PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Proc. R. Soc. Lond. A 1938 **166**, 501-528
doi: 10.1098/rspa.1938.0107

probably an electron, although 4 or 5 secondaries would on the average be expected from an electron of this energy. The most energetic particle in the photon-produced shower has an energy of about 10^8 e-volts.

FIG. 12. Production of a fast secondary in the gas of the chamber (argon + oxygen). The primary particle is very energetic ($E \sim 10^{10}$ e-volts) and the (negative) secondary has an energy of 6×10^6 e-volts.

Magnification, figs. 7–11, $\times 0{\cdot}45$; fig. 12, $\times 0{\cdot}65$. Magnetic field, 10,000 gauss. Figs. 7–10, 2 cm. copper plate, $l = 1{\cdot}6\,\lambda_0$; fig. 11, 1 cm. lead plate, $l = 2{\cdot}5\,\lambda_0$.

On the theory of heavy electrons and nuclear forces

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(*Communicated by R. H. Fowler, F.R.S.—Received* 28 *February* 1938)

The view has been expressed by several authors, including Neddermeyer and Anderson (1937), and Street and Stevenson (1937 *a*, *b*), that the penetrating component of cosmic radiation consists largely of new particles of electronic charge and mass intermediate between those of the electron and proton, and I have shown in a recent paper (Bhabha 1938 *a*) that the shape of the transition curve for bursts and the latitude effect at sea level, which are experimentally well-established facts, also necessitate such a particle and are not compatible with a breakdown of the theory for electrons of very high energy. It has further been shown in the same paper that it does not seem to be sufficient just to postulate another particle behaving exactly like an electron of larger mass, but that the experimental evidence demands further that under certain circumstances a single heavy electron must be able to change its rest mass in the absence or presence of particles constituting ordinary matter. Indeed, the energy loss measurements of Blackett and Wilson indicate that most particles below about 2×10^8 e-volts are electrons, whereas most particles above this energy are heavy electrons, so that at about this energy there must be a large probability of a heavy electron changing or losing its identity.

Now since one may assume that charge is always conserved, it follows that there are essentially only two ways in which a single heavy electron may disappear. If, for example, it has a negative charge, it may collide with a proton and communicate its charge to it, the proton changing into a neutron, or it may turn into an ordinary electron by changing its rest mass. In either of these two processes, a certain amount of energy is liberated and the spin and statistics to be attributed to the heavy electron depend on whether the liberation of this energy is accompanied by the simultaneous liberation of some particle having a half integral spin and obeying Fermi statistics or not. Unfortunately, so far there is no experimental evidence upon this point.

Now it was shown some time ago by Yukawa (1935) that the short range and magnitude of the proton-neutron exchange interaction could be explained satisfactorily by assuming the existence of a positively charged particle without spin which could be emitted and absorbed by the change of a proton into a neutron and vice versa. The proton and neutron are here considered as two states of the same particle.* The exchange interaction thus results from the virtual exchange of such a particle between a neutron and a proton and its dependence on the separation r of the neutron and proton is of the form $e^{-\lambda r}/r$, where λ is connected with the mass M_u of the new particle by the relation $\lambda = M_u c/\hbar$. To fit the fact that nuclear forces have a range of the order of 10^{-13} cm. Yukawa then deduced that the mass of this particle had to be of the order of a hundred times the electron mass. But this is just the order of magnitude of the mass which must be attributed to the heavy electrons in cosmic radiation, as was deduced from experiment in the paper mentioned above. This value of the mass also seems to be in agreement with what has been observed by Street and Stevenson (1937 *b*), and Nishina, Takeuchi and Ichimiya (1937) in Wilson chamber experiments, notwithstanding the uncertainty that attaches to these due to the extreme rarity of tracks of the required type.

It was further assumed by Yukawa that the new particle had a similar interaction with the electron and neutrino, being capable of being absorbed when an electron turned into a neutrino. On this basis it was shown by him that his theory would practically lead to the same results for the β-disintegration as the original Fermi theory. It must be mentioned, however, that in its present form it does not seem capable of explaining the asymmetrical spectrum of some of the β-decay elements. Moreover, on this theory it is possible for the new particle to disintegrate spontaneously into a positron and a neutrino, so that the possibility of identifying such particles with the heavy electrons of cosmic radiation is still open.

