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Forms of Relativistic Dynamics with World Line Condition and Separability

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The Dirac generator formalism for relativistic Hamiltonian dynamics is reviewed along with its extension to constraint formalism. In these theories evolution is with respect to a dynamically defined parameter, and thus time evolution involves an eleventh generator. These formulations evade the No-Interaction Theorem. But the incorporation of separability reopens the question, and together with the World Line Condition leads to a second no-interaction theorem for systems of three or more particles. Proofs are omitted, but the results of recent research in this area is highlighted.

1. INTRODUCTION

Newtonian physics describes instantaneous states of a dynamical system as the set of positions and velocities of all the particles. The equations of motions furnish the accelerations as prescribed functions of the positions and velocities. In general Lagrangian mechanics a suitable set of generalized coordinates and velocities replace the Cartesian positions and velocities, but the equations of motion still prescribe the generalized accelerations as functions of generalized coordinates and velocities. If it turns out that there are one or more relations among the velocities (which depend on the coordinates possibly) which cannot be integrated to find constraints between the coordinates, this method of choosing independent generalized coordinates cannot be applied to them. It does not help to attempt to proceed to canonical (Hamiltonian) mechanics since there are not enough independent functions of velocities and coordinates that can supply expressions for

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canonical momenta. It is to deal with such constrained dynamical systems that the Dirac theory[1] of constraints was developed. This marks the major advance in canonical mechanics[2] since the classical work of Lagrange and Hamilton.

It turns out that the Dirac theory of constraints is an essential ingredient in the formulation of relativistic particle interactions. The use of constraints in this context[3] is somewhat more general than the framework of canonical mechanics in which it was originally introduced.

2. IDENTITY IN DIFFERENCE: THE PHYSICS OF BECOMING

The dynamics of interacting particles must satisfy new criteria when the requirements of the special theory of relativity are imposed. The traditional specification of the instantaneous state of a canonical system of N interacting particles is to give N triplets of (Cartesian) coordinates q and their conjugate momenta p. The time evolution would be described by a Hamiltonian function of the $3N$ canonical pairs $\underset{\sim}{q}_r, \underset{\sim}{p}_r$. The *irrelevance* of time origin would be reflected in the *independence* of the Hamiltonian $H(\underset{\sim}{q}, \underset{\sim}{p})$ on time; and the irrelevance of orientation of the coordinate frame in 3-space would be reflected in the *immutability* of the angular momentum $\Sigma \underset{\sim}{q}_r \times \underset{\sim}{p}_r$ which in turn implies the rotation *invariance* of the Hamiltonian. The irrelevance of the space origin implies the invariance of the Hamiltonian under the common translation of all the (Cartesian coordinates and the associated invariance of the total momentum $\Sigma \underset{\sim}{p}_r$. Thus we have a seven-parameter set of canonical transformations implementing time translation, space translation and space rotations; the (infinitesimal) generators of these canonical transformations are H, $\underset{\sim}{P}$, and $\underset{\sim}{J}$, respectively. They satisfy the canonical Poisson bracket relations:

$$[H, [\underset{\sim}{P} \cdot \underset{\sim}{a}, F]] - [\underset{\sim}{P} \cdot \underset{\sim}{a}, [H, F]] = 0$$

$$[H, [\underset{\sim}{J} \cdot \underset{\sim}{\theta}, F]] - [\underset{\sim}{J} \cdot \underset{\sim}{\theta}, [H, F]] = 0$$

$$[\underset{\sim}{P} \cdot \underset{\sim}{a}, [\underset{\sim}{P} \cdot \underset{\sim}{b}, F]] - [\underset{\sim}{P} \cdot \underset{\sim}{b}, [\underset{\sim}{P} \cdot \underset{\sim}{a}, F]] = [\underset{\sim}{P} \cdot \underset{\sim}{\theta} + \underset{\sim}{a}, F]$$

