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Pedagogy

On the mass-radius relation for neutron stars

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Abstract. Classical equation of state for a nonrelativistically degenerate gas containing free fermions gives an upper limit to the radius of a neutron star while the Schwarzschild radius gives the lower limit. The lower mass limit for a neutron star is given by the Oppenheimer-Volkoff solution. These three curves define a triangular region in the mass-radius plane which is available for a real neutron star. If we represent the equation of state for interacting neutrons by a polytropic relation we can delimit the region where neutron stars would be found. Realistic values of the polytropic index give a mass between 1.8 and 2.8 $M_{\odot}$ and radius between 10 and 11 km for the neutron star.

1. Introduction

It is well known that white dwarfs, neutron stars, and black holes represent the ultimate remanants of evolved stars which have exhausted all their possible nuclear fuels. The structure of white dwarfs was considered by Chandrasekhar (1935) who gave a mass-radius relation for them and obtained the limiting mass for a helium white dwarf. The mass-radius relation for an elementary black hole was obtained earlier by Schwarzschild (1916) as a solution of the general relativistic equations for an isolated point mass. The mass-radius relation for neutron stars containing free neutrons was derived by Oppenheimer & Volkoff (1939) again as a solution of the general relativistic equations. Since then many attempts have been made for obtaining the limiting mass of a neutron star (*cf.* Cox & Giuli 1968) by taking into account the interactions between neutrons by an appropriate equation of state. But there has been no definite answer to this question so far. We present here a simplistic approach by skirting around the problem of a correct equation of state for the neutron gas.

2. Delimitation in the mass-radius plane

At lower densities ($\leqslant 10^{14}$ gm cm^{-3}) the neutrons which can be considered as free particles are nonrelativistically degenerate. In this case the matter behaves like a polytrope of index $n = 1.5$. Now for a white dwarf containing nonrelativistically degenerate electrons we have the relation (Chandrasekhar 1931)

$$M\mu_e^2 = \left[\frac{h^2}{3.85\ Gm}\right]^3 \left(\frac{3}{8\pi}\right)^2 \left(\frac{17.97}{4\pi m_H}\right)^5 \left(\frac{1}{R\mu_e}\right)^3 \quad \ldots(1)$$

Replacing m, the mass of the electron, by m_n, mass of the neutron, and putting $\mu_e = 1$ we obtain for neutron stars the relation

$$R = \frac{15.46}{(M/M_\odot)^{1/3}}\ \text{km}. \quad \ldots(2)$$

The curve marked 'NR' in figure 1 represents this relation; it represents an upper limit to the radius of a neutron star of a given mass. In the same figure the curve marked $R = R_s$ corresponds to the Schwarzschild radius for a black hole given by

$$R_S = \frac{2GM}{c^2} = 2.96\ \frac{M}{M_\odot}\ \text{km}. \quad \ldots(3)$$

This curve represents the lower limit for the radius of a neutron star of a given mass. It is seen that the two curves cross at the point where $M = 3.4\ M_\odot$ and $R = 10.0$ km; hence the mass of the neutron star must be less than $3.4\ M_\odot$.

So far we have neglected two effects : (i) We may not have complete degeneracy throughout the star; (ii) the short range forces between neutrons will make the star more compact. The first effect was taken into account by Oppenheimer & Volkoff (1939); their mass-radius relation for neutron stars is shown by the curve marked 'OV' in figure 1. It gives a maximum mass of $0.71\ M_\odot$ with a radius of 9.5 km.

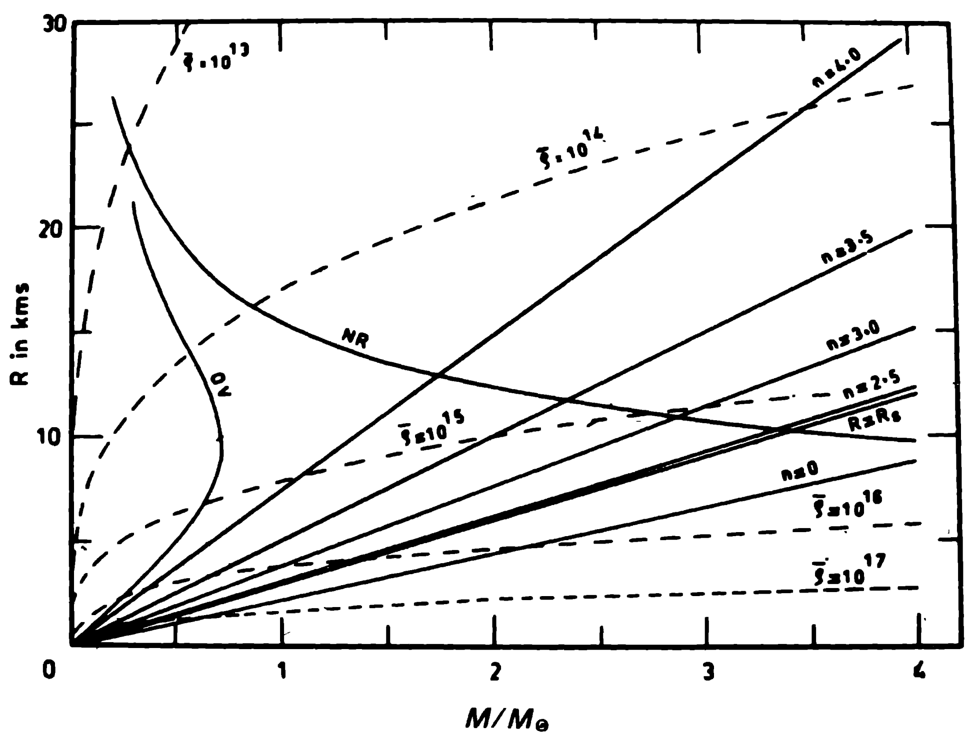

Figure 1. Mass-radius relation for neutron stars.

The real situation lies somewhere in the region delimited by the three curves marked 'NR', 'R = Rs' and 'OV'.

3. Use of polytropic configurations

The mutual interaction between neutrons gives rise to an equation of state for the neutron gas. Let us represent its effect by a polytropic configuration with $n > 1.5$. Now for a polytrope of index n the central pressure and density are given by

$$P_c = a_n \frac{GM^2}{R^4}, \quad \rho_c = b_n \frac{3M}{4\pi R^3}, \qquad \ldots(4)$$

were M is the mass and R the radius of the star. We are particularly concerned with the structure of a star at high density when the neutrons would become relativistically degenerate at the centre. In that case, since both the pressure and density are contributed by the neutrons, we can write

$$P_c = \frac{1}{3} U_c = \frac{1}{3} \rho_c c^2. \qquad \ldots(5)$$

Substituting in equation (4) we get

$$R = \frac{4\pi MG}{c^2} \frac{a_n}{b_n} = 18.62 \frac{a_n}{b_n} \frac{M}{M_\odot} \text{ km}. \qquad \ldots(6)$$

The mass-radius relations given by equation (6) for various values of n are also shown in figure 1. It is evident that $n \geqslant 2.5$ in order to avoid a black hole configuration. The classical value of n for relativistically degenerate gas of free particle is 3 which gives a maximum mass of 2.86 $M_\odot$ with $R = 10.2$ km. A higher central condensation corresponding to a more realistic value of $n = 4$ would give a limiting mass of 1.75 $M_\odot$ with $R = 10.6$ km.

Hence we conclude that the maximum mass of a neutron star would lie between 1.8 and 2.8 $M_\odot$ with radius close to 10.5 km.

References

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