightarrow P_{12} * f = f * P_{12} = f
\end{aligned} \tag{33}$$

In the WWM description this is transcribed as follows:

$$\begin{aligned}
&f(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_N, \mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_N)\\
&\quad= (P_{12} * f)(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_N, \mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_N)\\
&\quad= \exp\left[\frac{i\hbar}{2}\sum_{\alpha=1}^{N}\sum_{j=1}^{n}\left(\frac{\partial}{\partial q_{\alpha j}}\frac{\partial}{\partial p'_{\alpha j}} - \frac{\partial}{\partial p_{\alpha j}}\frac{\partial}{\partial q'_{\alpha j}}\right)\right]\\
&\qquad\cdot (2\pi\hbar)^n\,\delta^{(n)}(\mathbf{q}_1 - \mathbf{q}_2)\,\delta^{(n)}(\mathbf{p}_1 - \mathbf{p}_2)\\
&\qquad\cdot f(\mathbf{q}'_1, \mathbf{q}'_2, ..., \mathbf{q}'_N, \mathbf{p}'_1, \mathbf{p}'_2, ..., \mathbf{p}'_N)|_{\mathbf{q}'=\mathbf{q}, \mathbf{p}'=\mathbf{p}}
\end{aligned} \tag{34}$$

The right-hand side of this equation is really nonlocal in the arguments of f, and by putting in Fourier representations for the delta functions we find

$$\begin{aligned}
&\hat{P}_{12}F=F\\
&\Leftrightarrow f(\mathbf{q}_1, \mathbf{q}_2,\dots, \mathbf{q}_N, \mathbf{p}_1, \mathbf{p}_2,\dots, \mathbf{p}_N)\\
&=\int \frac{d^nq'\,d^np'}{(2\pi\hbar)^n}\exp\left[\frac{i}{\hbar}(\mathbf{p}'\cdot(\mathbf{q}_1-\mathbf{q}_2)-\mathbf{q}'\cdot(\mathbf{p}_1-\mathbf{p}_2))\right]\\
&\quad\cdot f(\mathbf{q}_1-\tfrac{1}{2}\mathbf{q}', \mathbf{q}_2+\tfrac{1}{2}\mathbf{q}', \mathbf{q}_3,\dots, \mathbf{q}_N,\\
&\quad \mathbf{p}_1-\tfrac{1}{2}\mathbf{p}', \mathbf{p}_2+\tfrac{1}{2}\mathbf{p}', \mathbf{p}_3,\dots, \mathbf{p}_N)\\
&\Leftrightarrow f(\mathbf{q}_1,\dots, \mathbf{q}_N, \mathbf{p}_1,\dots, \mathbf{p}_N)\\
&=\int \frac{d^nq'\,d^np'}{(2\pi\hbar)^n}\exp\left[\frac{i}{\hbar}(\mathbf{p}'\cdot(\mathbf{q}_1-\mathbf{q}_2)-\mathbf{q}'\cdot(\mathbf{p}_1-\mathbf{p}_2))\right]\\
&\quad\cdot f\left(\frac{\mathbf{q}_1+\mathbf{q}_2}{2}-\frac{1}{2}\mathbf{q}', \frac{\mathbf{q}_1+\mathbf{q}_2}{2}+\frac{1}{2}\mathbf{q}', \mathbf{q}_3,\dots, \mathbf{q}_N,\right.\\
&\quad\left.\frac{\mathbf{p}_1+\mathbf{p}_2}{2}-\frac{1}{2}\mathbf{p}', \frac{\mathbf{p}_1+\mathbf{p}_2}{2}+\frac{1}{2}\mathbf{p}', \mathbf{p}_3,\dots, \mathbf{p}_N\right)
\end{aligned}\tag{35}$$

The second step follows by a simple translation of $\mathbf{q}'$ and $\mathbf{p}'$. It follows that given this primitive nonlocal relation for f, essentially a double use of it will lead to a local but nonprimitive relation

$$\begin{aligned}
&f(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3,\dots, \mathbf{q}_N, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3,\dots, \mathbf{p}_N)\\
&\quad=f(\mathbf{q}_2, \mathbf{q}_1, \mathbf{q}_3,\dots, \mathbf{q}_N, \mathbf{p}_2, \mathbf{p}_1, \mathbf{p}_3,\dots, \mathbf{p}_N)
\end{aligned}\tag{36}$$

and similarly for the other elements of S_N.

4. BOSE SYMMETRY OF THE HIGHER-ORDER CORRELATION FUNCTIONS

Let us now consider the classical and quantum diagonal correlation functions $\Gamma^{(N,N)}$ and $G^{(N,N)}$ as defined in Eqs. (3) and (4). We suppress the time components of the arguments and imagine the space components to be n-dimensional vectors with $n=2$ or 3 (paraxial or not). Then, as noted before, both $\Gamma^{(N,N)}$ and $G^{(N,N)}$ have the same dimensions and, formally, *though not in terms of physical dimensions*, we can say there is a similarity between them and the configuration space matrix elements of density operators $\hat{\rho}^{(N)}$ for N Bose particles in quantum mechanics:

$$\Gamma^{(N,N)}(\mathbf{x}_1, \mathbf{x}_2,..., \mathbf{x}_N; \mathbf{y}_1, \mathbf{y}_2,..., \mathbf{y}_N)$$

or

$$G^{(N,N)}(\mathbf{x}_1, \mathbf{x}_2,..., \mathbf{x}_N; \mathbf{y}_1, \mathbf{y}_2,..., \mathbf{y}_N) \sim \langle \mathbf{x}_1, \mathbf{x}_2,..., \mathbf{x}_N | \hat{\rho}^{(N)} | \mathbf{y}_1, \mathbf{y}_2,..., \mathbf{y}_N \rangle$$

within the framework of the Hilbert space $H^{(N)} = L^2(R^n \times R^n \times \cdots \times R^n)$. Now this Hilbert space is appropriate for N particles, indistinguishable or not. So the position eigenkets $|\mathbf{x}_1, \mathbf{x}_2,..., \mathbf{x}_N\rangle$ are nontrivially acted upon by $\hat{P}_{12}, \hat{P}_{13},...,$ any $\hat{P} \in S_N$. On $H^{(N)}$ a general operator $\hat{O}$ has a general kernel $O(...\mathbf{x}...; ...\mathbf{y}...) \equiv \langle ...\mathbf{x}...| \hat{O} |...\mathbf{y}...\rangle$, with no simple behavior under S_N. For such quantities the WWM methods must be introduced independently for each "degree of freedom." Having done so, we then impose the Bose condition! To this end, regard $\Gamma^{(N,N)}$ as the kernel of an operator $\hat{\Gamma}^{(N)}$ on $H^{(N)}$:

$$\begin{aligned} &\Gamma^{(N,N)}(\mathbf{x}_1, \mathbf{x}_2,..., \mathbf{x}_N; \mathbf{y}_1, \mathbf{y}_2,..., \mathbf{y}_N) \\ &\qquad \equiv \langle \mathbf{x}_1, \mathbf{x}_2,..., \mathbf{x}_N | \hat{\Gamma}^{(N)} | \mathbf{y}_1, \mathbf{y}_2,..., \mathbf{y}_N \rangle \\ &G^{(N,N)}(\mathbf{x}_1, \mathbf{x}_2,..., \mathbf{x}_N; \mathbf{y}_1, \mathbf{y}_2,..., \mathbf{y}_N) \\ &\qquad \equiv \langle \mathbf{x}_1, \mathbf{x}_2,..., \mathbf{x}_N | \hat{G}^{(N)} | \mathbf{y}_1, \mathbf{y}_2,..., \mathbf{y}_N \rangle \end{aligned} \tag{37}$$

