Geometric Phases for $SU(3)$ Representations and Three Level Quantum Systems

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A comprehensive analysis of the pattern of geometric phases arising in unitary representations of the group $SU(3)$ is presented. The structure of the group manifold, convenient local coordinate systems and their overlaps, and complete expressions for the Maurer–Cartan forms are described. Combined with a listing of all inequivalent continuous subgroups of $SU(3)$ and the general properties of dynamical phases associated with Lie group unitary representations, one finds that nontrivial dynamical phases arise only in three essentially different situations. The case of three level quantum systems, which is one of them, is examined in further detail and a generalization of the $SU(3)$ solid angle formula is developed.

1. INTRODUCTION

The discovery of the geometric phase in quantum mechanics has led to an enormous amount of work clarifying its nature and properties as well as exploring

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various applications. By now a good basic understanding of its structure has been built up from many points of view [1]. In addition many of the conditions assumed in the original discovery have been relaxed. Thus from a situation wherein the geometric phase was defined in adiabatic, cyclic, unitary evolution described by the Schrodinger equation, it is now known that the phase can be defined for non-adiabatic, non-cyclic, non-unitary evolution. It has even been shown that the geometric phase can be reduced to a kinematic level without reference to the Schrodinger equation [2].

The most frequently considered illustrative examples of geometric phase relate to coherent states in quantum mechanics and to the quantum mechanics of two level systems, basically governed by the group *SU(2).* In the latter case, as is well known, the ray space for a two level system is the Poincare sphere S^2; and the geometric phase for cyclic evolution in ray space turns out to be one-half of the solid angle. In this context, one can also calculate the geometric phase associated with a general irreducible representation of *SU(2),* and the result turns out to be a certain multiple of the above mentioned solid angle [3].

It has been shown elsewhere that the properties of geometric phases associated with unitary representations of Lie groups can be systematically studied as a special case of the general theory of the geometric phase [4]. On account of the many specific features associated with Lie groups and their representations — Lie algebra generators, invariant vector fields, one forms, and coset spaces — the geometric phase in this case can be reduced to a maximally simplified form in which the algebraic and representation aspects are clearly separated from the differential geometric ones. It is useful to very briefly outline the structures involved at this stage.

Given the complex Hilbert space $\mathscr{H}$ appropriate for some quantum mechanical system, and given any continuous (possibly open) curve C in the ray space of the system, a geometric phase $\varphi_g[C]$ associated with C is immediately defined. It is the difference of a total phase and a dynamical phase, each of which is a functional of a continuous curve $\mathscr{C}$ in Hilbert space, which is a lift of the ray space curve C:

$$\varphi_g[C] = \varphi_p[\mathscr{C}] - \varphi_{\text{dyn}}[\mathscr{C}] \tag{1.1}$$

The relationship between $\mathscr{C}$ and C is infinitely many to one; any $\mathscr{C}$ projecting onto C may be used to calculate the individual terms on the right handside above, but the difference is independent of this choice. Clearly, once a choice of $\mathscr{C}$ is made, the calculation of the term $\varphi_p[\mathscr{C}]$ is trivial. Therefore the calculation of the geometric phase $\varphi_g[C]$ reduces to that of $\varphi_{\text{dyn}}[\mathscr{C}]$. For this reason, while our interest is in the geometric phase, we will often be concerned with computation of the dynamical phase.

In the application to unitary Lie group representations we are indeed mainly concerned with the properties of the dynamical phase, $\varphi_{\text{dyn}}[\mathscr{C}]$. We shall therefore in the sequel not refer much to the ray space and curves therein. Given a connected Lie group G, a faithful, unitary representation $U(.)$ of G on $\mathscr{H}$, and some chosen unit vector ψ_0 in $\mathscr{H}$, we are interested in curves $\mathscr{C}$ starting at ψ_0 and produced continuously by unitary group action. It turns out that such curves may be regarded as lying in the orbit $\mathscr{O}(\psi_0)$ produced by the action of *U(g)* for all *g e G* on ψ_0; and equally well as lying in the coset space $G/H_0(\psi_0)$, where $H_0(\psi_0)$ is the stability group of the vector ψ_0. This is the way in which we are able to exploit the rich geometric structures available with coset spaces of Lie groups. As stated above, and with the help of the Wigner–Eckart theorem of quantum mechanics, one is able to effect a clean separation between the dependences on $\mathscr{C}$ on the one hand and on the chosen vector ψ_0 and generators of the representation *U(.)* on the other.

The purpose of this paper is to provide a comprehensive analysis of all these aspects in the case of the group *SU(3).* We wish to build up the basic machinery which would enable calculation of the geometric phase associated with any unitary representation of *SU(3),* and in particular for three level quantum systems corresponding to the defining representation of *SU(3).* In this process we shall pay attention to the following important aspects: global ways of describing and handling elements of, and bringing out the manifold structure of, $SU(3)$; a catalogue of all possible Lie subgroups of *SU(3)* upto conjugation; the descent from *SU(3)* to its various coset spaces; and the calculation of the basic Maurer–Cartan one-forms over *SU(3)* along with their behaviour under pullback to coset spaces.

The contents of the paper are arranged as follows. Section 2 recounts the structure of the defining representation of *SU(3)* and its Lie algebra $\underline{SU(3)}$, bringing in the λ-matrices. We then provide a systematic catalogue of all possible continuous Lie subgroups H_0 in *SU(3)* upto conjugation [5]. We find there are infinitely many inequivalent one-dimensional cyclic Abelian subgroups having the structure of $U(1)$, and denoted by $U_{(p,q)}(1)$ where *p, q* are two relatively prime integers. There are also one-dimensional Abelian non cyclic subgroups having the structure of the real line $\mathscr{R}$, but these turn out to be irrelevant for geometric phase computations. Next there is one two-dimensional Abelian subgroup $U(1) \times U(1)$; and one each of the forms *SU(2), U(2)* and *SO(3).* The principal features of unitary irreducible representations (UIR's) of *SU(3)* are recapitulated. With this information we are able to analyse in general terms the kinds of stability subgroups H_0, and stability subgroups upto phases *H,* that can arise with general vectors ψ_0 in Hilbert spaces $\mathscr{H}$ carrying unitary representations (UR's) of *SU(3).* It is noteworthy that with a modest amount of effort we are able to obtain complete information on these aspects of *SU(3)* representations.

Section 3 turns to a study of the detailed topological and manifold structure of *SU(3).* For this we find it useful to begin with the five dimensional coset space manifold $\mathscr{M}$ = *SU(3)/SU(2)* (and the four-dimensional manifold *ffl* = $\mathscr{M}/U(1) = SU(3)/U(2)$) and then work our way upto *SU(3)* by carefully chosen coset representatives. It proves convenient to regard each of *SU(3),* $\mathscr{M}$ and $\mathscr{R}$ as the unions of three open overlapping subsets, over each of which a singularity-free coordinate description can be given. The transition rules in the overlaps are also determined. With these ingredients we define and compute (the essential parts of) the left-invariant Maurer-Cartan one-forms on *SU(3),* to the extent that one goes beyond the known expressions for $SU(2)$. For this purpose we introduce a set of five angle variables as coordinates on almost all of $\mathscr{M}$.

The main aim of Section 4 is to work out expressions for $SU(3)$ dynamical phases in various situations depending on the natures of the subgroups H_0, H determined by ψ_0. It is quite remarkable that we can prove that nontrivial dynamical phases arise in only three distinct cases: the generic case of arbitrary ψ_0 in an arbitrary UR with $H_0 = \{e\}$; the case $H_0 = U_{(p,q)}(1)$; and the case $H_0 = SU(2)$. In all other cases we can show that the dynamical phase vanishes. Thus the variety of situations that arise is much simpler and more tractable than may have been anticipated.

Section 5 describes a generalisation of the Poincare sphere representation for pure state density matrices for two-level quantum systems, to three-level systems. Here the d-symbols of $SU(3)$ play an important role. It turns out that the Poincare sphere gets replaced by a certain four-dimensional region embedded within the unit sphere S^7 in real eight-dimensional Euclidean space. The calculation of geometric phases for noncyclic or cyclic evolution of a three-level system is carried to the stage where the generalisation of the Poincare sphere solid angle formula can be clearly displayed. The relationship to a specific coadjoint orbit in the Lie algebra of $SU(3)$ and to the symplectic structure on this orbit, is explained. Section 6 contains some concluding remarks.

2. DEFINING AND GENERAL REPRESENTATIONS, LIE ALGEBRA, CONTINUOUS SUBGROUPS OF $SU(3)$

The defining representation of the group $SU(3)$ consists of all unitary unimodular matrices in three complex dimensions [6]:

$$SU(3) = \{A = 3\times 3 \text{ matrix} \mid A^{\dagger}A = I,\ \det A = 1\}. \tag{2.1}$$

The generators in this representation are hermitian, traceless, three dimensional matrices. The number of such independent generators, hence the dimension of $SU(3)$, is eight. We may choose them as the familiar λ matrices, generalising the Pauli matrices for $SU(2)$ [7]:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.2}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The commutation relations among these matrices involve the structure constants f_{rst} of *SU(3)* which are totally antisymmetric in *r, s, t* and whose independent non-zero components are given below [6, 7]:

$$
\begin{aligned}
&[\lambda_r, \quad \lambda_s] = 2if_{rst}\lambda_t\,, \\
&f_{123} = 1, \; f_{458} = f_{678} = \sqrt{3}/2, \; f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = 1/2
\end{aligned}
\tag{2.3}
$$

This choice of a basis for the Lie algebra *SU(3)* is appropriate to its use in particle physics. We shall generally denote specific generators in the defining representation with a $^{(0)}$, and omit the superscript in a general representation. The third component of isotopic spin and the hypercharge are:

$$
\begin{aligned}
I_3^{(0)} &= \lambda_3/2, \\
Y^{(0)} &= \frac{1}{\sqrt{3}}\,\lambda_8\,.
\end{aligned}
\tag{2.4}
$$

It will also be convenient to deal with two other linear combinations of the diagonal generators because they have integer eigenvalues in any representation:

$$
\begin{aligned}
H_1^{(0)} &= I_3^{(0)} + \tfrac{3}{2}Y^{(0)} = \mathrm{diag}(1,\, 0,\, -1), \\
H_2^{(0)} &= -I_3^{(0)} + \tfrac{3}{2}Y^{(0)} = \mathrm{diag}(0,\, 1,\, -1).
\end{aligned}
\tag{2.5}
$$

In a general representation the generators corresponding to $\frac{1}{2}\lambda_r$ will be denoted by F_r, so they obey

$$
[F_r,\, F_s] = if_{rst}F_t\,. \tag{2.6}
$$

For the *U(2)* subgroup generators we also use the notation $F_1 = I_1$, $F_2 = I_2$, $F_3 = I_3$, $F_8 = (\sqrt{3}/2)\,Y$; while the combinations corresponding to $H_1^{(0)}$ and $H_2^{(0)}$ are denoted by H_1 and H_2.