$$[\underset{\sim}{J} \cdot \underset{\sim}{\theta}, [\underset{\sim}{J} \cdot \underset{\sim}{\psi}, F]] - [\underset{\sim}{J} \cdot \underset{\sim}{\psi}, [\underset{\sim}{J} \cdot \underset{\sim}{\theta}, F]] = [\underset{\sim}{J} \cdot \underset{\sim}{\psi} \cdot \underset{\sim}{\theta}, F]$$

where $F = F(q, p)$ is any canonical variable. Making use of the freedom in redefining the quantities H, $\underset{\sim}{P}$, $\underset{\sim}{J}$ to within constant additive quantities we see that these relations could be simplified into the canonical realization[2] of a Lie algebra:

$$[H, \underline{P}] = 0, \qquad [H, \underline{J}] = 0$$

$$[\underline{P} \cdot \underline{a}, \underline{P} \cdot \underline{b}] = 0$$

$$[\underline{P} \cdot \underline{a}, \underline{J} \cdot \underline{\theta}] = \underline{\theta} \times \underline{a} \cdot \underline{P}$$

$$[\underline{J} \cdot \underline{\theta}, \underline{J} \cdot \underline{\psi}] = \underline{\psi} \times \underline{\theta} \cdot \underline{J}$$

In all this the canonical Poisson bracket relations all dynamical variables that are defined at a single time—the fundamental Poisson brackets

$$[\underline{q}_j \cdot \underline{a}, \underline{p}_k \cdot \underline{b}] = \delta_{jk} \underline{a} \cdot \underline{b}$$

$$[\underline{q}_j \cdot \underline{a}, \underline{q}_j \cdot \underline{b}] = 0$$

$$[\underline{p}_j \cdot \underline{a}, \underline{p}_j \cdot \underline{b}] = 0$$

are time independent as well as all the above Poisson brackets. The fundamental time evolution equation is given by

$$\dot{F}(q, p) = [F, H]$$

for any dynamical variables. It is then immediately clear that H, $\underline{P}$ and $\underline{J}$ have no change in time expressing the conservation law of energy, momentum, and angular momentum.

3. RELATIVISTIC DYNAMICS

All these expressions look very nonrelativistic. They employ a clock time t and dynamical variables at the *same time*. But special relativity tells us that distant simultaneity is not independent of the frame of reference.

Dirac[4] showed that nevertheless the invariance of the physical laws under relativistic frame changes can be implemented if we adjoing a set of a 3-parameter boost transformation to moving inertial frames. The corresponding infinitesimal generators should satisfy:

$$[\underline{K} \cdot \underline{u}, H] = -\underline{P} \cdot \underline{u}$$

$$[\underline{K} \cdot \underline{u}, \underline{P} \cdot \underline{a}] = -\underline{u} \cdot \underline{a} H$$

$$[\underline{K} \cdot \underline{u}, \underline{J} \cdot \underline{\theta}] = \underline{\theta} \times \underline{u} \cdot \underline{K}$$

$$[\underline{K} \cdot \underline{u}, \underline{K} \cdot \underline{v}] = \underline{u} \times \underline{v} \cdot \underline{J}$$

This gives the ("instant") form of relativistic dynamics for any canonical system. One immediate realization is for a collection of free particles: we need only identify the quantity $\underline{K}$ with the relativistic moment of energy.

These equations then imply that this moment changes in time and the rate of change is the momentum of the system, and that the moment depends on the spatial origin and spatial orientation in a geometrical fashion. The last equation is more characteristic of special relativity in that it points out that the change in the value of the moment of energy in going to a moving frame is proportional to the angular momentum.

These equations also alert us to the fact that the nature of the Hamiltonian of a system of *interacting* particles automatically demands that either the relativistic *boost* or the three-momentum *should be different* from that for free particles; for the instant form of relativistic dynamics it is the boost which must be different.