Now we know that $\Gamma^{(N,N)}$ is separately symmetric in each set of arguments, and so is $G^{(N,N)}$. In operator form this means, at the primitive level,

$$\begin{aligned} &\hat{P}_{\alpha\beta} \hat{\Gamma}^{(N)} = \hat{\Gamma}^{(N)} \\ &\hat{P}_{\alpha\beta} \hat{G}^{(N)} = \hat{G}^{(N)}, \qquad 1 \leqslant \alpha, \beta \leqslant N \end{aligned} \tag{38}$$

We now define the ray density functions for the classical and quantum fields, ω_N and W_N. For the rays we will use position vector and wave vector description, rather than position and momentum, so $\hbar$ will be absent. Patterned after Eq. (24) but rearranging the arguments, we define the N-fold joint classical generalized ray distribution function for one ray with parameters $\mathbf{x}_1, \mathbf{k}_1$, another with $\mathbf{x}_2, \mathbf{k}_2,...,$ and the final one with $\mathbf{x}_N, \mathbf{k}_N$ as

$$\begin{aligned} &\omega_N(\mathbf{x}_1, \mathbf{k}_1; \mathbf{x}_2, \mathbf{k}_2; ...; \mathbf{x}_N, \mathbf{k}_N) \\ &\quad \equiv (2\pi)^{-nN} \int \prod_{\alpha=1}^{N} d^n x'_\alpha \exp\left(i \sum_{\alpha=1}^{N} \mathbf{k}_\alpha \cdot \mathbf{x}'_\alpha\right) \\ &\qquad \cdot \Gamma^{(N,N)}(\mathbf{x}_1 - \tfrac{1}{2}\mathbf{x}'_1,..., \mathbf{x}_N - \tfrac{1}{2}\mathbf{x}'_N; \mathbf{x}_1 + \tfrac{1}{2}\mathbf{x}'_1,..., \mathbf{x}_N + \tfrac{1}{2}\mathbf{x}'_N) \end{aligned} \tag{39}$$

Similarly, we define the N-fold joint quantum generalized ray distribution function

$$W_N(\mathbf{x}_1, \mathbf{k}_1; \mathbf{x}_2, \mathbf{k}_2; ...; \mathbf{x}_N, \mathbf{k}_N)$$
$$\equiv (2\pi)^{-nN} \int \prod_{\alpha=1}^{N} d^n x'_\alpha \exp\left(i \sum_{\alpha=1}^{N} \mathbf{k}_\alpha \cdot \mathbf{x}'_\alpha\right)$$
$$\cdot G^{(N,N)}(\mathbf{x}_1 - \tfrac{1}{2}\mathbf{x}'_1, ..., \mathbf{x}_N - \tfrac{1}{2}\mathbf{x}'_N; \mathbf{x}_1 + \tfrac{1}{2}\mathbf{x}'_1, ..., \mathbf{x}_N + \tfrac{1}{2}\mathbf{x}'_N) \quad (40)$$

It is true that both ω_N and W_N are invariant under any permutation of complete sets of ray arguments, i.e., for example

$$(\mathbf{x}_1, \mathbf{k}_1) \leftrightarrow (\mathbf{x}_2, \mathbf{k}_2), \text{ etc.}$$

But the more basic conditions (38) tell us that generalized rays, *whether classical or quantum*, are correlated in a *nonlocal way*, which must thus be a *partial* rendering of the Bose nature of light: from (35)

$$(\omega_n \text{ or } W_N)(\mathbf{x}_1, \mathbf{k}_1; \mathbf{x}_2, \mathbf{k}_2; ...; \mathbf{x}_N, \mathbf{k}_N)$$
$$= \int \frac{d^n x' \, d^n k'}{(2\pi)^n} e^{i\mathbf{k}' \cdot (\mathbf{x}_1 - \mathbf{x}_2) - i\mathbf{x}' \cdot (\mathbf{k}_1 - \mathbf{k}_2)}$$
$$\cdot (\omega_N \text{ or } W_N)(\tfrac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2) - \tfrac{1}{2}\mathbf{x}', \tfrac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2) - \tfrac{1}{2}\mathbf{k}';$$
$$\tfrac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2) + \tfrac{1}{2}\mathbf{x}', \tfrac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2) + \tfrac{1}{2}\mathbf{k}'; \mathbf{x}_3, \mathbf{k}_3; ...; \mathbf{x}_N \cdot \mathbf{k}_N) \quad (41)$$

and similarly for any other pairs of rays. It is then a *consequence* of this *nonlocal* correlation (41) that we have *local* relations like

$$(\omega_N \text{ or } W_N)(\mathbf{x}_1, \mathbf{k}_1; \mathbf{x}_2, \mathbf{k}_2; \mathbf{x}_3, \mathbf{k}_3; ...; \mathbf{x}_N, \mathbf{k}_N)$$
$$= (\omega_N \text{ or } W_N)(\mathbf{x}_2, \mathbf{k}_2; \mathbf{x}_1, \mathbf{k}_1; \mathbf{x}_3, \mathbf{k}_3; ...; \mathbf{x}_N, \mathbf{k}_N)$$

This relation holds for both "bosons" and "fermions," so (41) is the basic result!

The nonlocal relation (41) contains within it the Bose effect which in a normal Bose gas exhibits itself as distance correlations.[20] In addition to this, we know from photon number fluctuations[21] that the quantum fluctuations must include both the particle fluctuation $\sim \langle n \rangle$ and the wave fluctuations $\sim \langle n \rangle^2$. We shall take up these two questions in the following two sections. But before that we wish to make a number of observations:

(1) For any *fixed* $N \geqslant 2$, the set of all $\Gamma^{(N,N)}(\mathbf{x}; \mathbf{y})$ is a proper subset of the set of all $G^{(N,N)}(\mathbf{x}, \mathbf{y})$ = the set of all (unnormalized) N-particle boson density matrices $\langle \mathbf{x} | \hat{\rho}^{(N)} | \mathbf{y} \rangle$. This is so for the following reason: While

$\Gamma^{(N,N)}(\mathbf{x}, \mathbf{y})$, $G^{(N,N)}(\mathbf{x}, \mathbf{y})$, and $\langle \mathbf{x} | \hat{\rho}^{(N)} | \mathbf{y} \rangle$ are all hermitian, positive semi-definite, and Bose symmetric, $\Gamma^{(N,N)}(\mathbf{x}; \mathbf{y})$ has to be an ensemble over realizations of the special form

$$\psi^*(\mathbf{y}_1) \cdots \psi^*(\mathbf{y}_N)\, \psi(\mathbf{x}_1) \cdots \psi(\mathbf{x}_N)$$

For instance, an allowed $G^{(2,2)}$ and $\hat{\rho}^{(2)}$, but not $\Gamma^{(2,2)}$, is

$$\phi^*(\mathbf{y}_1, \mathbf{y}_2)\, \phi(\mathbf{x}_1, \mathbf{x}_2)$$

$$\phi(\mathbf{x}_1, \mathbf{x}_2) = u(\mathbf{x}_1)\, v(\mathbf{x}_2) + u(\mathbf{x}_2)\, v(\mathbf{x}_1)$$

(2) We recall that $G^{(N,N)}$ and $\Gamma^{(N,N)}$ have eigenmode decompositions; but while an eigenmode that enters in $\Gamma^{(1,1)}$ must enter all $\Gamma^{(N,N)}$ it is not necessary that the eigenmodes that enter $G^{(1,1)}$ be present in $G^{(N,N)}$.[22]

(3) The set of all $\Gamma^{(N,N)}(\mathbf{x}; \mathbf{y})$ can be realized as moments of a probability functional; but the corresponding generalized diagonal weight functional is not pointwise positive for the set $G^{(N,N)}$.[23,16]

Many of these questions would benefit from a detailed analysis of the multivariate ray density functions. In this paper we shall attempt only the implications for the pair correlations of light rays.