* The possible existence of a Bose particle was also suggested by Stueckelberg (1937). Kemmer (1938) has recently investigated nuclear forces on these lines, and Fröhlich and Heitler (1938) the magnetic moments of the neutron and proton.

A theory of the β-decay and nuclear forces in which spinless particles obeying Bose statistics are assumed to exist has also been proposed by Wenzel (1937 *a*, *b*). On this theory both the proton and neutron can change into both charged or uncharged particles without spin by the emission or absorption of an electron or a neutrino, whichever is consistent with the conservation of charge.* This theory seems to be so far capable of explaining the general features of the β-decay and nuclear forces. The Wenzel theory however has the great disadvantage from our point of view that the new particles must have masses near those of the proton or neutron, which in our opinion definitely excludes their being identified with heavy electrons. Nor is it possible to reformulate the theory in such a way as to allow these particles to have masses corresponding to that of the heavy electron (about a tenth of the proton mass) for this would make all nuclei unstable, inasmuch as every proton, for example, could disintegrate into a positive "Bose" particle and a neutrino.

The theory of Yukawa therefore has in our opinion the following advantages over other theories. It separates the theory of nuclear forces from that of the β-decay in such a way that the former only depends on the interaction of the proton and neutron with the heavy electron, while the latter also depends on its interaction with the electron and neutrino, so that by making the former interaction strong and the latter weak it is possible to explain the large magnitude of nuclear forces and the weakness of the β-decay. In all other theoretically self-consistent theories which have been put forward so far the two phenomena have been directly connected, so that any interaction weak enough to give a β-decay of the right magnitude has always given nuclear forces which were too small. It gives, then, an explanation of the proton-neutron exchange interaction which in non-relativistic approximation takes the form $e^{-\lambda r}/r$, where λ is connected directly with the mass of the new particle, whereas in the Fermi and Wenzel theories the force falls away inversely as a more or less high power of the distance, which it is difficult to believe is the correct description of such a fundamental thing as the force between a proton and neutron. In consequence it demands the existence of a particle of about a hundred times the electron mass, that is, of the same type as the heavy electron required in cosmic radiation. It is capable of explaining the β-decay to the same extent as the Fermi theory though not the Wenzel theory, and finally, it allows various processes which may play some role in cosmic radiation, as

* The part which the collision of such particles with nuclei may play in creating showers has also been mentioned by Professor Wenzel at the Congres du Palais de la Decouverte, Paris (1937).

for example the absorption of heavy electrons by one of the particles in a nucleus, and their spontaneous disintegration into an electron and a neutrino. Since the original theory of Yukawa was not in a relativistically invariant form, it seems worth while to find a proper relativistically invariant formulation and generalization of it so that its consequences may be investigated in detail with special reference to the cosmic ray phenomenon.*

Following Yukawa then, the proton and neutron are considered as two states of the same particle and we assume that a proton may change into a neutron by the emission of a charged particle of mass M_u which we shall call a U-particle. It follows at once that this U-particle must have an integral spin and obey Einstein Bose statistics. A consequence of this is that the Hamiltonian for the free U-particle must be a positive definite form, since if this were not so, it would have eigenvalues corresponding to states of negative energy, which could not now be avoided by an assumption similar to that made by Dirac in the theory of the positron. Under these circumstances, as will appear below, the choice of a Hamiltonian for a free particle is very limited. A relativistic formulation of Yukawa's original theory in which the motion of the heavy electron is described by a scalar function leads just to the problem treated by Pauli and Weisskopf, but does not give the correct form of nuclear forces.† The simplest generalization is therefore to describe the motion of U-particles by four wave-functions transforming like a four-vector. This is the case that is here treated in detail, and it will appear below that the possible forms of the interaction of the heavy electron with the proton-neutron are very limited and contain only two arbitrary constants. This treatment already gives the forces we require, so that we have not investigated equations of higher tensor form.‡

It is obvious that the U-particles, being charged, can never lead to a proton-proton force except as a fourth order process, corresponding to the emission of U-particles by each of two protons and their absorption by

* I am very much indebted to Dr W. Heitler for a discussion on cosmic radiation in which he drew my attention to the theory of Yukawa. This discussion formed the starting point of the considerations of this paper.