Subgroups of SU(3) up to Conjugation

We now discuss the possible non-trivial Lie subgroups in *SU(3)* upto conjugation by working with the Lie algebra in the defining representation. We begin with possible one-dimensional subgroups, which are necessarily abelian. Since the hermitian generator of such a subgroup can always be diagonalised by an *SU(3)* matrix, we may assume the subgroup to be made up of diagonal matrices. Let us write the generator as:

$$
H = \mathrm{diag}(p,\, q,\, r), \tag{2.7}
$$

where *p, q, r* are real numbers. The element of the abelian subgroup with parameter θ is therefore:

$$
A(\theta) = \exp(i\theta H) = \mathrm{diag}(e^{ip\theta}, e^{iq\theta}, e^{ir\theta}). \tag{2.8}
$$

We first consider the case when the subgroup is cyclic, and we assume without loss of generality that $9 = 2\pi$ is the smallest parameter value at which we return to the identity. This combined with the unimodularity property implies:

$$\begin{aligned} &(p,\ q,\ r) = \text{relatively prime integers}, \\ &p + q + r = 0. \end{aligned} \tag{2.9}$$

We shall denote such a $U(1)$ subgroup within $SU(3)$ by $U_{(p,\,q)}(1)$, with the understanding that p and q are relatively prime integers in the case that they are both non-vanishing. (In case one of them vanishes, we have the subgroup $U_{(1\ 0)}(1)$). Within the defining representation the generator of this subgroup is the combination:

$$\text{Generator}(\,U_{(p,q)}(1)) = pH_1^{(0)} + qH_2^{(0)}. \tag{2.10}$$

Since we wish to regard conjugate subgroups as equivalent, we realise that all pairs $(p,\ q)$, (q, p), $(p, -q-p)$, $(-q-p, p)$, $(q, -q-p)$, $(-q-p, q)$ denote equivalent $U(1)$ type subgroups within $SU(3)$. This just corresponds to six different ways in which the diagonal entries in Eqs. (2.8) and (2.10) could be ordered. It is important to realise that two pairs $(p,\ q)$ and (p', q') not related in the above manner denote inequivalent subgroups of $SU(3)$. As examples we identify a few of these subgroups by their generators.

$$\begin{aligned} &H_1^{(0)} - U_{(1,\,0)}(1), \\ &H_2^{(0)} - U_{(0,\,1)}(1), \\ &2I_3^{(0)} - U_{(1,\,-1)}(1), \\ &3\,Y^{(0)} - U_{(1,\,1)}(1). \end{aligned} \tag{2.11}$$

Of these, the first three are equivalent.

Another type of one dimensional abelian subgroup within $SU(3)$ arises if there is no value of the parameter 0 other than zero, for which $A(0)$ in Eq. (2.8) becomes the unit matrix. Such subgroups of $SU(3)$ are isomorphic to the real line R, and have as generators linear combinations of $H_1^{(0)}$ and $H_2^{(0)}$ (or $I_3^{(0)}$ and $Y^{(0)}$) with relatively irrational coefficients, and are not closed in the topological sense. Upon closure they lead to the two dimensional torus subgroup of $SU(3)$. For reasons which will be clear shortly, such subgroups of $SU(3)$ cannot arise as stability groups of vectors ψ_0 in unitary representations of $SU(3)$ and will therefore not be further considered.

We next turn to possible two dimensional abelian subgroups in $SU(3)$. Any such subgroup is generated by two commuting generators which can therefore be simultaneously diagonalised by a single $SU(3)$ transformation. The tracelessness condition means that there are only two independent traceless diagonal generators, which we may take to be $H_1^{(0)}$ and $H_2^{(0)}$ of Eq. (2.5). We conclude that any two-dimensional abelian subgroup in *SU(3)* is, up to reparametrisation, conjugate to the torus or $U(1) \times U(1)$ subgroup defined by the elements

$$A(\theta_1, \theta_2) = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{-i\theta_1 - i\theta_2}). \tag{2.12}$$

It is clearly not possible to accomodate three dimensional abelian subgroups within *SU(3)*.

Now we move on to non-abelian subgroups and their possible local products with $U(1)$ factors. The simplest non-abelian possibility is *SU(2)*. Within the defining representation of *SU(3)* we see that there are only two ways in which *SU(2)* could be accomodated: (a) as a direct sum of its defining two dimensional representation and the trivial one dimensional representation, and (b) via the three dimensional adjoint representation which is the defining representation of $SO(3)$. Up to conjugation we may identify the former case with the subgroup of *SU(3)* which does not act upon the third dimension:

$$SU(2) = \left\{ A(a) = \begin{pmatrix} a & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a \in SU(2) \right\} \subset SU(3). \tag{2.13}$$

The generators of this subgroup are $\lambda_1/2$, $\lambda_2/2$, and $\lambda_3/2$, with standard normalisation. Turning to the second possibility we have:

$$50(3) = \{A = \text{real} \mid A^t A = I,\ \det A = 1\} \subset SU(3). \tag{2.14}$$

The generators for this subgroup are purely imaginary, antisymmetric matrices; we can take them to be λ_2, λ_5, λ_7 with standard normalisation.

We may ask whether either of the two possibilities above can be extended by adjoining a commuting *U(l)* factor. This is indeed possible in the first case and it leads to the *U(2)* subgroup:

$$U(2) = \left\{ A(u) = \begin{pmatrix} u & & 0 \\ & & 0 \\ 0 & 0 & (\det u)^{-1} \end{pmatrix} \;\middle|\; u \in U(2) \right\} \subset SU(3). \tag{2.15}$$

This is generated by $\lambda_1/2$, $\lambda_2/2$, $\lambda_3/2$, $\lambda_8/2$ and is sometimes incorrectly referred to as $SU(2) \times U(1)$. The $SO(3)$ subgroup of $SU(3)$ cannot however be extended in this way since it is already irreducibly represented in the defining representation of *SU(3)*.

One may easily convince oneself by comparing dimensionalities that there are no other inequivalent Lie subgroups in *SU(3)*. It is not possible to embed $SO(n)$ for $n \geqslant 4$, *SU(n)* for $n \geqslant 3$, *USp(2n)* for $n \geqslant 2$ or any of the compact exceptional Lie groups into *SU(3)*. We list our results in Table I giving the dimensions of the subgroups and their generators.

TABLE I

Subgroups of $SU(3)$ up to Conjugation

Subgroup	Dimension	Generators	Remarks
(a) $U_{(p,q)}(1)$	one	$pH_1^{(0)} + qH_2^{(0)}$	p, q relatively prime integers
(b) R	one	$pH_1^{(0)} + qH_2^{(0)}$	p/q irrational
(c) £7(1) x £7(1)	two	$H_1^{(0)}$, $H_2^{(0)}$	Torus Subgroup
(d) $SU(2)$	three	$\frac{1}{2}\lambda_1$, $\frac{1}{2}\lambda_2$, $\frac{1}{2}\lambda_3$	Isospin Subgroup
(e) £7(2)	four	$\frac{1}{2}\lambda_1$, $\frac{1}{2}\lambda_2$, $\frac{1}{2}\lambda_3$, $\frac{1}{2}\lambda_8$	Isospin, Hypercharge Subgroup
(f) 50(3)	three	λ_2, λ_5, λ_7	Spatial Rotations

General Representations of $SU(3)$

Since $SU(3)$ is compact, every representation may be assumed to be unitary, and a direct sum of unitary, irreducible representations (UIR's). We briefly recall some important features of the latter [8]. A general UIR of $SU(3)$ is denoted by (m, n) and is of dimension $(m+1)(n+1)(m+n+2)/2$, where m and n are non-negative integers. The defining representation is $(1, 0)$, while its complex conjugate is $(0, 1)$ [in general, the complex conjugate of (m, n) is (n, m)]. Within a UIR (m, n), an orthonormal basis can be set up as the simultaneous eigenvectors of the generators I_3 and Y and the square of the isospin, i.e. the quadratic Casimir operator of $SU(2)$:

$$
\begin{aligned}
I_3 \,|(m,n); I' I_3' Y'\rangle &= I_3' \,\backslash(m, n); I' I_3' Y'\rangle, \\
Y \,|(m,n); I' I_3' Y'\rangle &= Y' \,|(m,n); I' I_3' Y'\rangle, \\
I^2 \,|(m,n); I' I_3' Y'\rangle &= I'(I'+1) \,|(m,n); I' I_3' Y'\rangle, \\
I^2 &= /\mathrm{f} + I_2^2 + I_3^2, \\
I_3' &= I', I'-1, \ldots, -I'.
\end{aligned}
\tag{2.16}
$$

The spectrum of I-Y multiplets is given as follows:

$$
\begin{aligned}
I' &= \tfrac{1}{2}(r+s), \\
Y' &= r - s + \tfrac{2}{3}(n-m), \\
0 &\leqslant r \leqslant m, \; 0 \leqslant s \leqslant n.
\end{aligned}
\tag{2.17}
$$

Thus for each pair of integers (r, s) in the above ranges we have one I-Y multiplet with I_3' running over the usual set of values. The diagonal generators I_3 and Y may or may not have integer eigenvalues. The eigenvalues of Y differ from $\frac{2}{3}(n-m)$ by integers. However the combinations introduced earlier,

$$
\begin{aligned}
H_1 &= I_3 + \tfrac{3}{2} Y, \\
H_2 &= -I_3 + \tfrac{3}{2} Y,
\end{aligned}
\tag{2.18}
$$

always have integral eigenvalues; we may write their eigenvalues H'_1, H'_2 in such a way as to make this evident:

$$H'_1 = I'_3 - I' + 2r - s + n - m,$$
$$H'_2 = I' - I'_3 + r - 2s + n - m. \qquad (2.19)$$

(It is understood in the above that $I' = \frac{1}{2}(r + s)$). This integral nature of the eigenvalues of H_1 and H_2 obviously holds even in reducible representations of $SU(3)$ and we may always assume that they are diagonal.