5. CORRELATIONS OF LIGHT RAYS

For the case of $N = 2$ we get the bivariate light ray distribution

$$\begin{aligned} W_2(\mathbf{x}_1, \mathbf{k}_1; \mathbf{x}_2, \mathbf{k}_2) = \int \frac{d^n x'\, d^n k'}{(2\pi)^n}\, & e^{i\mathbf{k}' \cdot (\mathbf{x}_1 - \mathbf{x}_2)}\, e^{-i\mathbf{x}' \cdot (\mathbf{k}_1 - \mathbf{k}_2)} \\ & W_2(\tfrac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2) - \tfrac{1}{2}\mathbf{x}', \tfrac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2) - \tfrac{1}{2}\mathbf{k}'; \\ & \tfrac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2) + \tfrac{1}{2}\mathbf{x}', \tfrac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2) + \tfrac{1}{2}\mathbf{k}') \end{aligned}$$

An identical relation obtains for ω_N. This somewhat untidy relationship would look much more satisfactory if we use the average and relative coordinates

$$\mathbf{x} = \tfrac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2), \qquad \mathbf{k} = \tfrac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2)$$

$$\boldsymbol{\xi} = (\mathbf{x}_1 - \mathbf{x}_2), \qquad \boldsymbol{\eta} = (\mathbf{k}_1 - \mathbf{k}_2)$$

and abuse notation by writing

$$W(\mathbf{x}, \mathbf{k}; \boldsymbol{\xi}, \boldsymbol{\eta}) = W_2(\mathbf{x}_1, \mathbf{k}_1; \mathbf{x}_2, \mathbf{k}_2)$$

Then the Bose property demands

$$W(\mathbf{x}, \mathbf{k}; \boldsymbol{\xi}, \boldsymbol{\eta}) = \int \frac{d^n\xi' \, d^n\eta'}{(2\pi)^n} e^{+i\boldsymbol{\eta}' \cdot \boldsymbol{\xi} - i\boldsymbol{\xi}' \cdot \boldsymbol{\eta}} W(\mathbf{x}, \mathbf{k}; \boldsymbol{\xi}', \boldsymbol{\eta}') \tag{42}$$

For each $\mathbf{x}$, $\mathbf{k}$ the distribution in the relative coordinates and momenta $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ is self-conjugate in the sense that

$$f(\boldsymbol{\xi}, \boldsymbol{\eta}) = \int \frac{d^n\xi' \, d^n\eta'}{(2\pi)^n} e^{-i(\boldsymbol{\xi}' \cdot \boldsymbol{\eta} - \boldsymbol{\xi} \cdot \boldsymbol{\eta}')} f(\boldsymbol{\xi}', \boldsymbol{\eta}') \tag{43a}$$

This symplectic-invariant transformation would take us from the Wigner–Moyal function to the ambiguity function[24] in the normal case, but here the two coincide.

There are many solutions to this functional equation. If $g(\xi, \eta)$ is any solution, then so is the function

$$g_1(\xi, \eta) = \left(\pm\beta^2 \frac{\partial^2}{\partial\xi^2} + \beta^{-2}\xi^2 \right) \left(\pm\beta^{-2} \frac{\partial^2}{\partial\eta^2} + \beta^2\eta^2 \right) g(\xi, \eta)$$

A set of distinct solutions is provided by

$$f(\xi, \eta) = \sum_{m,n} g_{mn} \phi_m(\xi/\beta) \, \phi_n(\eta\beta) \tag{43b}$$

$$\phi_n(\xi) = \exp\left(-\frac{1}{2} \xi^2 \right) H_n(\xi) \tag{43c}$$

$$g_{nm} = (-1)^{(m+n)/2} g_{mn} \quad \text{if } m+n = \text{even}, \qquad g_{mn} = 0 \quad \text{if } m+n = \text{odd}$$

Here $H_n(\xi)$ are the Hermite polynomials and β has the dimensions of a length.

All these functions $\phi_n(\xi)$ are bell shaped and show that the beam spreads extend over a phase cell. This is the spread in the relative wave vectors k and position x and is over and above the beam spread inherent in the wave nature of light.

This additional correlation between the light rays is a manifestation of their inherent Bose symmetry and is an alternate manifestation of the positive distance correlation in an ideal Bose gas.[20] Since it depends only on the symmetry of the (N, N) order correlation function valid for both $\Gamma^{(N,N)}$ and $G^{(N,N)}$, we see that this aspect of the Bose symmetry of light rays is valid for both classical wave optics and quantum wave optics.

It is now useful to consider the relation between the various multivariate many-ray correlation functions especially since this would bring out essential quantum field theoretic differences. As a preliminary to this, we need to consider the WWM formalism for many degrees of freedom in its complex form and its limit for dealing with operator analytic signals.

6. WWM IN COMPLEX FORM FOR MANY DEGREES OF FREEDOM

We start with $2M$ hermitian operators

$$\hat{q}_r, \hat{p}_r, \qquad 1 \leqslant r \leqslant M$$

obeying

$$[\hat{q}_r, \hat{p}_s] = i\hbar\, \delta_{rs}, \qquad r, s = 1,..., M \tag{44}$$

Then, using M masses $m_1, m_2,..., m_M$ and M frequencies $\omega_1, \omega_2,..., \omega_M$ (for keeping dimensions correct) we arrive at $\hat{a}_r$ and $\hat{a}_r^\dagger$ obeying

$$[\hat{a}_r, \hat{a}_s^\dagger] = \hbar\, \delta_{rs}, \qquad [\hat{a}_r, \hat{a}_s] = [\hat{a}_r^\dagger, \hat{a}_s^\dagger] = 0 \tag{45}$$

Then we generalize Eqs. (19)–(20) thus: If $\underset{\sim}{a} \equiv (a_1, a_2,..., a_M) =$ a set of M complex numbers, $\underset{\sim}{\xi} \equiv (\xi_1, \xi_2,..., \xi_M) =$ similar set, with $\underset{\sim}{a}$ having dimensions (action)$^{1/2}$, and $\underset{\sim}{\xi}$ having dimensions (action)$^{-1/2}$,

$$\hat{W}(\underset{\sim}{a}; \underset{\sim}{a}^*) = \int \prod_{r=1}^{M} \left(\frac{\hbar}{\pi}\, d^2 \xi_r\right) e^{\underset{\sim}{\xi} \cdot \hat{\underset{\sim}{a}}^\dagger}\, e^{-\underset{\sim}{\xi}^* \cdot \hat{\underset{\sim}{a}}}\, e^{-(\hbar/2)\underset{\sim}{\xi}^* \cdot \underset{\sim}{\xi} + \underset{\sim}{\xi}^* \cdot \underset{\sim}{a} - \underset{\sim}{\xi} \cdot \underset{\sim}{a}^*} \tag{46}$$

The dot means product and sum over M terms. Then any operator F has a WWM representative f:

$$\begin{aligned} f(\underset{\sim}{a}; \underset{\sim}{a}^*) &= \operatorname{Tr} F\hat{W}(\underset{\sim}{a}; \underset{\sim}{a}^*) \\ F &= \int \prod_{r=1}^{M} \left(\frac{d^2 a_r}{\pi\hbar}\right) f(\underset{\sim}{a}; \underset{\sim}{a}^*)\, \hat{W}(\underset{\sim}{a}; \underset{\sim}{a}^*) \\ \operatorname{Tr} FG &= \int \prod_{r=1}^{M} \left(\frac{d^2 a_r}{\pi\hbar}\right) f(\underset{\sim}{a}; \underset{\sim}{a}^*)\, g(\underset{\sim}{a}; \underset{\sim}{a}^*) \end{aligned} \tag{47}$$