† Since these calculations were completed, a new paper has appeared by Yukawa and Sakata (1937) in which just this case has been treated. As their results show, the theory gives a Heisenberg exchange force giving only a stable singlet state to the deuteron, which is contrary to observation.

‡ Taking the wave functions to be antisymmetrical in all the suffices, these reduce in essence to the cases dual to those treated by Yukawa and ourselves. An analysis of all the various cases in relation to nuclear forces has been carried out by Dr N. Kemmer, and I am indebted to him for informing me that all the other cases except the one treated here also give the wrong nuclear forces.

the other. Now the proton-proton force as deduced by Breit, Condon and Present (1936) from experiments by Tuve, Heydenberg and Hafstad (1936) appears to be the same as the neutron-proton force to within about 30 %. In order to have a proton-proton interaction of the same order as the proton-neutron force, it appears necessary to introduce a neutral particle N obeying Bose statistics which may be emitted or absorbed when a proton jumps from one energy state to another. I do not believe, however, that it is yet possible to say that the introduction of a neutral particle is *necessary*, since the usual approximation methods of quantum mechanics, upon which this conclusion is based, consisting in a development in powers of those constants here which correspond to the fine-structure constant $e^2/\hbar c$ of the radiation theory, is not valid inasmuch as these constants have a value which lies between a third and a tenth, and are not small compared to unity. However, I consider the existence of such a particle as not improbable, since it corresponds to a symmetrical state of things in which all masses fall into three groups of the order of magnitude of the electron mass, the U-particle mass, and the proton mass, and that we would have positive, negative and neutral particles in each group.* It must be emphasized, however, that the introduction of such particles has nothing to do formally with the theory of U-particles which can be formulated without them, and is in addition to it. The theory of neutral Bose particles can be formulated exactly as the theory of U-particles with trivial modifications.

Finally, it may be mentioned that this theory also leads to showers of Heisenberg's type, with this difference that these showers would consist mostly of heavy electrons and some proton-neutrons, but few electrons. The existence of these showers still further invalidates the usual methods of calculation, since it shows that for high energies higher order processes are just as important as the first-order process.

1. The equations of motion

In all problems where equations of motion are to be found for some system it is convenient to derive the equations by the variation of some relativistically invariant Lagrange function to ensure that the conservation laws should automatically hold. We shall adopt this procedure, treating all the quantities classically, and then proceed to the quantization in the next section.

* The theory of Wenzel also contains two new particles, one charged and one uncharged.

The simplest assumption is to suppose that the motion of the U-particles is described by some invariant wave function U. This would lead to the scalar relativistic wave equation for U which has been quantized by Pauli and Weisskopf (1934). This assumption has been made by Yukawa and Sakata (1937) in a recent paper and does not lead to the correct form for the neutron-proton interaction, as we have already mentioned in the introduction. We therefore make the next simplest assumption, namely that the motion of the U-particles is described by four wave functions, U_0, U_1, U_2, and U_3, which transform like a four-vector. In order to treat the time component on the same footing as the space components, following Proca (1936) we introduce a quantity ϵ such that $\epsilon^2 = -1$, and as usual define a quantity U_4 by

$$U_4 = \epsilon U_0.$$

But ϵ must not be confounded with i, the other root of -1 which appears in our equations, and in going over to the conjugate complex equations it must be remembered that whereas the sign of i is changed, the sign of ϵ is not.*

Now the simplest relativistically invariant Lagrange function which contains only the U_μ† and their derivatives with regard to the co-ordinates x_k and $x_4 \equiv \epsilon ct$ is, according to Proca, of the form

$$\left|\left(\frac{\partial U_\nu}{\partial x_\mu} - \frac{\partial U_\mu}{\partial x_\nu}\right)\right|^2 + \text{const. } \overline{U_\mu} U_\mu.$$

The $\partial/\partial x_\mu$ here, as usual, play the part of the momentum operator p_μ, so that to generalize the above Lagrangian when a field is present we replace

$$\frac{\partial}{\partial x_\mu} \quad \text{by} \quad \frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\phi_\mu,$$

where ϕ_μ are the electromagnetic potentials. We therefore introduce the quantities $G_{\mu\nu}$ defined by

$$G_{\mu\nu} = \left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\phi_\mu\right)U_\nu - \left(\frac{\partial}{\partial x_\nu} - \frac{ie}{\hbar c}\phi_\nu\right)U_\mu, \tag{1}$$