The matrix elements of I_1, I_2, I_3, Y in the above basis are standard and as in quantum angular momentum theory. Those of F_4, F_5, F_6 and F_7 may be found in refs. (9).

The UIR's of $SU(3)$ can be classified according to the notion of triality, namely the value of $(m-n)$ modulo 3. All UIR's with triality zero are faithful representations of the factor group $SU(3)/Z_3$ where Z_3 is the three element centre of $SU(3)$. Only in these UIR's the hypercharge generator Y has integer eigenvalues. Examples are the adjoint representation (1, 1), the decuplet (3, 0), the 27-plet (2, 2), etc. These are not faithful representations of $SU(3)$. The situation is similar to integer spin representations of $SU(2)$ which are faithful representations of $SO(3)$. Non-zero triality representations of $SU(3)$, such as (1, 0), (0, 1), (2, 0), (0, 2) are faithful UIR's of $SU(3)$; in each of these the generator Y has non-integral eigenvalues.

Survey of Stability Subgroups

With the information gathered above about the structure of general unitary representations of $SU(3)$, we can survey the kinds of stability groups that can arise for different kinds of vectors ψ_0 in different representation spaces. To motivate this, it is useful to briefly recall the preliminary steps involved in any calculation of $SU(3)$ geometric phase. Given a unit vector ψ_0 in the Hilbert space $\mathscr{H}$ carrying a unitary representation $\bar{U}(A)$ of $SU(3)$ (irreducible or otherwise), as mentioned in the Introduction (see refs. (2, 4) for further details) it is necessary to first determine its stability group H_0:

$$H_0 = \{A \in SU(3) \quad \bar{U}(A)\psi_0 = \psi_0\} \subset SU(3). \qquad (2.20)$$

All the generators of H_0 annihilate ψ_0. Next it is also necessary to determine the stability group upto a phase, denoted by H:

$$H = \{A \in SU(3) \quad \bar{U}(A)\psi_0 = \text{(phase factor)}\,\psi_0\} \subset SU(3). \qquad (2.21)$$

That is, each generator of H either annihilates ψ_0 or has ψ_0 as an eigenvector with real nonzero eigenvalue. Both H and H_0 are subgroups of $SU(3)$, and furthermore the latter is an invariant subgroup of the former. General theory shows that three distinct situations or types can occur: $(A)\, H = H_0$, so H/H_0 is trivial; $(B)\, H/H_0$ is nontrivial but discrete; $(C)\, H/H_0 = U(1)$. At the Lie algebra level, Types (A) and

(B) coincide, and H_0 and H have the same generators. With type (C), H has an extra $U(1)$ generator, commuting with the generators of H_0, and for which ψ_0 is an eigenvector with a real non zero eigenvalue. Further steps involved in calculating dynamical phases are taken up in Section 4. We can now ask whether each of the Lie subgroups of *SU(3)* listed in Table 1 could appear as the stability group H_0 for some vector ψ_0 in some representation of *SU(3)*. (In the generic or most general case, of course, H_0 is the trivial subgroup of $SU(3)$). If the answer is in the affirmative we can next ask whether for that H_0, both possibilities $H \simeq H_0$ and $H \sim H_0 \times U(1)$ can be realised. For the latter case, as said above, clearly we need to have an *SU(3)* generator commuting with those of H_0, and having ψ_0 as eigenvector with some real nonzero eigenvalue. We can answer these questions systematically going down the list of choices *(a)*, ..., *(l)* for H_0 in Table I. We need to exploit the following general fact: given any pair of integers H'_1, H'_2, each positive or negative or zero, they can appear as simultaneous eigenvalues for H_1, H_2 in some suitable *UIR* of *SU(3)*.

Case (a): $H_0 = U_{(p,q)}(1)$. For a nonzero integer *m*, consider the eigenvalue pair $H'_1 = qm$, $H'_2 = -pm$. We can definitely find a corresponding simultaneous eigenvector ψ_0 of H_1 and H_2 in some *UIR* of *SU(3)*. Such ψ_0 is annihilated by $pH_1 + qH_2$, so Case (a) is definitely realisable (It is easy to ensure that H_0 is not larger than $U_{(p\,q)}(1)$). Further, the combination $qH_1 - pH_2$ has ψ_0 as its eigenvector with nonzero eigenvalue $(p^2 + q^2)m$, so we have $H \simeq H_0 \times U(1)$.

Next suppose we superpose two such vectors for two different nonzero m and m', taken from two different *UIR*'s if necessary; H_0 remains the same. But one can now check that no combination of H_1 and H_2 has ψ_0 as eigenvector with a nonzero eigenvalue, so we have $H \sim H_0$. Thus both $H \sim H_0$ and $H \sim H_0 \times U(1)$ can occur.

Case (b): $H_0 = R$. The generator of H_0 is a combination of H_1 and H_2 with relatively irrational coefficients. Since however H_1 and H_2 have only integer eigenvalues, we see immediately that this case cannot occur.

Case (c): $H_0 = U(1) \times U(1)$. The subgroup generators are H_1 and H_2. Now we can for example take ψ_0 to be an eigenvector *of* H_1 and H_2 within a *UIR of SU(3)*, having $H'_1 = H'_2 = 0$. (We can also easily ensure that H_0 is not larger than $U(1) \times U(1)$). Since this means $Y' = 0$ as well, the *UIR* must have triality zero. Now there are no *SU(3)* generators independent of, and commuting with, H_1 and H_2. Therefore we have $H \sim H_0$, never $H \sim H_0 \times U(1)$.

Case (d): $H_0 = SU(2)$. Every *UIR*(m,n) of *SU(3)* contains a unique *SU(2)* singlet state, carrying hypercharge $Y' = \frac{2}{3}(n - m)$. For $n \neq m$ we realise $H \simeq H_0 \times U(1)$, H_0 being no larger than *SU(2)*. By superposing such states from different *UIR*'s, and arranging the Y' values to be different, we realise $H \simeq H_0$.

Case (e): $H_0 = U(2)$. From what was said in the previous case, this can be realised only in the triality zero *UIR*'s(m, m). It is also clear that $H \simeq H_0$ always.

TABLE II

Existence of Vectors with Different Stability Groups

Case	Existence of ψ_0	$H \sim H_0$	$H \sim H_0 \times U(1)$
(a)	y	y	y
(b)	x	—	—
(c)	y	y	x
(d)	y	y	y
(e)	y	y	x
(f)	y	y	x

Case (f): $H_0 = SO(3)$. We need to look for $SU(3)$ UIR's containing $l = 0$ states, i.e. with vanishing angular momentum. Such a state is present for example in the UIR's (2, 0), (0, 2). However again as in Case (e) we have $H \simeq H_0$ always.

The above results are displayed in Table II, where the generic case H_0 = trivial is omitted.

3. STRUCTURE OF THE $SU(3)$ GROUP MANIFOLD AND THE MAURER-CARTAN ONE-FORMS

The purpose of this Section is to develop a description of the group $SU(3)$ using local coordinates, which discloses clearly the structure of the group manifold [10]. We shall do this in such a way as to preserve a kind of cyclic symmetry, and also so that the passages to the two coset spaces $SU(3)/SU(2)$ and $SU(3)/U(2)$ are simple. This will be useful when we discuss three-level quantum systems in Section 5.

Structures of SU(3) and Coset Space Manifolds

We shall work with the $SU(2)$ and $U(2)$ subgroups of $SU(3)$ identified in Eqs. (2.13, 15). We denote the corresponding coset spaces by $\mathscr{M}$ and $\mathscr{R}$:

$$\begin{aligned} SU(3)/SU(2) &= \mathscr{M}, \\ SU(3)/U(2) &= \mathscr{R}. \end{aligned} \tag{3.1}$$

These are manifolds of dimension five and four respectively. As is intuitively clear and will be soon seen explicitly, the relation between them is

$$\mathscr{R} = \mathscr{M}/U(1). \tag{3.2}$$

The three projection maps among $SU(3)$, $\mathscr{M}$ and $\mathscr{R}$ will be denoted thus:

$$\begin{aligned} &\pi_1 : SU(3) \to \mathscr{M}, \ \pi_2 : \mathscr{M} \to \mathscr{R}, \\ &\pi = \pi_2 \circ \pi_1 : SU(3) \to \mathscr{R}. \end{aligned} \tag{3.3}$$

If a general matrix A e $SU(3)$ is multiplied on the right by a matrix $A(a)$, $a \in SU(2)$, given by Eq. (2.13), it is clear that the third column of A is unchanged. One can easily convince oneself that this column uniquely and unambiguously corresponds to a single $SU(2)$ left coset in $SU(3)$. Let us write the elements of this column of A as α_j:

$$A_{j3} = \alpha_j, \ j = 1, 2, 3, \qquad \alpha^\dagger \alpha = 1. \tag{3.4}$$

Thus each point $m \in \mathscr{M}$ corresponds uniquely to one complex three-dimensional unit vector a:

$$m \in \mathscr{M} : m = m(\alpha) = \left\{ \begin{pmatrix} \cdot & \cdot & \alpha_1 \\ \cdot & \cdot & \alpha_2 \\ \cdot & \cdot & \alpha_3 \end{pmatrix} A(a) \ \middle| \ \alpha \text{ fixed}, a e SU(2) \right\} \subset SU(3). \tag{3.5}$$

From here it is clear that $\mathscr{M}$ is essentially the unit sphere S^5 in six-dimensional real Euclidean space $\mathscr{R}^6$:

$$\mathscr{M} = \{ \alpha \ \ \alpha^\dagger \alpha = 1 \} \simeq S^5 \subset \mathscr{R}^6. \tag{3.6}$$

If next a general matrix $A \in SU(3)$ is multiplied on the right by a matrix $A(u)$, ue $U(2)$, given by Eq. (2.15), we see that a gets altered by an overall phase:

$$A' = AA(u) \Rightarrow \alpha' = (\det u)^{-1} \alpha. \tag{3.7}$$

Therefore to pass to the quotient or coset space $\mathscr{R} = SU(3)/U(2)$, we have to identify any two unit complex three-dimensional vectors which only differ by a phase, so we have:

$$\begin{aligned} \mathscr{R} &= \{ \rho(\alpha) = \alpha\alpha^\dagger \ \ \alpha^\dagger\alpha = 1 \} \\ &= \{ \rho \mid \rho^\dagger = \rho \geqslant 0, \ \rho^2 = \rho, \ \mathrm{Tr}\, \rho = 1 \}. \end{aligned} \tag{3.8}$$

The relation (3.2) between $\mathscr{R}$ and $\mathscr{M}$ is also now transparent.