In the above we can connect to real q's and p's by

$$\begin{aligned} \hat{a}_r &= \frac{m_r\omega_r\hat{q}_r + i\hat{p}_r}{\sqrt{2m_r\omega_r}}, \qquad & \hat{a}_r^\dagger &= \frac{m_r\omega_r\hat{q}_r - i\hat{p}_r}{\sqrt{2m_r\omega_r}} \\ a_r &= \frac{m_r\omega_r q_r + ip_r}{\sqrt{2m_r\omega_r}}, \qquad & a_r^* &= \frac{m_r\omega_r q_r - ip_r}{\sqrt{2m_r\omega_r}} \end{aligned} \tag{48}$$

Now the WWM rule associates, for any $\underset{\sim}{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_M)$,

$$e^{\underset{\sim}{\lambda} \cdot \underset{\sim}{\hat{a}}^\dagger - \underset{\sim}{\lambda}^* \cdot \underset{\sim}{\hat{a}}} \overset{\text{WWM}}{\longleftrightarrow} e^{\underset{\sim}{\lambda} \cdot \underset{\sim}{a}^* - \underset{\sim}{\lambda}^* \cdot \underset{\sim}{a}}$$

or

$$e^{\underset{\sim}{\lambda} \cdot \underset{\sim}{\hat{a}}^\dagger} e^{-\underset{\sim}{\lambda}^* \cdot \underset{\sim}{\hat{a}}} \longleftrightarrow e^{(\hbar/2)\underset{\sim}{\lambda}^* \cdot \underset{\sim}{\lambda}} e^{\underset{\sim}{\lambda} \cdot \underset{\sim}{a}^* - \underset{\sim}{\lambda}^* \cdot \underset{\sim}{a}}$$
$$= \exp\left[-\frac{\hbar}{2}\left(\sum_1^M \frac{\partial}{\partial a_r^*}\frac{\partial}{\partial a_r}\right)\right] e^{\underset{\sim}{\lambda} \cdot \underset{\sim}{a}^* - \underset{\sim}{\lambda}^* \cdot \underset{\sim}{a}}$$

Therefore, expanding the exponentials and comparing terms, we see that

$$\hat{a}_{r_1}^\dagger \cdots \hat{a}_{r_m}^\dagger \hat{a}_{s_1} \cdots \hat{a}_{s_n} \overset{\text{WWM}}{\longleftrightarrow} \exp\left[\frac{-\hbar}{2} \sum_{r=1}^M \frac{\partial}{\partial a_r^*}\frac{\partial}{\partial a_r}\right] a_{r_1}^* \cdots a_{r_m}^* a_{s_1} \cdots a_{s_n} \tag{49}$$

Thus, for any operator F with WWM representative f, we have

$$\mathrm{Tr}(F\hat{a}_{r_1}^\dagger \cdots \hat{a}_{r_m}^\dagger \hat{a}_{s_1} \cdots \hat{a}_{s_n})$$
$$= \int \prod_{r=1}^M \left(\frac{d^2 a_r}{\pi\hbar}\right) f(\underset{\sim}{a}; \underset{\sim}{a}^*) \exp\left[-\frac{\hbar}{2} \sum_{r=1}^M \frac{\partial^2}{\partial a_r^* \partial a_r}\right] a_{r_1}^* \cdots a_{r_m}^* a_{s_1} \cdots a_{s_n} \tag{50}$$

In the case $F = \hat{\rho}$ = density operator for some state, we will write W instead of f, and associate the factors of $\hbar$ in the volume element with W: i.e., we will say

$$\hat{\rho} \overset{\text{WWM}}{\longleftrightarrow} \hbar^M W(\underset{\sim}{a}; \underset{\sim}{a}^*)$$

$$\mathrm{Tr}(\hat{\rho}\hat{a}_{r_1}^\dagger \cdots \hat{a}_{r_m}^\dagger \hat{a}_{s_1} \cdots \hat{a}_{s_n})$$
$$\equiv \langle \hat{a}_{r_1}^\dagger \cdots \hat{a}_{r_m}^\dagger \hat{a}_{s_1} \cdots \hat{a}_{s_n} \rangle$$
$$= \int \prod_{r=1}^M \left(\frac{d^2 a_r}{\pi}\right) W(\underset{\sim}{a}; \underset{\sim}{a}^*) \exp\left[\frac{-\hbar}{2} \sum_1^M \partial^2/\partial a_r^* \partial a_r\right] a_{r_1}^* \cdots a_{r_m}^* a_{s_1} \cdots a_{s_n}$$

$$W(\underset{\sim}{a}; \underset{\sim}{a}^*) = \frac{1}{\hbar^M} \mathrm{Tr}(\hat{\rho}\hat{W}(\underset{\sim}{a}; \underset{\sim}{a}^*)) \tag{51}$$

This is arranged so that the volume element $\prod(d^2 a_r/\pi)$ in this case remains acceptable classically.

Now we can set up the formulas for the spinless scalar field. Start with the classical Lagrangian, then proceed. Just to have all factors in place, we write

$$\mathscr{L} = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi = \frac{1}{2c^2}\dot{\phi}^2 - \tfrac{1}{2}(\nabla\phi)^2$$

$$g_{00} = -1$$

$$x^0 = ct$$

$$\phi(x) \equiv \phi(\mathbf{x}, t) = \text{real scalar} \tag{52}$$

$$\pi(x) = \text{canonical momentum} = \frac{\partial\mathscr{L}}{\partial\dot{\phi}} = \frac{1}{c^2}\dot{\phi}(\mathbf{x}, t) \tag{53}$$

The equation of motion and equal time Poisson brackets are

$$\begin{aligned} &\Box^2\phi(x) \equiv \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \\ &\{\phi(\mathbf{x}, t), \pi(\mathbf{x}'; t\} = \delta^{(3)}(\mathbf{x} - \mathbf{x}'), \qquad \text{rest zero} \end{aligned} \tag{54}$$

The general solution is

$$\begin{aligned} \phi(x) &= \frac{\sqrt{c}}{(2\pi)^{3/2}}\int\frac{d^3k}{\sqrt{2k^0}}\,(a(\mathbf{k})\,e^{ik\cdot x} + \text{c.c.}), \qquad k^0 = |\mathbf{k}| = \omega/c \\ \pi(x) &= \frac{-i}{\sqrt{c}\,(2\pi)^{3/2}}\int d^3k\,\sqrt{\frac{k^0}{2}}\,(a(\mathbf{k})\,e^{ik\cdot x} - \text{c.c.}) \end{aligned} \tag{55}$$

So $\psi(x)$, the analytic signal positive frequency part of $\phi(x)$, is given by

$$\begin{aligned} \psi(x) &= \frac{\sqrt{c}}{(2\pi)^{3/2}}\int\frac{d^3k}{\sqrt{2k^0}}\,a(\mathbf{k})\,e^{ik\cdot x} \\ &= \frac{1}{2}\left(\phi(x) + \frac{ic}{\sqrt{-\nabla^2}}\,\pi(x)\right) \end{aligned} \tag{56}$$

$\psi(x)$ obeys a first-order equation of motion[22,23]:

$$\frac{\partial}{\partial t}\psi(x) = -ic\sqrt{-\nabla^2}\,\psi(x), \qquad \psi(\mathbf{x}, t) = e^{-ict\sqrt{-\nabla^2}}\,\psi(\mathbf{x}, 0) \tag{57}$$

It follows that

$$\psi(x) \xleftrightarrow{\text{one-to-one}} \psi(\mathbf{x}, 0) \equiv \psi(\mathbf{x}) \xleftrightarrow{\text{one-to-one}} a(\mathbf{k})$$

$$\underset{\text{rest zero}}{\{\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)\} = \delta^{(3)}(\mathbf{x} - \mathbf{x}')} \leftrightarrow \underset{\text{rest zero}}{\{a(\mathbf{k}), a^*(\mathbf{k}')\} = -i\delta^{(3)}(\mathbf{k} - \mathbf{k}')}$$

$$\begin{aligned} &\leftrightarrow \{\psi(\mathbf{x}), \psi^*(\mathbf{x}')\} = \frac{-ic}{(2\pi)^3}\int\frac{d^3k}{2k^0}\,e^{ik\cdot(\mathbf{x} - \mathbf{x}')} \\ &= \frac{-ic}{2}\frac{1}{\sqrt{-\nabla^2}}\,\delta^3(\mathbf{x} - \mathbf{x}') = \Delta^{(+)}(\mathbf{x} - \mathbf{x}'; 0), \qquad \text{say} \end{aligned} \tag{58}$$