* This is merely based on the elementary fact that if U_0, U_1, U_2, U_3, form a complex four-vector, U_1, U_2, U_3, and ϵU_0 form the corresponding Euclidean four-vector, and the Euclidean four-vector corresponding to the conjugate complex vector $\overline{U_0}$, $\overline{U_1}$, $\overline{U_2}$, $\overline{U_3}$, is just $\overline{U_1}$, $\overline{U_2}$, $\overline{U_3}$, and $\epsilon\overline{U_0}$.

† Greek suffices run from 1 to 4, Latin suffices from 1 to 3. A summation is understood over repeated suffices.

and take as the relativistically invariant Lagrange function for the U-particles in the presence of a field (Proca 1936)

$$L^U = -\tfrac{1}{2}\hbar^2 c^2 \overline{G}_{\mu\nu} G_{\mu\nu} - M_u^2 c^4 \overline{U}_\mu U_\mu, \tag{2}$$

where M_u is the mass of the U-particles, and $\hbar$ is Planck's constant divided by 2π.

As stated in the introduction, the neutron and proton are considered as two states of the same particle described by the eigenvalues $+1$ and -1 respectively of some operator τ say. We then introduce the matrices

$$\tau_N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \; \tau_P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \; \tau_{NP} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \; \tau_{PN} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3}$$

which satisfy the relations

$$\tau_P \tau_{PN} = \tau_{PN} \tau_N = \tau_{PN}; \; \tau_N \tau_{PN} = \tau_{PN} \tau_P = 0, \text{ etc.} \tag{3a}$$

and

$$\left.\begin{aligned} \tau_{NP}\tau_{PN} &= \tau_N, \\ \tau_{PN}\tau_{NP} &= \tau_P. \end{aligned}\right\} \tag{3b}$$

The above relations are easily remembered if it is noticed that all those products are zero in which dissimilar suffices appear in succession in two neighbouring matrices. ψ will denote the wave function for the proton-neutron, which now consists of two components belonging to the proton and neutron respectively. We assume that the motion of the free neutron or proton is given by the Dirac equation, so that each of the two components is really composed of the usual four wave functions for such a particle. The objection to the use of the Dirac equation here on the ground that it does not give a correct description of the magnetic moment of the proton or neutron is not strictly valid, since the proton or neutron possess an additional magnetic moment due to their interaction with the U-particles, exactly as in the Fermi theory. The Lagrangian for the proton-neutron is assumed to be of the form

$$L^M = -\psi^+ \left[c\gamma^\mu \left(p_\mu - \frac{e}{c} \phi_\mu \tau_P \right) - ic^2 (M_N \tau_N + M_P \tau_P) \right] \psi. \tag{4}$$

Here p_k is the momentum and $p_4 = \epsilon E/c$, E being the energy, and M_N, M_P are the masses of the neutron and proton respectively. Denoting by α^k and β the four Dirac matrices and writing $\alpha^4 \equiv \epsilon$, we introduce the matrices γ^μ defined by

$$\gamma^\mu \equiv -i\beta\alpha^\mu, \tag{5a}$$

and define ψ^+ by

$$\psi^+ = \epsilon \overline{\psi} \gamma^4 = i \overline{\psi} \beta. \tag{5b}$$

Then, as is well known, $\psi^+\gamma^\mu\psi$ form a Euclidian four-vector, and $\psi^+\gamma^\mu\gamma^\nu\psi$ an antisymmetrical Euclidean tensor with all components real,* so that

$$\overline{\psi^+\gamma^\mu\psi} = \psi^+\gamma^\mu\psi$$

and

$$\overline{\psi^+\gamma^\mu\gamma^\nu\psi} = \psi^+\gamma^\mu\gamma^\nu\psi. \tag{5c}$$

