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On the Theory of Point-Particles

H. J. Bhabha and Harish-Chandra

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Example 3

10. Figures 9 and 10 (corresponding with figures 7 and 8) show the same problem solved for a stress system which is partly plastic, partly elastic, so that *the contours exhibit 'refraction' at a plastic-elastic interface*. The stress system is that induced by tension applied along the horizontal centre-line of a symmetrical specimen having two semicircular notches which form a 'waist'. The computed stress-components, being based on a particular hypothesis regarding plastic strain, are open to question; but this is of no importance to the present paper, which is concerned solely with the graphical representation of a *specified* stress.

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On the theory of point-particles

By H. J. Bhabha, F.R.S. and Harish-Chandra

Cosmic Ray Research Unit, Indian Institute of Science, Bangalore

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It is deduced from the conservation of the energy-momentum tensor that if the flow of energy and momentum into a tube surrounding a time-like world-line, on which the field is singular, become singular as the size of the tube is contracted to zero, then the singular terms are necessarily perfect differentials of quantities on the world-line with respect to the proper time along the world-line. The same can be proved of any other tensor, as, for example, the angular-momentum tensor, which is conserved. It is proved from this that for any *point*-particle whatever having charge, spin or other properties, which need not be specified, it is always possible to deduce exact equations of motion which are finite.

It is proved further that if the energy-momentum tensor is altered by the addition of $\partial K^{\mu\nu\sigma}/\partial x^{\sigma}$, where $K^{\mu\nu\sigma}$ is any tensor antisymmetric in ν and σ, then the equations of motion are unaltered, but it is possible to choose $K^{\mu\nu\sigma}$ in such a way as to make the flow of energy and momentum into a given tube non-singular.

By a point-particle is understood a particle whose field-producing and inertial properties are all located at a point. The particle may have a finite charge and a finite mass, but the charge density and mass density are exactly zero at every point of space other than the point at which the particle is located at that instant of time. The motion of the particle through space-time is therefore described by a time-like world-line. If the particle possesses a dipole or a higher multipole moment, then this is described by a suitable co-ordinate having a given value at each point of the world-line, but it is again assumed that the dipole density is exactly zero at all points of space-time not lying on the world-line. Thus the motion of the particle through space-time is completely described by a time-like world-line with the values of the co-ordinates describing the spin and other properties of the particle given at each point of it.

It has been shown by Dirac (1938) that an exact theory of a point electron moving in an electromagnetic field can be set up free from singularities, and Bhabha (1940) and Bhabha & Corben (1941) have shown that a similar theory free from singularities can be set up for a point dipole. It has been shown further by one of us (Bhabha 1939, 1941) that the theory can be extended to point-particles interacting with meson fields. These cover all the cases of practical interest, but a general demonstration that it is always possible to set up an exact classical theory for point-particles interacting with any field has not so far been given. This will be done in the present paper.

In all the work referred to above, the method used for finding the equations of motion of the point-particle is the same. From the field equations we calculate the field produced by the point-particle, or more exactly, we take that solution of the *homogeneous* field equations which has a singularity of the required type on the world-line. We now surround a finite length of the world-line by a world tube whose radius is ultimately made to tend to zero, and calculate the flow of energy, momentum and angular momentum into the world tube from outside by using the usual energy-momentum and angular-momentum tensors of the field. For brevity we shall refer to the quantities so calculated as the inflow. The equations of motion are now found from the condition that conservation of energy, momentum and angular momentum require that the flow of all these quantities into the tube must only depend on conditions at the two ends of the tube, that is, it must only be a function of the co-ordinates of the particle and their higher derivatives and also possibly of the field quantities at the two ends of the tube. Since the field is singular on the world-line, the usual energy-momentum tensor is also singular, and in consequence the flow of energy and momentum into the tube is likewise singular. That in spite of this it has always been possible to derive finite equations for the motion of the point-particles has always depended on the circumstance that the singular parts of the inflow over an infinitesimal length of the tube are always perfect differentials. We shall prove that this is a general property which is a consequence only of the conservation of the energy-momentum tensor and therefore that finite equations for the motion of a point-particle of any type whatsoever can always be derived.

We shall also show that it is always possible to modify the energy-momentum tensor by the addition of the divergence of a tensor of higher rank in the manner suggested by Pryce (1938) so that the inflow over a tube of given shape becomes finite. But the inflow would not necessarily be finite over a tube of any other shape. That it is always possible to modify the tensor in the same way as to make finite the inflow over tubes of any arbitrary shape, as also the energy and momentum integrals over arbitrary space-like surfaces is shown by one of us (H.-C.) in the paper which immediately follows this.