In the spirit of an earlier paper[25] by two of us, we define a space S of solutions $\psi(x)$ as follows:

$$S=\left\{\psi(x) \,\middle|\, \|\psi\|^2 \equiv \int d^3k\,|a(\mathbf{k})|^2<\infty\right\} \tag{59}$$

This norm $\|\psi\|^2$ can be written as

$$\begin{aligned}
\|\psi\|^2 &= \frac{2i}{c}\int d^3x\,\psi^*(x)\,\partial_0\psi(x)\\
&= -\frac{2i}{c}\int d^3x(\partial_0\psi^*(x))\,\psi(x)\\
&= \frac{2}{c}\int d^3x\,\psi^*(\mathbf{x})(-\nabla^2)^{1/2}\,\psi(\mathbf{x})\\
&= \frac{2}{c}\int d^3x((-\nabla^2)^{1/2}\,\psi(\mathbf{x}))^*\,\psi(\mathbf{x})
\end{aligned}$$

$$\begin{aligned}
(\psi',\psi) &= \int d^3k\,a'(\mathbf{k})^*\,a(\mathbf{k})\\
&= \frac{2}{c}\int d^3x\,\psi'(\mathbf{x})^*\,(-\nabla^2)^{1/2}\,\psi(\mathbf{x})\\
&= \frac{2}{c}\int d^3x((-\nabla^2)^{1/2}\,\psi'(\mathbf{x}))^*\,\psi(\mathbf{x})
\end{aligned} \tag{60}$$

So

$$S=\left\{\psi(\mathbf{x}) \,\middle|\, \|\psi\|^2 \equiv \frac{2}{c}\int d^3x\,\psi(\mathbf{x})^*\,(-\nabla^2)^{1/2}\,\psi(\mathbf{x})<\infty\right\} \tag{61}$$

Let us introduce a complete orthonormal basis in S, based on $\{f_r(\mathbf{k})\}$, $r=1, 2,\ldots, \infty$:

$$\int d^3k\,f_r(\mathbf{k})^*\,f_s(\mathbf{k})=\delta_{rs},\quad \sum_{r=1}^{\infty} f_r(\mathbf{k})\,f_r(\mathbf{k}')^*=\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

$$\psi_r(\mathbf{x})=\frac{\sqrt{c}}{(2\pi)^{3/2}}\int\frac{d^3k}{\sqrt{2k^0}}\,f_r(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$(\psi_r,\psi_s)\equiv\frac{2}{c}\int d^3x\,\psi_r(\mathbf{x})^*\,(-\nabla^2)^{1/2}\,\psi_s(\mathbf{x})=\delta_{rs}$$

$$\sum_{r=1}^{\infty}\psi_r(\mathbf{x})\,\psi_r(\mathbf{x}')^*=\frac{c}{(2\pi)^3}\int\frac{d^3k}{2k^0}\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}=i\Delta^+(\mathbf{x}-\mathbf{x}';0) \tag{62}$$

So $\{\psi_r(\mathbf{x})\}$ is an orthonormal basis for S. If a general $\psi \in S$ is expanded as

$$\begin{aligned} \psi(\mathbf{x}) &= \sum_{r=1}^{\infty} a_r \psi_r(\mathbf{x}):\ a_r = (\psi_r, \psi) \\ \|\psi\|^2 &= \sum_{r=1}^{\infty} |a_r|^2, \qquad \{a_r, a_s^*\}_{P.B.} = -i\delta_{rs} \end{aligned} \tag{63}$$

To quantize, we promote a_r, a_r^* to operators and set up the analytic signal field operator

$$\begin{aligned} \hat{\psi}(\mathbf{x}) &= \sum_{r=1}^{\infty} \hat{a}_r \psi_r(\mathbf{x}), \qquad \hat{a}_r = (\psi_r, \hat{\psi}) \\ \hat{\psi}^\dagger(\mathbf{x}) &= \sum_{r=1}^{\infty} \hat{a}_r^\dagger \psi_r^*(\mathbf{x}), \qquad \hat{a}_r^\dagger = (\hat{\psi}, \psi_r) \\ [\hat{a}_r, \hat{a}_s^\dagger] &= \hbar\, \delta_{rs};\ [\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x}')] = i\hbar\, \Delta^{(+)}(\mathbf{x}-\mathbf{x}'; 0) \end{aligned} \tag{64}$$

It is consistent that $\hat{a}_r$, $\hat{a}_r$, $a_r^\dagger$, a_r^* all have the dimensions (action)$^{1/2}$.

Now, to set up the WWM machinery, we pick a sequence of complex numbers $\{a_r\}$ and define formally

$$\begin{aligned} \hat{W}[\{a_r\}; \{a_r^*\}] = \lim_{M\to\infty} \int \prod_{r=1}^{M} \left(\frac{\hbar}{\pi} d^2\xi_r\right) \exp\left(\sum_1^M \xi_r \hat{a}_r^\dagger\right) \exp\left(-\sum_1^M \xi_r^* \hat{a}_r\right) \\ \cdot \exp\left[-\frac{\hbar}{2}\sum_1^M (\xi_r^* \xi_r) + \sum_1^M (\xi_r^* a_r - \xi_r a_r^*)\right] \end{aligned} \tag{65}$$

This is nothing but (46). Formally taking the limit $M \to \infty$, we identify

$$\begin{aligned} \psi(\mathbf{x}) &= \sum_r a_r \psi_r(\mathbf{x}) \in S \\ \xi(\mathbf{x}) &= \sum_r \xi_r \psi_r(\mathbf{x}) \in S \\ \sum_r \xi_r^* \xi_r &= (\xi, \xi) \\ \sum_r \xi_r \hat{a}_r^\dagger &= (\hat{\psi}, \xi), \text{ etc.} \end{aligned} \tag{66}$$

Now we define in a heuristic way

$$\begin{aligned} \hat{W}[\psi(\cdot); \psi^*(\cdot)] &= \text{an operator functional of } \psi(\cdot) \in S \\ &= \int_S D\xi^*\, D\xi \exp(\hat{\psi}, \xi) \exp[-(\xi, \hat{\psi})] \\ &\quad \cdot \exp\left[\frac{-\hbar}{2}(\xi, \xi) + (\xi, \psi) - (\psi, \xi)\right] \end{aligned} \tag{67}$$

$\hat{W}$ is a functional of $\psi(\cdot) \in S$, and the integration variable $\xi(\cdot)$ also runs over S. If F is an operator,

$$F \overset{\text{WWM}}{\longleftrightarrow} f[\psi(\cdot); \psi^*(\cdot)] = \text{functional on } S$$
$$= \text{Tr}\, F\hat{W}[\psi(\cdot); \psi^*(\cdot)] \tag{68}$$

Also the WWM treats elementary exponentials as follows. For any $\lambda(\cdot) \in S$,

$$\exp(\hat{\psi}, \lambda) \exp[-(\lambda, \hat{\psi})] \overset{\text{WWM}}{\longleftrightarrow} \exp\left[+\frac{\hbar}{2}(\lambda, \lambda)\right] \exp[(\psi, \lambda) - (\lambda, \psi)]$$

$$\exp\left[\frac{\hbar}{2}(\lambda, \lambda)\right] = \text{limit of } \exp\left[\frac{\hbar}{2}\sum_r \lambda_r^* \lambda_r\right]$$
$$= \exp\left[\frac{-\hbar}{2}\sum_r \frac{\partial^2}{\partial a_r^* \, \partial a_r}\right] \text{acting on } \exp[(\psi, \lambda) - (\lambda, \psi)]$$
$$= \exp\left(\frac{-\hbar}{2}\Delta\right) \text{acting on } \exp[(\psi, \lambda) - (\lambda, \psi)]$$