Some assumption must now be made about the interaction between the proton-neutron and U-particles. The simplest interaction that can be assumed is of the type $\psi^+\gamma^\mu(U_\mu\tau_{PN}+\overline{U}_\mu\tau_{NP})\psi$ consisting of the product of two four-vectors, as I have already indicated in a preliminary communication (1938*b*). The next simplest interaction is of the form

$$\psi^+\gamma^\mu\gamma^\nu(G_{\mu\nu}\tau_{PN}+\overline{G_{\mu\nu}}\tau_{NP})\psi$$

consisting of the products of two antisymmetrical tensors. Now, as will appear below, there is a certain symmetry in the equations for the G's and U's, and the two interactions given above are really on the same footing.† Consideration is therefore restricted to these two terms only, and we assume for the interaction between the U-particles and the proton-neutron a term of the form

$$L^I = -g_1\psi^+\gamma^\mu(U_\mu\tau_{PN}+\overline{U_\mu\tau_{NP}})\psi \tag{6}$$
$$-\tfrac{1}{2}g_2\psi^+\gamma^\mu\gamma^\nu(G_{\mu\nu}\tau_{PN}+\overline{G_{\mu\nu}}\tau_{NP})\psi,$$

where g_1 and g_2 are two independent constants.

A similar Lagrange function must be added for the electron neutrino, but we shall omit this until a later section for brevity. The Lagrange function for the electromagnetic field is as usual

$$L^{\text{Max}} = -\frac{1}{16\pi}F_{\mu\nu}F_{\mu\nu} = \frac{1}{8\pi}(\mathbf{E}^2-\mathbf{H}^2), \tag{7}$$

E and **H** being electric and magnetic force, where

$$E_k = F_{0k}, \quad H_{lm} = F_{lm}$$

and

$$F_{\mu\nu} = \frac{\partial}{\partial x_\mu}\phi_\nu - \frac{\partial}{\partial x_\nu}\phi_\mu. \tag{8}$$

The equations of motion for the whole system may then be derived as usual from the principle that for arbitrary variations of ψ, U_μ, ϕ_μ and their

* Cf. W. Pauli (1933, p. 221). In the usual formulation without the quantity ϵ, the space components are purely real whereas the time components are purely imaginary.

† I am indebted to Dr N. Kemmer for drawing my attention to this fact.

conjugates the change in the proper Lagrangian $\mathscr{L}$ for the whole system shall be zero, where

$$\mathscr{L} = \iiiint dx_1 dx_2 dx_3 dx_4 (L^M + L^U + L^I + L^{\text{Max}}). \tag{9}$$

Varying $\overline{\psi}$ gives

$$\left[c\gamma^\mu\left(p_\mu - \frac{e}{c}\phi_\mu \tau_P\right) - ic^2(M_N\tau_N + M_P\tau_P)\right]\psi$$
$$+ [g_1\gamma^\mu(U_\mu\tau_{PN} + \overline{U_\mu}\tau_{NP}) + \tfrac{1}{2}g_2\gamma^\mu\gamma^\nu(G_{\mu\nu}\tau_{PN} + \overline{G_{\mu\nu}}\tau_{NP})]\,\psi = 0, \tag{10a}$$

which, eliminating ϵ, reads

$$\left[c\alpha^k\left(p_k - \frac{e}{c}\phi_k\tau_P\right) - \left(E_0 - \frac{e}{c}\phi_0\tau_P\right) + \beta c^2(M_N\tau_N + M_P\tau_P)\right.$$
$$+ g_1\{\alpha^k(U_k\tau_{PN} + \overline{U_k}\tau_{NP}) - U_0\tau_{PN} - \overline{U_0}\tau_{NP}\}$$
$$\left. + \tfrac{1}{2}g_2\{i\beta\alpha^k\alpha^l(G_{kl}\tau_{PN} + \overline{G_{kl}}\tau_{NP}) + 2i\beta\alpha^l(G_{0l}\tau_{PN} + \overline{G_{0l}}\tau_{NP})\}\right]\psi. \tag{11}$$

Variation of ψ gives the conjugate equation

$$\psi^+\left[c\gamma^\mu\left(p_\mu - \frac{e}{c}\phi_\mu\tau_P\right) - ic^2(M_N\tau_N + M_P\tau_P)\right.$$
$$\left. + g_1\gamma^\mu(U_\mu\tau_{PN} + \overline{U_\mu}\tau_{NP}) + \tfrac{1}{2}g_2\gamma^\mu\gamma^\nu(G_{\mu\nu}\tau_{PN} + \overline{G_{\mu\nu}}\tau_{NP})\right], \tag{10b}$$

where p_μ acting backwards on ψ^+ is as usual to be understood as the operator $i\hbar\partial/\partial x_\mu$ instead of $-i\hbar\partial/\partial x_\mu$.