1. We use the same notation as in the previous papers by one of us (H.J.B.). We take the metric tensor to have the form $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$ with $g_{\mu\nu} = 0$ for $\mu \neq \nu$. The world-line of the particle is described by four co-ordinates $z^\mu(\tau)$ which are functions of the proper time τ measured from some point along it. The other co-ordinates of the particle, if any, need not be specified for the present work. A dot denotes differentiation with respect to τ, and $V^\mu = \dot{z}^\mu$ is used to denote the velocity of the particle. x^μ denotes a point of space, $s^\mu \equiv x^\mu - z^\mu(\tau_0)$ the distance from any point of space to the retarded point τ_0 on the world-line defined by

$$s_\mu s^\mu = 0. \tag{1}$$

If τ_0 be kept fixed then equation (1) is also the equation of the light cone whose apex is at τ_0. We further introduce a quantity κ as a function of the co-ordinates x^μ defined by

$$\kappa \equiv s_\mu v^\mu(\tau_0). \tag{2}$$

The energy-momentum tensor is denoted by $T^{\mu\nu}$. It satisfies the conservation equation

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0. \tag{3}$$

It is necessary to find the generalization of Gauss's theorem to four-dimensional hyperbolic space. Let

$$\zeta^0(x^\mu) = \alpha \tag{4}$$

be a closed three-dimensional surface S surrounding a four-dimensional volume V such that the surface $\zeta^0(x^\mu) = \alpha - \Delta\alpha$ for positive $\Delta\alpha$ is contained inside (4) and lies wholly in the volume V. Let ζ^1, ζ^2 and ζ^3 be parameters defining the position of a point on the surface S. Then, if X^μ be any tensor, the generalization of Gauss's theorem reads

$$\int_V \frac{\partial X^\nu}{\partial x^\nu} dx^0 dx^1 dx^2 dx^3 = \int_S X^\nu \frac{\partial \zeta^0}{\partial x^\nu} |D|^{-1} d\zeta^1 d\zeta^2 d\zeta^3, \tag{5}$$

where D is the determinant of the transformation from the x's to the ζ's and the surface is covered in such a way that $d\zeta^1$, $d\zeta^2$, $d\zeta^3$ are always positive:

$$D \equiv \begin{vmatrix} \dfrac{\partial \zeta^0}{\partial x^0} & \dfrac{\partial \zeta^1}{\partial x^0} & \dfrac{\partial \zeta^2}{\partial x^0} & \dfrac{\partial \zeta^3}{\partial x^0} \\ \dfrac{\partial \zeta^0}{\partial x^1} & \dfrac{\partial \zeta^1}{\partial x^1} & \dfrac{\partial \zeta^2}{\partial x^1} & \dfrac{\partial \zeta^3}{\partial x^1} \\ \dfrac{\partial \zeta^0}{\partial x^2} & \dfrac{\partial \zeta^1}{\partial x^2} & \dfrac{\partial \zeta^2}{\partial x^2} & \dfrac{\partial \zeta^3}{\partial x^2} \\ \dfrac{\partial \zeta^0}{\partial x^3} & \dfrac{\partial \zeta^1}{\partial x^3} & \dfrac{\partial \zeta^2}{\partial x^3} & \dfrac{\partial \zeta^3}{\partial x^3} \end{vmatrix} \tag{6}$$

$\partial\zeta^0/\partial x^\mu$ is the normal to the surface S. For a displacement $(\partial\zeta^0/\partial x^\mu)\Delta l$ in the direction of this normal with positive Δl the change $\Delta\zeta^0$ of ζ^0 is

$$\Delta\zeta^0 = \frac{\partial\zeta^0}{\partial x^\mu}\frac{\partial\zeta^0}{\partial x_\mu}\Delta l.$$