4. THE NO-INTERACTION THEOREM

Such interacting systems can be constructed by choosing

$$H = H^0 + V$$

where H^0 is appropriate for a collection of free particles and V is a rotationally and translationally invariant momentum-dependent "potential."[5]

The potential can be chosen to implement a natural separability requirement.[6] If we consider a system with $N = N_1 + N_2$ particles, the N_1 particles are far separated from N_2 particles, and each group must have an autonomous and fully relativistic description. Within the instant form of relativistic dynamics this is quite simple to implement: the potential V must have such a separable property. In more sophisticated forms of relativistic dynamics separability is a serious restriction.

While the instant form of relativistic dynamics circumvents the problem of distant simultaneity, the latter presses its claim in another format by requiring the existence of invariant world lines. This is to say that the three canonical coordinates of *each* particle together with its time label describe trajectories in space-time which must be seen as invariant world lines. This requirement[7] is not automatically met since we saw that the boost generators $\underline{K}$ did not have the free particle form. In the instant form of relativistic dynamics the World Line Condition (WLC) takes the simple but nonlinear form[8]:

$$[\underline{q}_j, \underline{K} \cdot \underline{u}] = [\underline{q}_j, H]\underline{q}_j \cdot u$$

If we try to find out the interactions which are compatible with the WLC we find that there are none that we can use: *all accelerations must* *vanish* if we have *canonical realization* of the *relativistic transformation group satisfying WLC*. This is the No-Interaction Theorem.[8]

One could take three points of view at this juncture. One can accept the result and proceed to a field formalism as the sole vehicle for particle interactions. Or one could abandon differential equations of motion and replace them by integrodifferential equations.[9] Perhaps these two methods are not essentially different but only different realizations of the same picture.[2]

Yet a third way is to seek a more geometrical and explicitly invariant formalism from the start. This is where the constraint formalism of Dirac enters in an essential way. In these descriptions one uses an explicitly invariant set of configuration variables to describe the system and guarantee that the system does describe particles by imposing suitable constraints on the system. A number of such descriptions have been developed during recent years.[10]

5. CONSTRAINT DYNAMICS OF INTERACTING PARTICLES

In the constraint formalism one starts from a description of the N particle system in terms of a redundant set of variables and imposes sufficient number of constraints to assure the $3N$ degrees of freedom. One choice is to make use of a moving center with coordinates Q^μ and momenta P_μ ($\mu = 1, 2, 3, 0$) and consider 3-momenta and 3-coordinates for each particle orthogonal to P_μ. Eight constraints then relate P_μ, Q^μ to the other variables. Equally well we can choose a second world view in which each particle is equipped with four pairs of canonical variables q_j^μ, $p_{j\mu}$ and $2N$ constraints are used to obtain an N-particle system. A third world view[11] is obtained by introducing $3N$ pairs of canonical variables defined in an arbitrary Lorentz Lorentz frame which is itself defined dynamically in terms of six pairs of dynamical variables; twelve constraints finally get imposed. Yet, other ways of generating invariant dynamics may be formulated.[11]

In all these models to describe a true physical system of N particles with $6N$ degrees of freedom it is essential to introduce the constraints. At least one of the constraints should involve a time evolution parameter τ explicitly so that as τ varies different kinematical states are realized. In other words, *motion is generated by the constraints*. The constraints must be of such a form that the degrees of freedom are reduced. So they must constitute a "second class" set in the terminology of Dirac.[1] For purposes of illustration we restrict ourselves to the DVKT approach, that is, the second world view.

The δN-dimensional phase space Γ is spanned by $2N$ four-factors q_j^{μ}, $p_{j\mu}$ under the Poincaré group fulfilling the Poisson bracket relations

$$[q_j^{\mu}, p_{k\nu}] = \delta_{jk}\,\delta^{\mu}{}_{\nu}$$
$$[q_j^{\mu}, q_k^{\nu}] = 0, \qquad [p_{j\mu}, p_{k\nu}] = 0$$

In this space we impose N Poincaré invariant constraints

$$K_j = p_j^2 - m_j^2 - v_j(q, p) = 0$$

which have vanishing Poisson brackets amongst themselves

$$[K_j, K_k] = 0$$

We supplement them with the N constraints

$$\chi_j = q_j^0 - \tau = 0$$

Together the K_j, χ_j form a set of $2N$ second class constraints whose matrix of Poisson brackets is nonsingular. Since it is possible to satisfy the first class conditions imposed on K_j amongst themselves with the "potentials" $V_j(q, p)$ separable[12] and since the χ_j are already separable, the dynamical description is separable.