Variation of $\overline{U_\mu}$ gives

$$\hbar^2c^2\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\phi_\mu\right)G_{\mu\nu} - M_u^2c^4U_\nu - g_1\psi^+\gamma^\nu\tau_{NP}\psi$$
$$+ g_2\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\phi_\mu\right)\{\psi^+\gamma^\mu\gamma^\nu\tau_{NP}\psi\}_{\mu\neq\nu} = 0, \tag{12a}$$

and variation with respect to U_μ gives the equations conjugate to (12), namely

$$\hbar^2c^2\left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}\phi_\mu\right)\overline{G_{\mu\nu}} - M_u^2c^4\overline{U_\nu} - g_1\psi^+\gamma^\nu\tau_{PN}\psi$$
$$+ g_2\left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}\phi_\mu\right)\{\psi^+\gamma^\mu\gamma^\nu\tau_{PN}\psi\}_{\mu\neq\nu} = 0. \tag{12b}$$

Introducing a quantity $G'_{\mu\nu}$ defined by*

$$G'_{\mu\nu} = \hbar c\, G_{\mu\nu} + \frac{g_2}{\hbar c} \psi^+ \gamma^\mu \gamma^\nu \tau_{NP} \psi \tag{13}$$

and its conjugate, and using (1) we get

$$G'_{\mu\nu} = \hbar c \left\{ \left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c} \phi_\mu \right) U_\nu - \left(\frac{\partial}{\partial x_\nu} - \frac{ie}{\hbar c} \phi_\nu \right) U_\mu \right\} + \frac{g_2}{\hbar c} \psi^+ \gamma^\mu \gamma^\nu \tau_{NP} \psi, \tag{14a}$$

and $$\overline{G'_{\mu\nu}} = \hbar c \left\{ \left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c} \phi_\mu \right) U_\nu - \left(\frac{\partial}{\partial x_\nu} + \frac{ie}{\hbar c} \phi_\nu \right) U_\mu \right\} + \frac{g_2}{\hbar c} \psi^+ \gamma^\mu \gamma^\nu \tau_{PN} \psi. \tag{14b}$$

The equations (12) may now be written

$$\hbar c \left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c} \phi_\mu \right) G'_{\mu\nu} - M_u^2 c^4 U_\nu - g_1 \psi^+ \gamma^\nu \tau_{NP} \psi = 0, \tag{15a}$$

$$\hbar c \left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c} \phi_\mu \right) \overline{G'_{\mu\nu}} - M_u^2 c^4 \overline{U}_\nu - g_1 \psi^+ \gamma^\nu \tau_{PN} \psi = 0 \tag{15b}$$

It is now seen that there is complete symmetry between the G''s and U's, the equations (14) expressing the G''s in terms of the rotation of the U's, while the equations (15) express the U's in terms of the divergence of the G''s. These equations show us that the two interaction terms in (6) are on exactly the same footing.

Finally, varying ϕ_μ,

$$\frac{1}{4\pi} \frac{\partial}{\partial x_\nu} F_{\nu\mu} + ie(U_\nu \overline{G'_{\mu\nu}} - \overline{U}_\nu G'_{\mu\nu}) + e\psi^+ \gamma^\mu \tau_P \psi = 0. \tag{16}$$

These are just the four Maxwell equations.

$$\frac{1}{4\pi} \left(\operatorname{curl}_k \mathbf{H} - \frac{1}{c} \dot{\mathbf{E}}_k \right) = ie(U_\nu \overline{G'_{k\nu}} - \overline{U}_\nu G'_{k\nu}) + e\overline{\psi} \alpha^k \tau_p \psi, \tag{17a}$$

and $$\frac{1}{4\pi} \operatorname{div} \mathbf{E} = ie(U_k \overline{G'_{0k}} - \overline{U}_k G'_{0k}) + e\overline{\psi} \tau_P \psi. \tag{17b}$$