This is positive if $\partial\zeta^0/\partial x^\mu$ is a time-like vector and negative if it is space-like. Thus on the space-like portions of the surface the normal to the surface on the right-hand side of (5) is directed *outwards*, and on the time-like portions it is directed *inwards*. By writing $dS_\nu = \dfrac{\partial\zeta^0}{\partial x^\nu}\,|\,D\,|^{-1}\,d\zeta^1 d\zeta^2 d\zeta^3$ for an element of the surface S with its normal directed in the sense defined above, the right-hand side of (5) can be written

$$\int X^\nu dS_\nu. \tag{7}$$

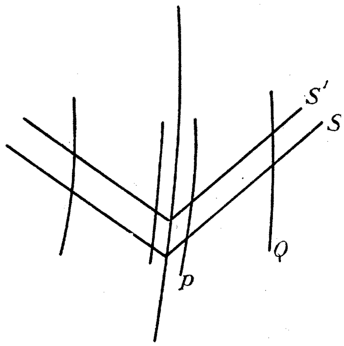

Figure 1

Consider a portion of the world-line between the points τ_1 and τ_2 and surround it by a world-tube of any arbitrary shape. We consider the ends of the tube to be given by the two two-dimensional surfaces where it intersects the future light cones from the points τ_1 and τ_2 respectively. The flow of energy and momentum into the tube is then given by

$$\int_{\tau_1}^{\tau_2} T^{\mu\nu} dS_\nu, \tag{8}$$

the normal to the surface being directed outwards. For conservation, (8) must be put equal to a function of the form $A^\mu(\tau_2) - A^\mu(\tau_1)$, where $A^\mu(\tau)$ is a function of v^μ and the other co-ordinates of the particle, possibly also of the field quantities, and their higher derivatives at the point τ only. On differentiating this equation

$$\frac{d}{d\tau}\int_{\tau_1}^{\tau} T^{\mu\nu} dS_\nu = \dot{A}^\mu(\tau). \tag{9}$$

This is not a mathematical identity but furnishes the equations of motion of the point-particle. It should be noted at once that the equations derived in this way are independent of the shape of the tube. For if the given tube is surrounded by another of some other shape, then it follows from (3) and (5) that the difference in the integral (8) over the two tubes is just equal to the integral over the portions of the light cones at τ_1 and τ_2 intercepted between the two tubes. In the limit when the size of both tubes is made to tend to zero, the integrals over the light cones just become functions of the conditions at τ_1 and τ_2 respectively, and hence the difference in (8) calculated over two tubes of different shape is identically of the form $A'^{\mu}(\tau_2)-A'^{\mu}(\tau_1)$. This difference can be absorbed into the right-hand side of (9), and hence it does not affect the equation of motion which can be so derived. One may therefore work with whatever tube is most convenient.

Consider now a volume V lying between the two light cones S and S' starting from the points τ' and $\tau'+d\tau'$ respectively, and between the two tubes P and Q defined by $\kappa=\epsilon$ and $\kappa=\eta$, as shown in figure 1. The boundary of V so defined can be characterized by the equation $\zeta^0=0$, where ζ^0 is a discontinuous function chosen in the following way:

$$\zeta^0 = \begin{cases} \tau-(\tau'+d\tau') & \text{on } S', \\ \kappa-\eta & \text{on } Q, \\ \epsilon-\kappa & \text{on } P, \\ \tau'-\tau & \text{on } S. \end{cases}$$

Since (3) holds inside this volume V it follows from (5) that

$$\int_{\tau=\tau'+d\tau'} T^{\mu\nu}dS_\nu - \int_{\tau=\tau'} T^{\mu\nu}dS_\nu = \int_{\kappa=\epsilon} T^{\mu\nu}dS_\nu - \int_{\kappa=\eta} T^{\mu\nu}dS_\nu, \tag{10}$$

the normals to the light cones being taken in the direction of increasing τ in both cases, and towards the world-line on the two world-tubes. To evaluate the second integral on the left it is convenient to take $\zeta^1=\kappa$, $\zeta^2=s^2$ and $\zeta^3=s^3$ in addition to $\zeta^0=\tau$. As shown in the previous papers, it follows from (1) and (2) that

$$\frac{\partial\tau}{\partial x^\mu} = \frac{s_\mu}{\kappa}, \tag{11}$$

$$\frac{\partial\kappa}{\partial x^\mu} \equiv \kappa_\mu = v_\mu - \frac{s_\mu}{\kappa}(1-\kappa'), \quad \text{where} \quad \kappa' \equiv \dot{v}_\mu s^\mu, \tag{12}$$

$$\frac{\partial s^2}{\partial x^\mu} = \delta^2_\mu - v^2\frac{s_\mu}{\kappa}, \tag{13}$$

$$\frac{\partial s^3}{\partial x^\mu} = \delta^3_\mu - v^3\frac{s_\mu}{\kappa}. \tag{14}$$