In detail:

$$\exp(\hat{\psi}, \lambda) \exp[-(\lambda, \hat{\psi})] \overset{\text{WWM}}{\longleftrightarrow} \exp\left(-\frac{\hbar}{2}\Delta\right) \exp[(\psi, \lambda) - (\lambda, \psi)],$$
$$\Delta = i \int d^3y \int d^3y' \, \Delta^{(+)}(\mathbf{y} - \mathbf{y}'; 0) \frac{\delta}{\delta\psi^*(\mathbf{y})} \frac{\delta}{\delta\psi(\mathbf{y}')} \tag{69}$$

Comparing coefficients, we have

$$\hat{\psi}^\dagger(\mathbf{y}_1) \cdots \hat{\psi}^\dagger(\mathbf{y}_m) \hat{\psi}(\mathbf{x}_1) \cdots \hat{\psi}(\mathbf{x}_n)$$
$$\overset{\text{WWM}}{\longleftrightarrow} \exp\left(\frac{-\hbar}{2}\Delta\right) \psi^*(\mathbf{y}_1) \cdots \psi^*(\mathbf{y}_m) \psi(\mathbf{x}_1) \cdots \psi(\mathbf{x}_n) \tag{70}$$

We can use this to express the $G^{(n,m)}$ for a quantum state $\hat{\rho}$. Notwithstanding troublesome infinite powers of $\hbar$, we can formally define the Wigner distribution function for the whole quantum field as a functional on S:

$$\bar{W}[\psi(\cdot); \psi^*(\cdot)] = \text{Tr}\, \hat{\rho} \hat{W}[\psi(\cdot); \psi^*(\cdot)] \tag{71}$$

Then the hierarchy of quantum correlation functions is

$$G^{(n,m)}(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n; \mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_m)$$
$$\equiv \text{Tr}(\hat{\rho}\hat{\psi}^\dagger(\mathbf{y}_1) \cdots \hat{\psi}^\dagger(\mathbf{y}_m) \hat{\psi}(\mathbf{x}_1) \cdots \hat{\psi}(\mathbf{x}_n))$$
$$= \int_S D\psi^* \, D\psi \; \bar{W}[\psi(\cdot); \psi^*(\cdot)]$$
$$\cdot \exp\left(-\frac{\hbar}{2}\Delta\right) \psi^*(\mathbf{y}_1) \cdots \psi^*(\mathbf{y}_m) \psi(\mathbf{x}_1) \cdots \psi(\mathbf{x}_n) \tag{72}$$

In this form, since the WWM method treats the two factors in the Trace FG symmetrically, we see that the quantum features reside in *two* places: in $\bar{W}[\psi(\cdot); \psi^*(\cdot)]$, which is a real but not necessarily nonnegative functional, and in $\exp((-\hbar/2)\Delta)$. The factor $\exp((-\hbar/2)\Delta)$ reflects the normal ordering in definition (4).

This is reminiscent of the "excess fluctuation" of the photon counts[21] including the particle-like Poisson noise in addition to the wave noise: $\langle n^2\rangle - \langle n\rangle^2 = \langle n\rangle^2 + \langle n\rangle$. We may compare (72) with the heuristic classical case:

$$\Gamma^{(n,m)}(\mathbf{x}_1,..., \mathbf{x}_n; \mathbf{y}_1,..., \mathbf{y}_m)$$
$$\equiv \langle \psi^*(\mathbf{y}_1)\cdots\psi^*(\mathbf{y}_m)\,\psi(\mathbf{x}_1)\cdots\psi(\mathbf{x}_n)\rangle$$
$$= \int_S D\psi^*\, D\psi\, P[\psi(\cdot); \psi^*(\cdot)]\, \psi^*(\mathbf{y}_1)\cdots\psi^*(\mathbf{y}_m)\,\psi(\mathbf{x}_1)\cdots\psi(\mathbf{x}_n) \qquad (73)$$

We see that $\bar{W}$ is replaced by P which is a "pointwise" nonnegative functional on S, and $\exp((-\hbar/2)\Delta)$ is of course absent. If, however, we use the diagonal representation[23] of Sudarshan, then in place of (72) we have the formally simpler expression

$$G^{(n,m)}(\mathbf{x}_1,..., \mathbf{x}_n; \mathbf{y}_1,..., \mathbf{y}_m)$$
$$= \int_S D\psi^*\, D\psi\, \Phi[\psi(\cdot); \psi^*(\cdot)]$$
$$\cdot\psi^*(\mathbf{y}_1)\cdots\psi^*(\mathbf{y}_m)\,\psi(\mathbf{x}_1)\cdots\psi(\mathbf{x}_n) \qquad (74)$$

so the factor $\exp((-\hbar/2)\Delta)$ is removed, and W gives place to the more singular (in general) functional Φ. Now comparing the classical (73) with the quantum (74), the only change is the replacement $P[\psi; \psi^*] \to \Phi[\psi; \psi^*]$. *All* quantum effects are now in Φ: again all this is because we have normal ordering in (4).

7. THE N-FOLD RAY DISTRIBUTION FUNCTIONS

In this notation what do the N-fold ray distributions ω_N and W_N look like? The expressions are quite nice. From (73) we have for the classical ray distribution function defined by (39)

$$\omega_N(\mathbf{x}_1, \mathbf{k}_1; \mathbf{x}_2, \mathbf{k}_2; ...; \mathbf{x}_N, \mathbf{k}_N)$$
$$= \int_S D\psi^*\, D\psi\, P[\psi(\cdot); \psi^*(\cdot)]\, \omega_1^{(\psi)}(\mathbf{x}_1, \mathbf{k}_1)\, \omega_1^{(\psi)}(\mathbf{x}_2, \mathbf{k}_2)\cdots\omega_1^{(\psi)}(\mathbf{x}_N, \mathbf{k}_N)$$
$$\equiv \langle \omega_1^{(\psi)}(\mathbf{x}_1, \mathbf{k}_1)\, \omega_1^{(\psi)}(\mathbf{x}_2, \mathbf{k}_2)\cdots\omega_1^{(\psi)}(\mathbf{x}_N, \mathbf{k}_N)\rangle_P \qquad (75)$$

the ensemble average being over the classical ensemble defined by $P[\psi(\cdot); \psi^*(\cdot)]$. Here

$$\omega_1^{(\psi)}(\mathbf{x}, \mathbf{k}) = (2\pi)^{-n} \int d^n x' \, e^{ik\cdot x'} \, \psi(\mathbf{x} - \tfrac{1}{2}\mathbf{x}') \, \psi^*(\mathbf{x} + \tfrac{1}{2}\mathbf{x}'), \qquad \psi \in S \tag{76}$$

is the random-valued generalized ray-distribution function we would have defined at the level of $\Gamma^{(1,1)}$ for a random-valued pure state $\psi \in S$! Moreover, from (74) the same $\omega_1^{(\psi)}$ suffices to express the quantum ray distributions W_N *provided* we use the weight functional Φ:

$$\begin{aligned} W_N(\mathbf{x}_1, \mathbf{k}_1; \mathbf{x}_2, \mathbf{k}_2; ...; \mathbf{x}_N, \mathbf{k}_N) &= \int D\psi^* \, D\psi \, \Phi[\psi(\cdot); \psi^*(\cdot)] \, \omega_1^{(\psi)}(\mathbf{x}_1, \mathbf{k}_1) \cdots \omega_1^{(\psi)}(\mathbf{x}_N, \mathbf{k}_N) \\ &\equiv \langle \omega_1^{(\psi)}(\mathbf{x}_1, \mathbf{k}_1) \cdots \omega_1^{(\psi)}(\mathbf{x}_N, \mathbf{k}_N) \rangle_\Phi \end{aligned} \tag{77}$$