The WLC is somewhat more involved. For any general choice of χ_j, not necessarily the ones made above the dynamical evolution of the system is determined as follows: First the choice of the N χ-constraints at $\tau = 0$ combined with the N K-constants gives a complete set of restrictions on Γ restricting it to a seix-dimensional variety. On the other hand by requiring that these constraints be maintained for all τ they determine a definite dynamical law, that is they define a generator of τ-evolution

$$\mathscr{H} = \sum_j v_j K_j$$

which guarantees

$$\frac{dK_j}{d\tau} = 0$$

$$\frac{d\chi_j}{d\tau} = \frac{\partial \chi_j}{\partial \tau} + [\chi_j, \mathscr{H}] = 0$$

by the choice

$$v_j = -\sum_k C_{jk} \cdot \frac{\partial \chi_k}{\partial \tau}$$

with

$$C_{jk}\{\chi_k, K_l\} = \delta_{jl}$$

The WLC itself however demands that under the infinitesimal Poincaré transformation

$$x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu + a^\mu$$

the coordinates q_j^μ must transform according to[13]

$$\delta q_j^\mu = \omega^\mu{}_\nu \cdot q_j^\nu + a^\mu + \{q_j^\mu, \mathscr{H}\}^* \, \delta\tau_j$$

where the $*$ denotes Dirac's modified brackets. Since the WLC involves all the particles in a nontrivial manner the separability question has to be carefully reexamined.

A scrutiny of the constraint formalism shows that this model involves an enlargement of the original Dirac framework: it is necessary to consider *dynamical* choices of temporal evolution variables rather than the kinematical choices offered by Dirac in his various forms of relativistic dynamics. This is true not only in the particular form of constraints that we have used but in all world views. We must use not ten generators H, $\underline{P}$, $\underline{J}$, $\underline{K}$ but an eleventh generator $\mathscr{H}$ describing τ evolution.[14]

In nonrelativistic particle mechanics if the interaction potential falls off "sufficiently fast" when subsets of particles are far removed from each other each subset follows its own dynamics unaffected by the presence of other groups. Such is the intuitive appeal to separability.

In the constraint formalism we have several potentials and a WLC. We have already mentioned that separable potentials leading to suitable K-constraints exist. But since the temporal evolution with respect to τ is determined dynamically and autonomously we should have for a system that breaks up into clusters each cluster should have a parameter *evolution independent of the other cluster.* As things stand this *cannot* be fulfilled.[15] What we must seek is the possibility of a transformation of the description based on the constraints $K_j = 0$, $\chi_j = 0$ to one based on $K_j = 0$, $\chi_j' = 0$ where χ_j' are the second set of constraints adapted to the relevant cluster. But this *change of description* should be achieved by a canonical transformation *without altering the world lines* of the particles.

For free particles we can explicitly demonstrate the existence of such "tilting" transformations. Emboldened by it we now seek the kind of interaction potentials which guarantee WLC and separability.

An elementary but tedious analysis shows that interactions satisfying this requirement exists for two-particle systems. But if there are nontrivial two-particle interactions between any two particles of a subsystem there is no three-particle system which satisfies separability![15]

This curious result is reminiscent of the EPR "paradox" in quantum theory,[16] but the above indicated circle of ideas suggests that correlations between distant objects need not always involve transport of material influences. It may rather depend upon the indecomposable nature of the dynamical system itself. In the present context this is brought about by the imposition of the apparently innocent WLC.

As has often been shown by Dirac there are surprising structural similarities between classical mechanics and quantum mechanics; and often ideas that were identified in quantum mechanics reappear from a deeper study of classical mechanics.

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