The right-hand sides of (11)–(14) have to be inserted as the columns of the matrix on the right of (6). Since the determinant of a matrix vanishes if it has any two columns identical, it follows that we may omit all except the first terms on the right-hand sides of (11)–(14) and get

$$D = \frac{s_0 v_1 - s_1 v_0}{\kappa}. \tag{15}$$

Therefore

$$\int_{\tau=\text{const.}} T^{\mu\nu} dS_\nu = \iiint_\epsilon^\eta T^{\mu\nu} \frac{s_\nu}{\kappa} \left| \frac{\kappa}{s_0 v_1 - s_1 v_0} \right| ds^2 ds^3 d\kappa = \iint_\epsilon^\eta T^{\mu\nu} \frac{s_\nu}{\kappa} d\Omega \, d\kappa, \tag{16}$$

$d\Omega$ being the element of surface of a sphere of radius κ in the 'rest system' in which $v_0 = 1$, $v_1 = v_2 = v_3 = 0$. Similarly,

$$\int_{\kappa=\epsilon} T^{\mu\nu} dS_\nu = \int d\tau \int_{\kappa=\epsilon} T^{\mu\nu} \kappa_\nu d\Omega. \tag{17}$$

(10) now becomes
$$\frac{d}{d\tau}\left\{\int_\epsilon^\eta \int_\kappa T^{\mu\nu} \frac{s_\nu}{\kappa} d\Omega \, d\kappa\right\} = \int_{\kappa=\epsilon} T^{\mu\nu} \kappa_\nu d\Omega - \int_{\kappa=\eta} T^{\mu\nu} \kappa_\nu d\Omega. \tag{18}$$

It is important to note that the left-hand side of (18) is *not* a perfect differential. The integral does not depend in general only on conditions on the world-line at the point τ. Differentiating this equation with respect to ϵ, we get

$$\frac{d}{d\kappa} \int_\kappa T^{\mu\nu} \kappa_\nu d\Omega + \frac{d}{d\tau} \int_\kappa T^{\mu\nu} \frac{s_\nu}{\kappa} d\Omega = 0. \tag{19}$$

As $\kappa \to 0$ the second term in (19) becomes a perfect differential, and hence if there are any singular terms in the first term of (19) these must be perfect differentials and be compensated identically by those in the second term. (19) is the exact statement that the singular part of the inflow is a perfect differential. In case the energy tensor can be expanded as a series in ascending powers of κ for sufficiently small values of κ starting with some negative power of κ, then (19) shows that all the terms except the term independent of κ in the inflow must be perfect differentials.*

The singular terms being perfect differentials can always be compensated by the addition of suitable terms to A^μ in (9) and therefore *play no part in determining the motion of the point-particle.* The term independent of κ is not a perfect differential, and by putting it equal to a suitable perfect differential as in (9) the equation of translational motion of the point-particle is obtained. It is seen that to determine the equation of a point-particle it is only necessary to calculate that part of the inflow which remains finite as $\kappa \to 0$. The rest can be ignored.

It is clear that the same argument holds if two of the boundaries of the volume V are taken as any two surfaces passing through the points τ and $\tau + d\tau$ instead of the light cones. The argument also holds for any other tensor which satisfies a conservation law of the type (3), and therefore for the angular-momentum tensor of the field. In finding the rotational equations of the particle it is therefore only necessary to calculate the finite part of the inflow of angular momentum and to put this equal to a suitable perfect differential.

2. Let $X^{\mu\nu}$ be a tensor antisymmetric in μ and ν. Let us suppose that the coordinates ζ^0, ζ^1, ζ^2 and ζ^3 of the last section are so chosen that $\zeta^1 = \beta$ with constant β

* This result is implicit in Mathisson's paper (1940) and can be derived from his variational equations, but it is much more cumbersome and indirect to prove it by his method.

represents a closed two-dimensional surface ω forming the boundary of a portion S' of the three-dimensional surface $\zeta^0 = \alpha$, and assume that $\zeta^1 = \beta - \Delta\beta$ ($\Delta\beta$ positive) is a two-dimensional surface lying entirely within the three-dimensional surface so enclosed. Then it is well known that

$$\int_{S'} \frac{\partial X^{\nu\sigma}}{\partial x^\sigma} dS_\nu \equiv \int_{S'} \frac{\partial X^{\nu\sigma}}{\partial x^\sigma} \frac{\partial \zeta^0}{\partial x^\nu} | D |^{-1} d\zeta^1 d\zeta^2 d\zeta^3 = \int_\omega X^{\nu\sigma} \frac{\partial \zeta^0}{\partial x^\nu} \frac{\partial \zeta^1}{\partial x^\sigma} | D |^{-1} d\zeta^2 d\zeta^3. \quad (20)$$