We now express the $SU(3)$ group manifold as the union of three open overlapping subsets $\mathscr{A}_j$, $j = 1, 2, 3$, and similarly for $\mathscr{M}$ and $\mathscr{R}$:

$$\begin{aligned} SU(3) &= U_{j=1}^3 \mathscr{A}_j; \\ \mathscr{M} &= U_{j=1}^3 \mathscr{M}_j, \quad \mathscr{M}_j = \pi_1(\mathscr{A}_j); \\ \mathscr{R} &= U_{j=1}^3 \mathscr{R}_j, \quad \mathscr{R}_j = \pi_2(\mathscr{M}_j) = \pi(\mathscr{A}_j). \end{aligned} \tag{3.9}$$

To define $\mathscr{A}_j$, we will temporarily use cyclic index notation—thus from here upto Eq. (3.13), jkl = 123, 231 or 312. We take $\mathscr{A}_j$ to be that subset of $SU(3)$ over which $\alpha_j = A_{j3}$ is nonzero:

$$\mathscr{A}_j = \{ A \in SU(3) \| \ |\alpha_j| > 0 \} \subset SU(3), \ j = 1, 2, 3. \tag{3.10}$$

The corresponding definitions of $\mathscr{M}_j$ and $\mathscr{R}_j$ follow immediately. Since $\alpha^\dagger\alpha = 1$, it follows that over $\mathscr{A}_j$, α_k and α_l are both strictly less than one in magnitude. Therefore if we define positive numerical factors N_j by

$$N_j = (1 - |\alpha_j|^2)^{-1/2}, \; j = 1, 2, 3, \tag{3.11}$$

then N_j is well-defined over $\mathscr{A}_k$ and $\mathscr{A}_l$.

Over each of the three open regions $\mathscr{M}_j$ in $\mathscr{M}$ one can define a smooth coset representative $l_j(\alpha)$ in a simple way. This is where the cyclic symmetry enters. The expressions are as follows:

$$\begin{aligned}
&\alpha \in \mathscr{M}_j : l_j(\alpha) \in \mathscr{A}_j, \; \pi_1(l_j(\alpha)) = \alpha; \\
&l_1(\alpha) = \begin{pmatrix} -N_2\alpha_1\alpha_2^* & -N_2\alpha_3^* & \alpha_1 \\ N_2^{-1} & 0 & \alpha_2 \\ -N_2\alpha_3\alpha_2^* & N_2\alpha_1^* & \alpha_3 \end{pmatrix}, \\
&l_2(\alpha) = \begin{pmatrix} -N_3\alpha_1\alpha_3^* & N_3\alpha_2^* & \alpha_1 \\ -N_3\alpha_2\alpha_3^* & -N_3\alpha_1^* & \alpha_2 \\ N_3^{-1} & 0 & \alpha_3 \end{pmatrix}; \\
&l_3(\alpha) = \begin{pmatrix} N_1^{-1} & 0 & \alpha_1 \\ -N_1\alpha_2\alpha_1^* & N_1\alpha_3^* & \alpha_2 \\ -N_1\alpha_3\alpha_1^* & -N_1\alpha_2^* & \alpha_3 \end{pmatrix}.
\end{aligned} \tag{3.12}$$

In the overlap $\mathscr{M}_j \cap \mathscr{M}_k$, the two coset representatives $l_j(\alpha)$ and $l_k(\alpha)$ are related by an α-dependent $SU(2)$ element on the right:

$$\begin{aligned}
&\alpha \in \mathscr{M}_j \cap \mathscr{M}_k : l_k(\alpha) = l_j(\alpha)\; A(a_{jk}(\alpha)), \; \text{no sum on } j; \\
&a_{jk}(\alpha) = -N_k N_l \begin{pmatrix} \alpha_k\alpha_l^* & \alpha_j^* \\ -\alpha_j & \alpha_l\alpha_k^* \end{pmatrix}.
\end{aligned} \tag{3.13}$$

To move up to choices of local coordinates over each of the regions $\mathscr{A}_j$ in $SU(3)$, we have to include a variable $SU(2)$ element on the right of the coset representative. We write

$$\begin{aligned}
&A \in \mathscr{A}_1 : A = l_1(\alpha)\, a(\xi); \\
&A \in \mathscr{A}_2 : A = l_2(\alpha)\, a(\xi'); \\
&A \in \mathscr{A}_3 : A = l_3(\alpha)\, a(\xi''); \\
&\quad \pi_1(A) = \alpha; \\
&\quad a(\xi) = \begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2^* & \xi_1^* \end{pmatrix} \in SU(2), \; \xi^\dagger\xi = 1.
\end{aligned} \tag{3.14}$$

Thus each $A \in \mathscr{A}_j \subset SU(3)$ is uniquely specified by a point $\alpha \in \mathscr{M}_j$ and a unique *SU(2)* element, $a(\xi)$ or $a(\xi')$ or $a(\xi'')$ for $j =$ 1, 2, 3 respectively. So the eight independent real local coordinates over $\mathscr{A}_j$ are supplied by α for $\alpha_j \neq 0$, and ξ or ξ' or ξ''. It is interesting to note that in this way each $SU(3)$ element $A \in \mathscr{A}_j$ is uniquely determined by one complex three-component unit vector a (with $\alpha_j \neq 0$), and one complex two-component unit vector ξ, ξ' or ξ''. The coordinate transformation rules in the overlaps are determined by combining Eqs. (3.13), (3.14):

$$
\begin{aligned}
A \in \mathscr{A}_1 \cap \mathscr{A}_2 &: a(\xi) = a_{12}(\alpha)\, a(\xi'); \\
A \in \mathscr{A}_2 \cap \mathscr{A}_3 &: a(\xi') = a_{23}(\alpha)\, a(\xi''); \\
A \in \mathscr{A}_3 \cap \mathscr{A}_1 &: a(\xi'') = a_{31}(\alpha)\, a(\xi).
\end{aligned} \tag{3.15}
$$

This gives a complete picture of the manifold structure of *SU(3),* based on a convenient description of $\mathscr{M} = SU(3)/SU(2)$.

We now focus on the subset $\mathscr{A}_3$ in *SU(3),* and correspondingly on $\mathscr{M}_3$ and $\mathscr{R}_3$. We ask for the portion of *SU(3)* not contained in $\mathscr{A}_3$. It consists of all those $A \in SU(3)$ for which $\alpha_3 = 0$:

$$
A \notin \mathscr{A}_3 \Leftrightarrow A = \begin{pmatrix} A_{11} & A_{12} & \alpha_1 \\ A_{21} & A_{22} & \alpha_2 \\ A_{31} & A_{32} & 0 \end{pmatrix} \in SU(3). \tag{3.16}
$$

By careful counting of parameters one can check that this is a six-dimensional region in *SU(3),* a region of vanishing measure. In that sense, $\mathscr{A}_3$ covers almost all of $SU(3)$, as do $\mathscr{A}_1$ and $\mathscr{A}_2$. Correspondingly, points in $\mathscr{M}$ outside of $\mathscr{M}_3$ have $\alpha_3 = 0$; they form a three-dimensional region determined by unit complex two-component vectors (α_1, α_2). The portion of $\mathscr{R}$ not contained in $\mathscr{R}_3$ is a two-dimensional region, essentially a Poincare sphere S^2.

For practical calculations, we largely restrict ourselves to the regions $\mathscr{A}_3$, $\mathscr{M}_3$, $\mathscr{R}_3$ in $SU(3)$, $\mathscr{M}$, $\mathscr{R}$ respectively, in the knowledge that in each case the omitted portion is a low-dimensional region. This will introduce expected coordinate type singularities at the boundaries of $\mathscr{A}_3$, $\mathscr{M}_3$, $\mathscr{R}_3$ in the well-known way, but the account given above shows us in principle how to circumvent such problems.

Maurer–Cartan one-forms on SU(3)

The calculation of the dynamical phase within any *SU(3)* representation requires in principle knowledge of the complete set of independent left-invariant Maurer-Cartan one-forms over *SU(3),* and pull-backs of suitable subsets of them to various coset spaces [11]. There are eight independent one-forms, each being globally well-defined over *SU(3).* We shall give expressions for them (modulo known results for $SU(2)$) over $\mathscr{A}_3$.

We may define the one-forms $\hat{\theta}_r^{(0)}$, $r = 1, 2, ..., 8$, by the symbolic formula

$$A^{-1}\, dA = -\frac{i}{2}\, \lambda_r \hat{\theta}_r^{(0)},$$
$$\hat{\theta}_r^{(0)} = i Tr\{\lambda_r A^{-1}\, dA\}, \tag{3.17}$$

where A is a variable matrix in *SU(3)*. For $A \in \mathscr{A}_3$ we write:

$$A^{-1}\, dA = a(\xi'')^{-1}\, da(\xi'') + a(\xi'')^{-1} \cdot l_3(\alpha)^{-1}\, dl_3(\alpha) \cdot a(\xi''). \tag{3.18}$$