The quantum nature is partially rendered in the relations between the different W_N's. From (75), since $P \geqslant 0$, if we have a nontrivial field of illumination so that the classical ensemble has *some* nonzero ψ, then

$$\omega_1(\mathbf{x}, \mathbf{k}) \neq 0 \rightarrow \omega_N(\mathbf{x}_1, \mathbf{k}_1; ...; \mathbf{x}_N, \mathbf{k}_N) \neq 0 \qquad \text{for all} \quad N \geqslant 2$$

But in the quantum case we can have a state $\hat{\rho}$ with a finite number of photons, and then W_N vanish for large N! Thus $W_1(\mathbf{x}, \mathbf{k}) \neq 0 \nrightarrow W_N(\mathbf{x}_1, \mathbf{k}_1; ...; \mathbf{x}_N, \mathbf{k}_N) \neq 0$ for $N \geqslant 2$. We come back to this later. But first let us develop generating functionals for ω_N and W_N!

These are quite easily set up, based on the expressions (75) and (77). To begin with, we deal with the classical ω_N: since $\omega_1^{(\psi)}$ as defined in (76) is a real function of $\mathbf{x}, \mathbf{k}$, it suffices to introduce a real "external source" function $\lambda(\mathbf{x}, \mathbf{k})$ and set up

$$\begin{aligned} &\left\langle \exp\left[i \int d^n x \, d^n k \, \lambda(\mathbf{x}, \mathbf{k}) \, \omega_1^{(\psi)}(\mathbf{x}, \mathbf{k}) \right] \right\rangle_P \\ &\quad = 1 + \sum_{N=1}^{\infty} \frac{i^N}{N!} \int d^n x_1 \, d^n k_1 \cdots d^n x_N \, d^n k_N \, \lambda(\mathbf{x}_1 \mathbf{k}_1) \cdots \\ &\qquad \lambda(\mathbf{x}_N \mathbf{k}_N) \, \omega_N(\mathbf{x}_1 \mathbf{k}_1; ...; \mathbf{x}_N \mathbf{k}_N) \end{aligned} \tag{78}$$

Thus the entire collection of classical generalized ray distributions can be handled compactly "up in the exponent." If we wish, the expression in the exponent could be rewritten in the spirit of (11) as

$$\begin{aligned} \int d^n x \, d^n k \, \lambda(\mathbf{x}, \mathbf{k}) \, \omega_1^{(\psi)}(\mathbf{x}, \mathbf{k}) &= (2\pi)^n \int d^n x \, d^n x' \, \tilde{\lambda}(\mathbf{x}; \mathbf{x}') \, \psi(\mathbf{x}') \, \psi^*(\mathbf{x}) \\ \tilde{\lambda}(\mathbf{x}; \mathbf{x}') &= \int d^n k \, \lambda\left(\frac{\mathbf{x} + \mathbf{x}'}{2}, \mathbf{k}\right) e^{i\mathbf{k}\cdot(\mathbf{x} - \mathbf{x}')} \end{aligned} \tag{79}$$

so that we have

$$\left\langle \exp\left[i(2\pi)^n \int d^n x\, d^n x'\, \tilde{\lambda}(\mathbf{x};\mathbf{x}')\, \psi(\mathbf{x}')\, \psi^*(\mathbf{x}) \right] \right\rangle_P$$
$$= 1 + \sum_{N=1}^{\infty} \frac{i^N}{N!} \int d^n x_1\, d^n k_1 \cdots d^n x_N\, d^n k_N\, \lambda(\mathbf{x}_1 \mathbf{k}_1) \cdots$$
$$\lambda(\mathbf{x}_N \mathbf{k}_N)\, \omega_N(\mathbf{x}_1 \mathbf{k}_1; \ldots; \mathbf{x}_N \mathbf{k}_N) \tag{80}$$

Now we can compare all this with the expression (77) for the quantum functions W_N, and immediately see that the ensemble averaging is to be done with respect to Φ rather than P: but in terms of $\hat{\rho}$ this just means normal ordering! Thus we have

$$1 + \sum_{N=1}^{\infty} \frac{i^N}{N!} \int d^n x_1\, d^n k_1 \cdots d^n x_N\, d^n k_N\, \lambda(\mathbf{x}_1 \mathbf{k}_1) \cdots$$
$$\lambda(\mathbf{x}_N \mathbf{k}_N)\, W_N(\mathbf{x}_1 \mathbf{k}_1; \ldots; \mathbf{x}_N \mathbf{k}_N)$$
$$= \mathrm{Tr}\left[\hat{\rho} : \exp\left\{ i(2\pi)^n \int d^n x\, d^n x'\, \tilde{\lambda}(\mathbf{x};\mathbf{x}')\, \hat{\psi}^\dagger(\mathbf{x})\, \hat{\psi}(\mathbf{x}') \right\} : \right] \tag{81}$$

Comparing expressions (80) and (81), we can say that in classical statistical optics the generalization in the concept of rays that has been necessitated already at the level of dealing with the two point function $\Gamma^{(1,1)}$, leading to bright and dark rays (just a consequence of WWM in QM!), is all that is needed in generalizing rays to handle $\Gamma^{(N,N)}$ for all N, apart from the Bose correlation of rays. This is because the averaging in (80) is over a positive semidefinite functional $P[..]$ describing a true statistical mixture. In quantum mechanics, on the other hand, the normal ordering needed in (81) tells us that when we go from $N=1$, i.e., $G^{(1,1)}$, to $N \geq 2$, quantum-generalized rays acquire "new properties" not seen at the level of $G^{(1,1)}$. Simply because $\Phi[..]$ is not a probability, that is all. Since the *entire* difference is caused by the normal ordering in (81), the situation is *qualitatively similar* to the difference between a classical state possessing first-order coherence and a quantum state possessing first-order coherence, as explained, for example, in Ref. 16. Suppose for some classical statistical state we know that in terms of some $\psi_0(\mathbf{x}) \in S$

$$\Gamma^{(1,1)}(\mathbf{x};\mathbf{y}) \equiv \langle \psi(\mathbf{x})\, \psi^*(\mathbf{y}) \rangle = \psi_0(\mathbf{x})\, \psi_0^*(\mathbf{y}) \tag{82}$$

Then the ensemble consists *only* of (complex) multiples of $\psi_0(\mathbf{x})$! That is, there is some probability distribution $p(z)$ over the complex plane, $z \in \mathbb{C}$, such that for any functional $f[\psi(\cdot); \psi^*(\cdot)]$ the ensemble average is

$$\langle f[\psi(\cdot); \psi^*(\cdot)] \rangle = \int_{\mathbb{C}} d^2z \, p(z) \, f\left[z \frac{\psi_0(\cdot)}{\|\psi_0\|}; z^* \frac{\psi_0^*(\cdot)}{\|\psi_0\|} \right] \tag{83}$$

$$\int_{\mathbb{C}} d^2z \, z^* z p(z) = \|\psi_0\|^2$$

Then all $\Gamma^{(n,m)}$ factorize:

$$\Gamma^{(n,m)}(\mathbf{x}_1 \cdots \mathbf{x}_n; \mathbf{y}_1 \cdots \mathbf{y}_m) = \gamma_{n,m} \psi_0(\mathbf{x}_1) \cdots \psi_0(\mathbf{x}_n) \, \psi_0^*(\mathbf{y}_1) \cdots \psi_0^*(\mathbf{y}_m)$$

$$\gamma_{n,m} = \frac{\int d^2z \, z^n (z^*)^m \, p(z)}{\|\psi_0\|^{n+m}}, \qquad \gamma_{1,1} = 1 \tag{84}$$