In this formula let us insert a tensor $K^{\mu\nu\sigma}$ antisymmetric in ν and σ in place of $X^{\nu\sigma}$, and take for the surface ζ^0 the surface of the tube $\kappa = \epsilon$ between the points τ_1 and τ_2. It is convenient in this case to take as the variables ζ, $\zeta^0 = \kappa$, $\zeta^1 = \tau$, $\zeta^2 = s^2$, $\zeta^3 = s^3$, and we obtain, as in the last section, remembering (11)–(15),

$$\int_{\tau_1}^{\tau_2} \int \frac{\partial K^{\mu\nu\sigma}}{\partial x^\sigma} \kappa_\nu d\Omega\, d\tau = \int_\kappa K^{\mu\nu\sigma} \kappa_\nu \frac{s_\sigma}{\kappa} d\Omega \Big|_{\tau_1}^{\tau_2}$$

$$= \int_\kappa K^{\mu\nu\sigma} v_\nu \frac{s_\sigma}{\kappa} d\Omega \Big|_{\tau_1}^{\tau_2}, \quad (21)$$

on account of the antisymmetry of $K^{\mu\nu\sigma}$ in ν and σ. Differentiating this with respect to τ,

$$\int_\kappa \frac{\partial K^{\mu\nu\sigma}}{\partial x^\sigma} \kappa_\nu d\Omega = \frac{d}{d\tau} \int_\kappa K^{\mu\nu\sigma} v_\nu \frac{s_\sigma}{\kappa} d\Omega. \quad (22)$$

The inflow of $\partial K^{\mu\nu\sigma}/\partial x^\sigma$ over an infinitesimal portion of a world-tube of any arbitrary shape can similarly always be shown to be identically a perfect differential. Hence, if the energy-momentum tensor $T^{\mu\nu}$ is replaced by a new one $T'^{\mu\nu}$ defined by

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{\partial K^{\mu\nu\sigma}}{\partial x^\sigma}, \quad (23)$$

which obviously also satisfies a conservation equation like (3), then the inflow is only altered by the addition of perfect differentials, and these do not alter the equations of motion since they can be compensated by the addition of suitable terms to A^μ in (9).

Denote by $\dot{B}^\mu$ that part of the inflow, calculated by using the original tensor $T^{\mu\nu}$, which is a perfect differential. B^μ depends only on the variables at τ. Then taking

$$K^{\mu\nu\sigma} = \frac{1}{4\pi} \frac{B^\mu}{\kappa^3} (v^\nu s^\sigma - v^\sigma s^\nu) \quad (24)$$

and inserting it into (22), we get, remembering (1), (2),

$$\int_\kappa \frac{\partial K^{\mu\nu\sigma}}{\partial x^\sigma} \kappa_\nu d\Omega = -\dot{B}^\mu. \quad (25)$$

Since it has been shown that the singular parts of the inflow are always perfect differentials they can be included in B^μ, and it has therefore been proved that it is always possible to find a tensor $K^{\mu\nu\sigma}$ such that the inflow calculated from the modified tensor (23) is finite over the particular world-tube chosen.

The tensor (24) is by no means uniquely defined. Thus

$$K^{\mu\nu\sigma} = \frac{1}{4\pi}\frac{B^\mu}{\kappa^3}e^{-b\kappa^n}(v^\nu s^\sigma - v^\sigma s^\nu) \tag{26}$$

could equally well have been taken, with b a positive constant and n a number greater than the highest order of singularity in B^μ. This tensor would in fact have the advantage over (24) that its integral over any surface at infinity would always vanish.

The integral of the modified tensor over any arbitrary three-dimensional surface is not necessarily finite, nor is the inflow finite over a tube of any other shape than the one chosen above. The general properties of the energy tensor $T^{\mu\nu}$ are investigated in more detail in the paper by one of us (H.-C.) which follows this, and it is shown there that it is always possible to find a tensor $K^{\mu\nu\sigma}$ such that the modified tensor has no singularities of order higher than the third.

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