The first piece here comes from *SU(2)* and is well known; it is a combination of λ_1, λ_2, λ_3 [12]. The essentially new part is the second piece. Here it is basically enough to compute $l_3(\alpha)^{-1}\, dl_3(\alpha)$ and express it as a linear combination of the λ_r, since the result of conjugation by $a(\xi'')$ is easily given. Omitting the double primes for simplicity we have:

$$a = \begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2^* & \xi_1^* \end{pmatrix}:$$

$$a^{-1} \lambda_j a = \mathscr{R}(a)_{jk} \lambda_k,$$

$$\mathscr{R}(a) = \begin{pmatrix} \mathrm{Re}(\xi_1^2 - \xi_2^2) & \mathrm{Im}(\xi_1^2 + \xi_2^2) & -2\,\mathrm{Re}(\xi_1 \xi_2) \\ \mathrm{Im}(\xi_2^2 - \xi_1^2) & \mathrm{Re}(\xi_1^2 + \xi_2^2) & 2\,\mathrm{Im}(\xi_1 \xi_2) \\ 2\,\mathrm{Re}(\xi_1 \xi_2^*) & 2\,\mathrm{Im}(\xi_1 \xi_2^*) & |\xi_1|^2 - |\xi_2|^2 \end{pmatrix} \in SO(3); \tag{3.19a}$$

$$a^{-1} \lambda_8 a = \lambda_8; \tag{3.19b}$$

$$a^{-1}\{\lambda_6 + i\lambda_7,\ \lambda_4 + i\lambda_5\}\, a$$
$$= \{\xi_1(\lambda_6 + i\lambda_7) + \xi_2(\lambda_4 + i\lambda_5), -\xi_2^*(\lambda_6 + i\lambda_7) + \xi_1^*(\lambda_4 + i\lambda_5)\}; \tag{3.19c}$$

$$a^{-1}\{\lambda_4 - i\lambda_5,\ -\lambda_6 + i\lambda_7\}\, a$$
$$= \{\xi_1(\lambda_4 - i\lambda_5) + \xi_2(-\lambda_6 + i\lambda_7), -\xi_2^*(\lambda_4 - i\lambda_5) + \xi_1^*(-\lambda_6 + i\lambda_7)\}. \tag{3.19d}$$

To complete the calculation using the above strategy, we need to parametrise a and ξ'' over $\mathscr{A}_3$ by suitable independent real variables. However, since results for *SU(2)* are well known, we avoid use of any particular system of real coordinates for ξ'', and concentrate on a. This means that we must choose five independent real coordinates for the region $\mathscr{M}_3$ in $\mathscr{M}$. It is convenient to take them to be angle and phase type variables. Recognizing that the triplet $(|\alpha_1|,\ \alpha_2|,\ \alpha_3|)$ is a real three dimensional unit vector with nonnegative components, we introduce five angles *9*, ϕ, χ_1, *X2*, *n* thus:

$$(\alpha_1,\ \alpha_2,\ \alpha_3) = e^{i\eta}(e^{i\chi_1} \cos 9,\ e^{i\chi_2} \sin 9 \cos \phi,\ \sin 9 \sin \phi);$$
$$0 < \theta,\ \phi \leqslant \pi/2; \tag{3.20}$$
$$0 \leqslant \eta,\ \chi_1,\ \chi_2 < 2\pi.$$

The limits on 9, ϕ ensure that $\alpha_3 \neq 0$; the overall phase 77 is then well defined all over $\mathscr{M}_3$, and disappears in the passage to $\mathscr{R}_3$. In fact the angles 9, ϕ, η are all unambiguously defined throughout $\mathscr{M}_3$; while χ_1 is undefined when $9 = \pi/2$ and χ_2 when $\phi = \pi/2$. The real triplet $|\alpha_j|$ lies in the first octant in three dimensional space, and the situation can be pictured as in Fig. 1. Corresponding coordinates for $\mathscr{R}_3$ are θ, ϕ, χ_1, χ_2 with the above ranges.

We mentioned that points in $\mathscr{M}$ outside of $\mathscr{M}_3$ form a three dimensional region, where $\alpha_3 = 0$. In Fig. 1 they can be taken as the limit $\phi = 0$, namely the arc *AB*. Sacrificing χ_2 we can parametrise these points of $\mathscr{M}$ as follows:

$$\text{Complement of } \mathscr{M}_3 . \alpha_1 — e \quad \cos 9,\ \alpha_2 = e^{i\eta} \sin 9,\ \alpha_3 = 0; \qquad (3.21)$$
$$0 \leqslant \theta \leqslant \pi/2, \quad 0 \leqslant \eta, \chi_1 < 2\pi.$$

Over $\mathscr{R}$, the complement of $\mathscr{R}_3$, which is essentially the Poincare sphere, is then parametrized by 9 and χ_1.

The calculation of $l_3(\alpha)^{\ 1}\, dl_3(\alpha)$ can now be completed by using the parametrisation (3.20) in the matrix $l_3(\alpha)$ in Eq. (3.12). If we write

$$l_3(\alpha)^{-1}\, dl_3(\alpha) = -\frac{i}{2} \lambda_r f_r, \qquad (3.22)$$

then each f_r is a linear expression in $d9$, $d\phi$, $d\eta$, $d\chi_1$, $d\chi_2$. It is convenient to display the results as in Table III, where in the rth row we give the coefficients of $d9$, ..., $d\chi_2$ appearing in f_r.

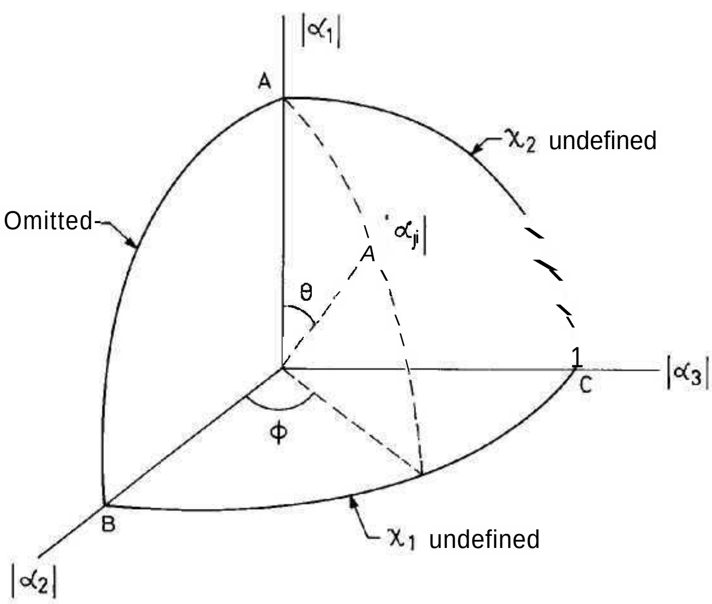

FIG. 1. Choice of angles and phases for $\mathscr{M}_3$ in *M*.

TABLE III

Coefficients of Independent Differentials in the Forms f_r

	$d\theta$	$d\phi$	$d\eta$	$d\chi_1$	$d\chi_2$
f_1	0	$2\cos 9\sin(\chi_1-\chi_2-\eta)$	0	0	$\cos\theta\sin 2\phi\cos(\chi_1-\chi_2-\eta)$
f_2	0	$2\cos 9\cos(\chi_1-\chi_2-\eta)$	0	0	$-\cos\theta\sin 2\phi\sin(\chi_1-\chi_2-\eta)$
f_3	0	0	-1	$\cos^2 9$	$-\cos^2\phi(1+\cos^2 9)$
f_4	$2\sin(\chi_1+\eta)$	0	0	$-\cos(\chi_1+\eta)\sin 29$	$\cos(\chi_1+\eta)\sin 2\theta\cos^2 0$
f_5	$2\cos(\chi_1+\eta)$	0	0	$\sin(\chi_1+\eta)\sin 29$	$-\sin(\chi_1+\eta)\sin 29\cos^2 0$
f_6	0	$2\sin\theta\sin(\chi_2+2\eta)$	0	0	$-\sin 9\sin 20\cos(\chi_2+2\eta)$
f_7	0	$2\sin\theta\cos(\chi_2+2\eta)$	0	0	$\sin 9\sin 20\sin(\chi_2+2\eta)$
f_8	0	0	$\sqrt{3}$	$\sqrt{3}\cos^2 9$	$\sqrt{3}\sin^2 9\cos^2 0$

To complete the calculation of the expression for $\hat{\theta}_r^{(0)}$ over $\mathscr{A}_3$, we need to combine these results for f_r with Eq. (3.18), calculate the results of conjugation by $a(\xi'')$ using Eqs. (3.19), and add the pure *SU(2)* contribution as well. While this is in principle straightforward, we do not present the details, since all the ingredients have been provided. In the same spirit, one can in principle do all this in each of the other two regions $\mathscr{A}_1$, $\mathscr{A}_2$, and it will be the case that the different expressions for $\hat{\theta}_r^{(0)}$ in regions $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$ will agree in the overlaps, if one switches coordinates according to the transition formulae in Eqs. (3.15).

4. SURVEY OF FORMS OF $SU(3)$ DYNAMICAL PHASES

We described in Section 2 the preliminary steps involved in calculating any $SU(3)$ geometric phase: finding the stability subgroups H_0, H going with a given vector ψ_0. In Table III we have listed the possible pairs of nontrivial subgroups H_0, H (upto equivalence) that can arise. To these of course must be added the generic case $H_0 = \{e\}$. We have also mentioned that the quotient group H/H_0 could be one of three types, namely trivial (Type A), nontrivial and discrete (Type B), or $U(1)$ (Type C). For our purposes, we may treat Types A and B together.