Then the generalized ray distributions are

$$\omega_N(x_1 k_1; \ldots; x_N k_N) = \gamma_N \omega_1^{(\psi_0)}(x_1 k_1) \cdots \omega_1^{(\psi_0)}(x_N k_N) \tag{85}$$

where $\gamma_N \equiv \gamma_{N,N}$ is nondecreasing:

$$1 = \gamma_{1,1} \equiv \gamma_1 \leqslant \gamma_2 \leqslant \gamma_3 \leqslant \cdots \tag{86}$$

On the other hand, if we have a quantum state $\hat{\rho}$ for which

$$G^{(1,1)}(\mathbf{x}; \mathbf{y}) \equiv \operatorname{Tr}(\hat{\rho} \hat{\psi}^\dagger(\mathbf{y}) \, \hat{\psi}(\mathbf{x})) = \psi_0(\mathbf{x}) \, \psi_0^*(\mathbf{y}) \tag{87}$$

then $\hat{\rho}$ has to have the following general form: we define

$$\begin{aligned} \hat{a}(\psi_0) &= (\psi_0, \hat{\psi}) = \frac{2}{c} \int d^3x \, \psi_0^*(\mathbf{x}) (-\nabla^2)^{1/2} \, \hat{\psi}(\mathbf{x}) \\ \hat{a}(\psi_0)^\dagger &= (\hat{\psi}, \psi_0) = \frac{2}{c} \int d^3x \, \hat{\psi}(\mathbf{x})^\dagger \, (-\nabla^2)^{1/2} \, \psi_0(\mathbf{x}) \\ [\hat{a}(\psi_0), \hat{a}(\psi_0)^\dagger] &= \hbar \, \|\psi_0\|^2 \\ [\hat{\psi}(\mathbf{x}), \hat{a}(\psi_0)^\dagger] &= \hbar \psi_0(\mathbf{x}) \end{aligned} \tag{88}$$

and the most general $\hat{\rho}$ obeying (87) is

$$\begin{aligned} \hat{\rho} &= \sum_{m,n=0}^{\infty} \rho_{mn} |m\rangle \langle n|, \qquad |m\rangle = \frac{1}{\sqrt{m!}} \left(\frac{\hat{a}(\psi_0)^\dagger}{\hbar^{1/2} \|\psi_0\|} \right)^m |0\rangle \\ &\sum_{m=0}^{\infty} m \rho_{mm} = \|\psi_0\|^2 / \hbar \end{aligned} \tag{89}$$

Then since

$$\hat{\psi}(\mathbf{x})\,\hat{\rho} = \psi_0(\mathbf{x})\,\frac{\hat{a}(\psi_0)}{\|\psi_0\|^2}\,\hat{\rho}$$
$$\hat{\rho}\hat{\psi}(\mathbf{x})^\dagger = \psi_0^*(\mathbf{x})\,\hat{\rho}\,\frac{\hat{a}(\psi_0)^\dagger}{\|\psi_0\|^2} \tag{90}$$

the other $G^{(n,m)}$ are

$$G^{(n,m)}(\mathbf{x}_1\cdots\mathbf{x}_n;\mathbf{y}_1\cdots\mathbf{y}_m) = g_{n,m}\psi_0(\mathbf{x}_1)\cdots\psi_0(\mathbf{x}_n)\,\psi_0^*(\mathbf{y}_1)\cdots\psi_0^*(\mathbf{y}_m)$$
$$g_{n,m} = \frac{\hbar^{(n+m)/2}}{\|\psi_0\|^{n+m}}\,\mathrm{Tr}\left[\hat{\rho}\left(\frac{\hat{a}(\psi_0)^\dagger}{\hbar^{1/2}\|\psi_0\|}\right)^m\left(\frac{\hat{a}(\psi_0)}{\hbar^{1/2}\|\psi_0\|}\right)^n\right] \tag{91}$$

The quantum generalized ray distributions are

$$W_N(x_1k_1;\ldots;x_Nk_N) = g_N\,\omega_1^{(\psi_0)}(x_1k_1)\cdots\omega_1^{(\psi_0)}(x_Nk_N)$$
$$g_N \equiv g_{N,N} \tag{92}$$

Comparing (92) with (85), we see that γ_N has been replaced by g_N:

$$\gamma_N = \frac{1}{\|\psi_0\|^{2N}}\int d^2z\,|z|^{2N}\,p(z)$$
$$\to g_N = \frac{\hbar^N}{\|\psi_0\|^{2N}}\,\mathrm{Tr}[\hat{\rho}\!:(\hat{b}^\dagger\hat{b})^N\!:]$$
$$\hat{b} = \frac{\hat{a}(\psi_0)}{\hbar^{1/2}\|\psi_0\|} \tag{93}$$

This is just the same normal ordering seen in (81) as compared to (80)! So its main general effect must be qualitatively similar to its effect in the special case of first-order coherent fields where while γ_N cannot decrease, g_N certainly can; and in fact for a state with at most N_0 quanta, g_{N_0+1} and all higher g's vanish: i.e., for such nonclassical states, $W_{N_0+1} = W_{N_0+2} = \cdots = 0$.

8. DISCUSSION

In this paper we have raised and partially resolved the question of the description of statistical wavefields in terms of bundles of rays. This ray description is an *exact transcription* of the wavefield phenomena, both classical and quantum, and not a short wavelength limit. For the standard optical phenomena described by the (1, 1) correlation function we need to generalize the notion of a bundle of rays to include *both light rays and dark rays*. There is a conjugacy between position and wave number so that the bundle has a *minimum spread* over a phase cell. When we go to the N-ray distribution function, there are additional Bose effects leading to *ray–ray correlations*: these obtain both for classical and quantum fields.

The distinction between the two comes when ray distribution functions of N_1 and N_2 rays are compared. In classical wave optics if N_1 ray distribution is nonvanishing, N_2 ray distribution is also nonvanishing as long as N_1 and N_2 are nonzero; but for quantum optics, this need not obtain. This in turn can be traced to the diagonal quasi-probability functional in quantum optics[23] not being necessarily pointwise positive.

In classical optics paraxial wave propagation is an important special case.[8–14,26] For paraxial propagation the wave equation can be rewritten[8]

$$\Box^2\phi=\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}-\frac{2\partial^2}{\partial\sigma\,\partial\tau}\right)\phi(x_1x_2\sigma\tau)=0$$

with

$$\sigma=ct-x_3,\qquad \tau=\tfrac{1}{2}(ct+x_3)$$

If we consider quasihenochromatic[8] light with wave number $\approx\mu$, the wave equation can be rewritten

$$i\frac{\partial}{\partial\tau}\phi=-\frac{1}{2\mu}(\nabla_T)^2\phi$$

This is now quite similar to a two-dimensional Schrödinger equation. First-order axial optical systems acting on such wave amplitudes realize the Sp(4, R) group which can in turn be realized by the fundamental representation in terms of the paraxial light rays. Elsewhere[11,27] we have elaborated on these questions.

The question now arises as to the paraxial nature of a statistical ensemble and, therefore, as to how the N-ray distributions are paraxial. For classical wave optics this is straightforward: if all the components of the ensemble are paraxial with approximately the same axis, the resulting wave field ensemble is also paraxial. The N-fold ray distributions will all deal with paraxial rays. Recall that no ray can be present which is not already present at the (1, 1) level!

For the quantum optics case also this continues to be true. *If the* (1, 1) *function describes a paraxial bundle of rays, all the N-fold ray distributions must be exclusively paraxial.*

It is unpardonable to conclude this essay without computing the N-fold ray distributions for a cavity at temperature T. But Erwin Schrödinger generally concentrated on the essential principles from which other conclusions, which were but natural consequences although important ones, generally were omitted. We use this as our excuse.

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