To proceed further, in Types A and B we need to find a basis for the Lie algebra $SU(3)$ made up of a basis for H_0 together with remaining generators belonging to various irreducible representations of H_0. For the latter we may need to use complex combinations of the hermitian generators F_r. Among the generators outside of H_0 we must then search for a complete independent set of H_0-scalars, $S_\rho^{(0)}$ say, none of which can of course annihilate ψ_0. The Maurer–Cartan one-forms $\theta^{(0)\rho}$ on $SU(3)$ which go with the $S_\rho^{(0)}$ can all be pulled back to the coset space $SU(3)/H_0$ and lead to globally defined one-forms θ^ρ therein. Then the formula for the dynamical phase becomes [4]:

$$\text{Types A, B}: \varphi_{\text{dyn}}[\mathscr{C}] = -(\psi_0, S_\rho^{(0)}\psi_0)\int_{\mathscr{C}\subset SU(3)/H_0} 6''. \quad (4.1)$$

In a type C situation the subgroup H has an extra $U(1)$ generator which we denote by Q (this was denoted by Y in Ref. (4)). This Q is an H_0-scalar so it is one of the $S_r^{(0)}$ mentioned above. It is also automatically an H-scalar. Apart from H_0 and Q we need to classify additional basis elements for *SU(3)* with respect to H, and search for H-scalars among them. If S_α is a complete independent set of such generators, Q included, then the dynamical phase is [4]:

$$\text{Type } C: \varphi_d \ [\mathscr{C}] = -(\psi_0, S_\alpha \psi_0) \int_{\mathscr{C} \subset SU(3)/H_0} \theta^\alpha. \tag{4.2}$$

To carry out the above tasks for *SU(3)* for each nontrivial H_0, it is useful to work with the tensor basis for *SU(3)* in its defining representation, involving complex combinations of the λ_r:

$$F_{jk}^{(0)} = F_{kj}^{(0)\dagger} : (F_{jk}^{(0)})_{lm} = \delta_{jm}\delta_{kl} - \tfrac{1}{3}\delta_{jk}\delta_{lm},$$
$$F_{jj}^{(0)} = 0,$$
$$Tr(F_{jk}^{(0)}) = 0; \tag{4.3a}$$

$$F_{12}^{(0)} = \tfrac{1}{2}(\lambda_1 - i\lambda_2),\ F_{23}^{(0)} = \tfrac{1}{2}(\lambda_6 + i\lambda_7),\ F_{31}^{(0)} = \tfrac{1}{2}(\lambda_4 + i\lambda_5),$$
$$F_{11}^{(0)} = \tfrac{1}{3}(2H_1^{(0)} - H_2^{(0)}), \qquad F_{22}^{(0)} = \tfrac{1}{3}(2H_2^{(0)} - H_1^{(0)}). \tag{4.3b}$$

In a general *SU(3)* representation the tensor basis elements are written as F_{jk}. We may now classify all the generators of *SU(3)* according to their behaviours under commutation with $H_1^{(0)}$ and $H_2^{(0)}$, choosing combinations T with definite weights:

$$[H_1^{(0)} \text{ or } H_2^{(0)},\ T] = (m_1 \text{ or } m_2)\, T. \tag{4.4}$$

The results are given in Table IV. Equipped with them we can proceed with searches for the generators $S_\rho^{(0)}$, S_α needed for dynamical phase calculations. Leaving aside the case $H_0 = \{e\}$, we find the results in Table V. In particular, in the $SO(3)$ case

TABLE IV

Generators of $SU(3)$
According to Weights with Respect to $H_1^{(0)}$, $H_2^{(0)}$

m_1	m_2	T
0	0	$H_1^{(0)}, H_2^{(0)}$
1	−1	$F_{21}^{(0)}$
-1	1	$F_{12}^{(0)}$
1	2	$F_{32}^{(0)}$
-1	−2	$F_{23}^{(0)}$
2	1	$F_{31}^{(0)}$
-2	−1	$F_{13}^{(0)}$

TABLE V

Possibilities for Nonzero $SU(3)$ Dynamical Phases for Nontrivial H_0

Case	Nontrivial stability subgroup H_0	Generators of H_0	Possible addition of $U(1)$ generator Q	Possible generators $S^{(0)}_{\rho}$, S_{α}	φ_{dyn}
(a)	$U_{(p,q)}(1)$	$pH_1 + qH_2$	None or $+ qH_1 - pH_2$	$+ qH_1 - pH_2$	Nonzero
(c)	£7(1) x £7(1)	H_1, H_2	None	None	Zero
(d)	$SU(2)$	I_1, I_2, I_3	None or F_8	F_8	Nonzero
(e)	£7(2)	I_1, I_2, I_3, Y	None	None	Zero
(f)	$SO(3)$	$i(F_{jk} - F_{kj})$	None	None	Zero

the vanishing of φ_{dyn} follows since the *SU(3)* generators outside $SO(3)$ form a quadrupole tensor.

From this analysis we see that nonzero *SU(3)* dynamical phases can arise in only three situations:

(i) $H_0 = \{e\}$. In this case the curve $\mathscr{C}$ is to be visualised as drawn in the *SU(3)* group manifold, and each of the eight forms $\bar{\theta}^{(0)}_r$ can in principle contribute:

$$\varphi_{\text{dyn}}[\mathscr{C}] = -(\psi_0, F_r\psi_0) \int_{\mathscr{C} \subset SU(3)} \hat{\theta}^{(0)}_r. \tag{4.5}$$

In case $H = U(1)$ rather than $\{e\}$ there could be simplifications.

(ii) $H_0 = U_{(p,q)}(1)$. Now we have a seven-dimensional coset space $SU(3)/U_{(p,q)}(1)$. By rewriting the terms $F_3\bar{\theta}^{(0)}_3 + F_8\bar{\theta}^{(0)}_8$ in $F_r\bar{\theta}^{(0)}_r$ as a linear combination of $pH_1 + qH_2$ and $qH_1 - pH_2$, and dropping the former, we see that we only need the pull back of $(q+p)\bar{\theta}^{(0)}_3 + (1/\sqrt{3})(q-p)\hat{\theta}^{(0)}_8$ from $SU(3)$ to $SU(3)/U_{(p,q)}(1)$:

$$\varphi_{\text{dyn}}[\mathscr{C}] = -\frac{1}{2(p^2+q^2)} (\psi_0, (qH_1 - pH_2)\psi_0) \times \int_{\mathscr{C} \subset SU(3)/U_{(p,q)}(1)} \left((q+p)\hat{\theta}_3 + \frac{(q-p)}{\sqrt{3}}\hat{\theta}_8\right) \tag{4.6}$$

(iii) $H_0 = SU(2)$. Here we could have $H \simeq H_0$ or $H \simeq H_0 \times U(1)$. In either case the only candidate for scalar generators $S^{(0)}_{\rho}$, S_α is the single generator F_8, so we have

$$\varphi_{\text{dyn}}[\mathscr{C}] = -(\psi_0, F_8\psi_0) \int_{\mathscr{C} \subset \mathscr{M}} \hat{\theta}_8, \tag{4.7}$$

where $\hat{\theta}_8$ is the pullback of $\hat{\theta}_8^{(0)}$ from $SU(3)$ to $\mathcal{M}$ and is globally well-defined on $\mathcal{M}$. Over the open subset $\mathcal{M}_3 \subset \mathcal{M}$ we find that $\hat{\theta}_8$ has the following form read off from Table III:

$$\mathcal{M}_3 : \theta_8 = \sqrt{3}(d\eta + \cos^2\theta\, d\chi_1 + \sin^2\theta \cos^2\phi\, d\chi_2). \tag{4.8}$$

A particular case of this arises for three level quantum systems when $H = U(2)$. Since this has several interesting and important features, we look at it in detail in the next Section.

5. DYNAMICAL PHASES FOR THREE-LEVEL QUANTUM SYSTEMS

For two-level quantum systems it is well known that pure-state density matrices can be represented in a faithful manner by points on the Poincare sphere S^2 of real three-dimensional unit vectors. This is immediately seen by exploiting the properties of the Pauli matrices $\underline{\sigma}$. For two dimensional density matrices ρ we have:

$$\begin{aligned} &\rho^\dagger = \rho^2 = \rho \geqslant 0, \qquad \mathrm{tr}\,\rho = 1 \Leftrightarrow \\ &\rho = \tfrac{1}{2}(1 + \underline{n}.\underline{\sigma}), \quad \underline{n}^* = \underline{n}, \quad \underline{n}.\underline{n} = 1. \end{aligned} \tag{5.1}$$

We shall in this Section first show how this generalises to three level systems, then turn to the general form for dynamical and geometric phases for them. We begin with some useful algebraic preliminaries for three dimensional matrices.

In addition to the commutation relations (2.3), the matrices λ_r obey an anticommutation relation involving a set of completely symmetric d-symbols [7]:

$$\begin{aligned} &\lambda_r \lambda_s + \lambda_s \lambda_r = \tfrac{4}{3}\delta_{rs} + 2 d_{rst}\lambda_t, \\ &d_{118} = d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}}, \\ &d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \tfrac{1}{2}, \\ &d_{448} = d_{558} = d_{668} = d_{778} = -1/2\sqrt{3}. \end{aligned} \tag{5.2}$$

Here only the independent nonzero components have been listed. So the product of two λ's can be written as

$$\lambda_r \lambda_s = \tfrac{2}{3}\delta_{rs} + (d_{rst} + i f_{rst})\lambda_t. \tag{5.3}$$

For any real eight-component vector n_r let us denote by $n.\lambda$ the hermitian three dimensional matrix $n_r\lambda_r$. Several trace and determinant properties follow:

$$\begin{aligned} &Tr\,\lambda_r = 0, \qquad Tr(\lambda_r\lambda_s) = 2\delta_{rs}; \\ &Tr(n.\lambda)^2 = 2n^2; \\ &\det(n.\lambda) = \tfrac{2}{3} d_{rst} n_r n_s n_t. \end{aligned} \tag{5.4}$$

This motivates the definition of a * product among eight-dimensional vectors, the result being another such vector:

$$
\begin{aligned}
n'_r &= (n * n)_r = \sqrt{3}\, d_{rst}\, n_s n_t: \\
n'_j &= 2 n_8 n_j + \frac{\sqrt{3}}{2}\,(n_4 + i n_5 \;\; n_6 + i n_7)\,\sigma_j \begin{pmatrix} n_4 - i n_5 \\ n_6 - i n_7 \end{pmatrix}, \qquad j = 1, 2, 3; \\
n'_4 - i n'_5 &= (\sqrt{3} n_3 - n_8)(n_4 - i n_5) + \sqrt{3}(n_1 - i n_2)(n_6 - i n_7), \qquad (5.5) \\
n'_6 - i n'_7 &= -(\sqrt{3} n_3 + n_8)(n_6 - i n_7) + \sqrt{3}(n_1 + i n_2)(n_4 - i n_5), \\
n'_8 &= n_j n_j - n_8^2 - \tfrac{1}{2}(n_4^2 + n_5^2 + n_6^2 + n_7^2).
\end{aligned}
$$

When $n.\lambda$ is conjugated by $A \in SU(3)$, n transforms by the octet or (1, 1) *UIR* of *SU(3)*, which is eight dimensional; and so does n' since d_{rst} is an invariant tensor. For the square of $n.\lambda$ and the determinant we can write:

$$
\begin{aligned}
(n.\lambda)^2 &= \frac{2}{3} n^2 + \frac{1}{\sqrt{3}} (K * n).\lambda, \\
\det(n.\lambda) &= \frac{2}{3\sqrt{3}}\, n.(n * n).
\end{aligned} \qquad (5.6)
$$

By an easy analysis one finds that the eigenvalues of $n.\lambda$ can be displayed as follows:

$$
\begin{aligned}
&\text{Spectrum of } n.\lambda = \sqrt{n^2}(\mu_1, \mu_2, \mu_3), \\
&\mu_1 \geqslant \mu_2 \geqslant \mu_3, \\
&\mu_1 + \mu_2 + \mu_3 = 0, \\
&\mu_1^2 + \mu_2^2 + \mu_3^2 = 2; \\
&\frac{2}{\sqrt{3}} \geqslant \mu_1 \geqslant \frac{1}{\sqrt{3}} \geqslant \mu_2 \geqslant -\frac{1}{\sqrt{3}} \geqslant \mu_3 \geqslant -\frac{2}{\sqrt{3}}, \qquad (5.7) \\
&\mu_{2,3} = -\tfrac{1}{2}\mu_1 \pm \sqrt{1 - 3\mu_1^2/4}, \\
&\mu_2 \mu_3 = \mu_1^2 - 1; \\
&\mu_1(\mu_1^2 - 1) = \frac{2}{3\sqrt{3}}\, \frac{K.(K * K)}{(n^2)^{3/2}}.
\end{aligned}
$$

It is understood that we choose the root of the cubic lying in the range $1/\sqrt{3} \leqslant \mu_1 \leqslant 2/\sqrt{3}$.

It is possible to describe these properties of a general hermitian traceless generator matrix $n.\lambda$ in another geometrically interesting manner, in terms of orbits in the Lie algebra under adjoint action. (Since the Lie algebra of *SU(3)* possesses a nonsingular invariant quadratic Killing Cartan form, the coadjoint and adjoint representations are equivalent). We may write the diagonal form of $n.\lambda$ as a multiple of a normalised real linear combination of λ_3 and λ_8:

$$\text{diagonal form of } n.\lambda = (n^2)^{1/2}(a\lambda_3 + b\lambda_8)$$
$$= (n^2)^{1/2} \operatorname{diag}\left(\frac{b}{\sqrt{3}} + a,\ \frac{b}{\sqrt{3}} - a,\ \frac{-2b}{\sqrt{3}}\right), \qquad (5.8)$$
$$a = \cos 0, \qquad b = \sin 9$$

The identification with the μ's in Eq. (5.7) is

$$\mu_1 = \frac{b}{\sqrt{3}} + a, \qquad \mu_2 = \frac{b}{\sqrt{3}} - a, \qquad \mu_3 = \frac{-2b}{\sqrt{3}}, \qquad (5.9)$$

and the inequalities $\mu_1 \geqslant \mu_2 \geqslant \mu_3$ translate into $\pi/6 \leqslant \theta \leqslant \pi/2$. The cubic invariant K.K * K is easily computed:

$$n.n * n = (n^2)^{3/2}\, b(3a^2 - b^2) \qquad (5.10)$$

The factor $b(3a^2 - b^2)$ decreases monotonically from $+1$ at $9 = \pi/6$ to -1 at $9 = \pi/2$. For $9 = \pi/6$ we have $\mu_1 = 2/\sqrt{3}$, $\mu_2 = \mu_3 = \sim 1/\sqrt{3}$, while at the other end for $6 = n/2$ we have $\mu_1 = \mu_2 = 1/\sqrt{3}$, $\mu_3 = -2/\sqrt{3}$. For $\pi/6 < \theta < \pi/2$ we have $\mu_1 > \mu_2 > \mu_3$, three distinct eigenvalues of $n.\lambda$. Now the orbit to which K belongs is generated by conjugating $n.\lambda$ with A for all A e $SU(3)$:

$$\text{Orbit of } n = \{n' \mid n'.\lambda = An.\lambda A^{-1},\ A \in SU(3)\}, \qquad (5.11)$$

and both invariants n^2, K.K * K are constant over an orbit. We can now see (leaving aside the case K = 0) that there are two distinct types of orbits, depending on whether the eigenvalues of K.λ are all distinct, or two are equal. The former is the generic case. Here we have a two parameter continuous family of orbits $\mathscr{I}(\sqrt{n^2}, 9)$ with $n^2 > 0$ and $n/6 < 9 < \pi/2$. Each such orbit $\mathscr{I}(\sqrt{n^2}, 9)$ is of dimension six, with representative element $\sqrt{n^2/3}\operatorname{diag}(\sin 9 + \sqrt{3}\cos 6,\ \sin 9 - \sqrt{3}\cos 6,\ -2\sin 6)$; it provides a realisation of the coset space $SU(3)/U(1) \times U(1)$. The remaining non-generic orbits comprise two distinct one-parameter continuous families $\mathscr{I}^{(\pm)}(\sqrt{n^2})$, with $n^2 > 0$ again. Each orbit $\mathscr{I}^{(\pm)}(\sqrt{n^2})$ is of dimension four, with representative element $\sqrt{n^2/3}(2, -1, -1)$, $\sqrt{n^2/3}(1, 1, -2)$, respectively; and each of these is a realisation of the coset space *SU(3)/U(2)*. Thus while the collection of orbits $\mathscr{I}(\sqrt{n^2}, 9)$ fills out an eight dimensional region in $\mathscr{R}^8$, over which $-(n^2)^{3/2} <$ K.K * K $< (n^2)^{3/2}$, each of the collections $\mathscr{I}^{(\pm)}(\sqrt{n^2})$ fills out only a region of dimension five, over which $n.n*n = \pm(n^2)^{3/2}$ respectively. We can regard $\mathscr{I}^{(\pm)}(\sqrt{n^2})$ as the singular limits, as $\theta \to \pi/6$ and $\pi/2$ respectively, of $\mathscr{I}(\sqrt{n^2}, \theta)$: singular because of the abrupt change in dimension and subgroup H.

Now we consider how to represent pure state density matrices for three level systems using this formalism. Keeping track of the trace condition let us write

$$\rho = \rho(n) = \tfrac{1}{3}(1 + \sqrt{3}\, n.\lambda). \tag{5.12}$$

Hermiticity of ρ results in n being a real eight-dimensional vector. The pure state condition $\rho^2 = \rho$ then becomes, upon use of Eq. (5.6) and simplification:

$$n^2 = 1, \; n * n = n. \tag{5.13}$$

The first condition means that n is a point on the unit sphere S^7 in eight-dimensional real Euclidean space. The second condition $n * n = n$ restricts n to the four dimensional singular orbit $\mathscr{I}^{(+)}(1)$ described above This region is preserved under the action of $SU(3)$ on n via the octet representation $(1, 1)$. The condition $n * n = n$ also means that $n.\lambda$ obeys

$$(n.\lambda)^2 = \frac{2}{3} + \frac{1}{\sqrt{3}}\, n.\lambda, \tag{5.14}$$

so it has eigenvalues $1/\sqrt{3}(2, -1, -1)$. This results in $\rho(n)$ having eigenvalues $(1, 0, 0)$ as is appropriate for a normalised pure state density matrix. We see from all this that pure state density matrices for three level systems, which constitute the coset space $\mathscr{R} = SU(3)/U(2)$, correspond one-to-one to points n in $\mathscr{I}^{(+)}(1)$. We may denote this region too by $\mathscr{R}$:

$$\begin{aligned} \mathscr{R} &= \{\alpha\alpha^\dagger \mid \alpha^\dagger\alpha = 1\} \\ &= \mathscr{I}^{(+)}(1) \\ &= \{n \mid n^2 = 1, \; n * n = n\} \subset S^7. \end{aligned} \tag{5.15}$$

This is the generalisation of the Poincare sphere S^2 from two to three level systems. The expression for n in terms of α is:

$$\begin{aligned} &\rho(n) = \alpha\alpha^\dagger : n_r = \frac{\sqrt{3}}{2}\alpha^\dagger\lambda_r\alpha, \\ &(n_6 - in_7, n_4 + in_5, n_1 - in_2) = \sqrt{3}(\alpha_2\alpha_3^*, \; \alpha_3\alpha_1^*, \; \alpha_1\alpha_2^*), \\ &n_3 = \frac{\sqrt{3}}{2}(|\alpha_1|^2 - |\alpha_2|^2), \qquad n_8 = \frac{1}{2}(1 - 3|\alpha_3|^2). \end{aligned} \tag{5.16}$$

On account of the importance of this particular coset space and singular orbit, we mention in passing yet another way of realising it. It is the complex projective space CP^2 obtained as follows. We start with triplets of complex numbers $C^3 = \{\underset{\sim}{z} = (z_1, z_2, z_3)\}$, exclude the point $0 = (0, 0, 0)$ and define the natural $SU(3)$ action:

$$A \in SU(3): \underset{\sim}{z} \to \underset{\sim}{z}' = Az \tag{5.17}$$

Next we introduce an equivalence relation among triplets corresponding to multiplication by nonzero complex scalars:

$$z' \sim z \Leftrightarrow z' = \lambda \underset{\sim}{z}, \qquad \lambda \neq 0. \tag{5.18}$$

The $SU(3)$ action (5.17) clearly respects this equivalence and so passes to an action on the quotient space $CP^2 = (C^3 - \{\underset{\sim}{0}\})/\sim$. Moreover the latter action is easily seen to be transitive, so CP^2 is a coset space. A representative point in CP^2 is the equivalence class containing the triplet (0, 0, 1); since this class is invariant under the $U(2)$ subgroup (2.15) of $SU(3)$, the identification $CP^2 = SU(3)/U(2)$ follows. Thus for three level quantum systems, we can also regard CP^2 as the generalisation of the Poincare sphere for two-level systems.

The part of $\mathscr{R}$ not contained in $\mathscr{R}_3$ corresponds, as we saw in Section 3, to $\alpha_3 = 0$. In the above description this appears as follows:

$$\begin{aligned} n \notin \mathscr{R}_3 \Leftrightarrow n_j &= \frac{\sqrt{3}}{2}\, \alpha^\dagger \sigma_j \alpha,\ j = 1, 2, 3, ; \\ n_4 = n_5 = n_6 = n_7 &= 0; \\ n_8 &= \tfrac{1}{2}. \end{aligned} \tag{5.19}$$

We see as expected that $2/\sqrt{3}n_j$ is a point on the Poincare sphere S^2.

Turning to the calculation of geometrical and dynamical phases, we can see that the three-level quantum system corresponds exactly to the case $H_0 = SU(2)$, $H = U(2)$ of the possibilities listed in the previous Section. This is because any three-component complex unit vector $\psi_0 e \mathscr{H}$, the three-dimensional complex Hilbert space, can be transformed by a suitable $SU(3)$ element to the form $\psi_0 = (0, 0, 1)$; and then the above results for H_0 and H follow. Now let $\mathscr{C}$ be any smooth curve of unit vectors $\psi(s) \in \mathscr{M}$for $s_1 \leqslant s \leqslant s_2$, starting out at $\psi(s_1) = \psi_0$. Here $\mathscr{M} \subset \mathscr{H}$ is the complex unit sphere, identified with $SU(3)/SU(2)$in Section 2. Let us make the explicit assumption that $\mathscr{C}$ is contained wholly within $\mathscr{M}_3$, so its ray space image C is wholly within $\mathscr{R}_3$. This means that if we write the components of $\psi(s)$ as $\alpha_j(s)$, then $\alpha_3(s) \neq 0$ throughout, and the vector $\alpha_j(s)|$ stays within the positive octant of Figure 1 avoiding the arc AB. From the general connection (1.1) the geometric, total and dynamical phases are:

$$\varphi_g[C] = \varphi_p[\mathscr{C}] - \varphi_{\text{dyn}}[\mathscr{C}],$$

$$\varphi_p[\mathscr{C}] = \arg(\psi(s_1),\ \psi(s_2)) = \arg \alpha_3(s_2) = \int_{\mathscr{C} \subset \mathscr{M}_3} d\eta;$$

$$\varphi_{\text{dyn}}[\mathscr{C}] = \operatorname{Im} \int_{s_1}^{s_2} ds(\psi(s),\ \dot{\psi}(s)) \tag{5.20}$$

$$= -(\psi_0, \tfrac{1}{2}\lambda_8 \psi_0) \int_{\mathscr{C} \subset \mathscr{M}_3} \hat{\theta}_8$$

$$= \int_{\mathscr{C} \subset \mathscr{M}_3} (d\eta + \cos^2 9\, d\chi_1 + \sin^2 9 \cos^2 \phi\, d\chi_2)$$

As a special case let us suppose that $\mathscr{C}$ is closed and that $\varphi_p[\mathscr{C}] = 0$. The image C of $\mathscr{C}$ in $\mathscr{R}_3$ is also closed. One can express $\varphi_g[C]$ either as a one-dimensional integral along $\mathscr{C}$ or as a two-dimensional "surface integral" over any surface $S \subset \mathscr{R}_3$ bounded by C:

$$\varphi_g[C] = -\int_{\mathscr{C} \subset \mathscr{M}_3} (\cos^2 9\, d\chi_1 + \sin^2 9 \cos^2 \phi\, d\chi_2)$$

$$= -2 \int_{S \subset \mathscr{R}_3,\, \partial S = C} (\sin 9 \cos 9\, d\chi_{1\wedge}\, d9 + \sin 9 \cos 9 \cos^2 \phi\, d\theta_{\wedge}\, d\chi_2$$

$$+ \sin^2 \theta \cos \phi \sin \phi \qquad \text{fifj},_{\text{A}} \qquad \text{fif\^{}}) \tag{5.21}$$

This is the three-dimensional analogue of the well known "solid angle formula" on the Poincare sphere for two-level problems. What is significant is the absence of terms $d\theta_{\wedge}\ d\phi,\ d\phi_{\wedge}$ fifjj, fifj$_{\text{lA}}$ fifj,.

If the curve $\mathscr{C}$ passes through the region of $\mathscr{M}$ outside of $\mathscr{M}_3$, then in principle one must work with the expressions for $\hat{\theta}_8$ over $\mathscr{M}_1$, $\mathscr{M}_2$ and use transition formulae in the overlaps etc.

To conclude this Section we point out the close connection between the geometric phase formula (5.21), and the natural symplectic structure carried by CP^2 and $\mathscr{S}^{(+)}(1)$. It is more convenient to work with the latter, as then the general theory of symplectic structures on co-adjoint orbits is available. A direct way to obtain this structure on $\mathscr{S}^{(+)}(1)$ is to start with a variable point $n \in \mathscr{S}^{(+)}(1)$ and define a system of generalised Poisson brackets (GPB) among the components n_r using the structure constants f_{rst} of Eq. (2.3) [13]:

$$\{n_r,\ n_s\} = f_{rst} n_t. \tag{5.22}$$

One can consistently set $n^2 = n.n * n = 1$ here without any algebraic conflict arising. Now we limit ourselves to local calculations over $\mathscr{R}_3 \subset \mathscr{S}^{(+)}(1)$, using the coordinates $9,\ \phi,\ \chi_1,\ \chi_2$. The expressions for n_r in these coordinates are given by Eq. (3.20, 5.16). Then easy algebra starting from Eq. (5.22) shows that there are only three nonvanishing GPB's among these coordinates. The complete antisymmetric 4×4 matrix of GPB's, with the rows and columns corresponding to 9, ϕ, χ_1, χ_2 in that sequence is:

$$\begin{pmatrix} 0 & \{\theta,\phi\} & \dots & \{\theta,\chi_2\} \\ \cdots & \cdots & & \\ \cdots & \cdots & & \\ \{\chi_2,\theta\} & \{\chi_2,\phi\} & \cdots & 0 \end{pmatrix} = -(2\sqrt{3}\,\sin^2 6\, cos\, 0 \sin\phi\cos\phi)^{-1}$$

$$\times \begin{pmatrix} 0 & \sin 0 \sin\phi\cos\phi & 0 \\ & \cos 0 \cos^2\phi & \cos t \\ \cdots\ \cdots & & \\ \cdots\ \cdots & 0 & \end{pmatrix} \qquad (5.23)$$

(The appearance here of singular denominators reflects only the use of local coordinates, the GPB's (5.22) are globally well defined). Inverting this matrix gives the components of the closed nondegenerate two-form *co* on $\mathscr{I}^{(+)}(1)$ which defines its symplectic structure:

$$\begin{pmatrix} 0 & \{\theta,\phi\} & \dots & \{\theta,\chi_2\} \\ \cdots & \cdots & & \\ \cdots & \cdots & & \\ \{\chi_2,\theta\} & \{\chi_2,\phi\} & \cdots & 0 \end{pmatrix}^{-1}$$

$$= 2\sqrt{3}\sin^2\theta \begin{pmatrix} 0 & \cos 0 & -\cos\theta\cos^2\phi \\ & 0 & \sin 9 \sin\phi \cos\phi \\ \cdots\ \cdots & & \\ \cdots\ \cdots & & 0 \end{pmatrix},$$

$$co = 2\sqrt{3}\sin\theta(\cos 0\, d\theta_\wedge d\chi_1 - \cos 0\cos^2\phi\, d\theta_\wedge d\chi_2 + \sin 0 \sin\phi\cos\phi\, d\phi_\wedge d\chi_2). \qquad (5.24)$$

Upto a numerical factor this is just what appears in the integrand of the expression (5.21) for $\varphi_g[C]$ when $\mathscr{C}$ is closed. Thus for such cyclic evolution we have the result

$$\varphi_g[C] = \frac{1}{\sqrt{3}}\int_{S\subset\mathscr{R}_3,\,\partial S = C} co \qquad (5.25)$$

establishing the connection we had sought.

6. CONCLUDING REMARKS

We have presented a detailed analysis of geometrical and dynamical phases that can arise within general representations of the group *SU(3).* This has involved compiling a complete list of connected Lie subgroups of $SU(3)$ upto conjugation. We have found one discrete infinite family of inequivalent $U(1)$ subgroups, and five other possible subgroups. Our final results seem to be more simple than may have been anticipated: nonzero dynamical phases can arise only in two of these cases, namely when $H_0 = U_{(p,q)}(1)$ and $SU(2)$. This is of course apart from the generic case with a trivial stability group H_0. Again, excluding this case when all eight Maurer Cartan one-forms can contribute to φ_{dyn}, it is interesting that whenever we have a non zero dynamical phase we need only work with the pullbacks of the one-forms $(q+p)\hat{\theta}_3^{(0)} + ((q-p)/\sqrt{3})\hat{\theta}_8^{(0)}$ or $\hat{\theta}_8^{(0)}$ from $SU(3)$ to the coset space. The case of three level quantum systems falls under case (d) of our classification with $H_0 = SU(2)$ and $H = U(2)$. Our treatment of this problem has led to the correct generalisation of the Poincare sphere method for two level systems, and the very often quoted result linking geometric phase to the solid angle on S^2 for two level systems.

It is interesting to compare these results with another problem in which again the coset spaces $SU(2)/H_0$ play an important role, namely the classification of nonabelian $SU(3)$ monopoles [14]. Here one has a (classical) nonabelian $SU(3)$ gauge theory along with a Higgs field multiplet Φ belonging to a suitable representation of $SU(3)$. The self interaction of Φ leads to spontaneous symmetry breakdown. In this situation, under suitable conditions, the manifold of values of Φ minimising the Higgs potential and leading to spontaneous symmetry breakdown is a coset space $\mathscr{M}_0 = SU(3)/H_0$, where H_0 is the Lie subgroup of $SU(3)$ leaving invariant a chosen configuration $\Phi_0 \in \mathscr{M}_0$. The topological classification of distinct monopole types is a classification of ways in which Φ maps spatial infinity, a sphere S^2, into $\mathscr{M}_0$; thus we are concerned with $\Pi_2(\mathscr{M}_0) = \Pi_2(SU(3)/H_0)$. Since $SU(3)$ is simply connected, $\Pi_2(SU(3)/H_0) = \Pi_1(H_0)$, the fundamental group of H_0. Thus nontrivial monopoles can arise only when $\Pi_1(H_0)$ is nontrivial, i.e. H_0 is multiply connected. Going through the list of possible H_0's in Table 1, we can see that in each of the cases $H_0 = U_{(p,q)}(1)$, $U(1) \times U(1)$, $U(2)$ and $SO(3)$ we in principle have nontrivial nonabelian $SU(3)$ monopoles. From the geometric phase viewpoint, however, we have seen that nonzero dynamical phases arise only in the two cases $H_0 = U_{(p,q)}(1)$ and $SU(2)$ (apart from the generic case $H_0 = \{e\}$). The only common case is thus $H_0 = U_{(p,q)}(1)$.

Our results indicate that geometric phases for $SU(n)$ can be handled similarly in a recursive manner without too much trouble.

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