The second term on the right-hand side is just the usual charge current S_μ for the protons

$$S_\mu = e\overline{\psi} \alpha^\mu \tau_P \psi, \tag{18}$$

while the first term gives the expression for the current due to the U-particles

$$\sigma_\mu = ie(U_\nu \overline{G'_{\mu\nu}} - \overline{U}_\nu G'_{\mu\nu}), \tag{19}$$

* We make the convention that $G'_{\mu\nu}$ is identically zero when the two suffices are equal.

as has already been found by Proca in the absence of the interaction (6). Applying the operator $\partial/\partial x_\mu$ to the equation (16), the left-hand side vanishes, so that

$$\frac{\partial}{\partial x_\mu}(S_\mu + \sigma_\mu) = 0. \tag{20}$$

However, the divergence of each part separately is *not* zero, which is obviously due to the fact that charge may be transferred from the proton to the U-particle. Using the equations (12) and (10), it can easily be shown that

$$\frac{\partial \sigma_\mu}{\partial x_\mu} = \frac{ie}{\hbar c}[g_1 \psi^+ \gamma^\nu (U_\nu \tau_{PN} - \overline{U}_\nu \tau_{NP}) \psi + \tfrac{1}{2} g_2 \psi^+ \gamma^\mu \gamma^\nu (G_{\mu\nu} \tau_{PN} - \overline{G_{\mu\nu}} \tau_{NP}) \psi] = -\frac{\partial S_\mu}{\partial x_\mu}, \tag{21}$$

which is an explicit expression for the rate at which the U-particles are created due to the proton-neutron alone. It should be noticed that (20) completely determines the correct sign of e in (1), since with the opposite sign of e the sign of the first expression on the left of equation (21) would be changed, thus violating (20) and leading to an inconsistency.

To find the Hamiltonian corresponding to the material part of the Lagrangian (9), we seek the energy-momentum tensor $T_{\mu\nu}$, a quantity which by definition satisfies the equation

$$\frac{\partial}{\partial x_\mu} T_{\mu\rho} = F_{\rho\nu}(S_\nu + \sigma_\nu). \tag{22}$$

The method here used follows that of Proca closely. Writing

$$L = L^M + L^U + L^I.$$

and using the well-known Lagrange equations,

$$\frac{\partial L}{\partial x_\rho} = \frac{\partial}{\partial x_\mu}\left[\frac{\partial L}{\partial\left(\frac{\partial U_\nu}{\partial x_\mu}\right)} \cdot \frac{\partial U_\nu}{\partial x_\rho} + \text{conj.} + \frac{\partial L}{\partial\left(\frac{\partial \psi}{\partial x_\mu}\right)} \cdot \frac{\partial \psi}{\partial x_\rho} + \text{conj.}\right] + \frac{\partial L}{\partial \phi_\mu} \cdot \frac{\partial \phi_\mu}{\partial x_\rho}. \tag{23}$$

Then, remembering (15),

$$F_{\rho\mu}(S_\mu + \sigma_\mu) = \left(\frac{\partial \phi_\mu}{\partial x_\rho} - \frac{\partial \phi_\rho}{\partial x_\mu}\right)\frac{\partial L}{\partial \phi_\mu} = -\frac{\partial}{\partial x_\mu}\left(\phi_\rho \frac{\partial L}{\partial \phi_\mu}\right) + \frac{\partial \phi_\mu}{\partial x_\rho}\frac{\partial L}{\partial \phi_\mu},$$

and using (18)

$$F_{\rho\mu}(S_\mu+\sigma_\mu) = -\frac{\partial}{\partial x_\mu}\left[\phi_\rho\frac{\partial L}{\partial\phi_\mu}+\frac{\partial L}{\partial\left(\dfrac{\partial U_\nu}{\partial x_\mu}\right)}\cdot\frac{\partial U_\nu}{\partial x_\rho}\right.$$
$$\left.+\text{conj.}+\frac{\partial L}{\partial\left(\dfrac{\partial\psi}{\partial x_\mu}\right)}\cdot\frac{\partial\psi}{\partial x_\rho}+\text{conj.}-\delta_{\mu\rho}L\right]$$
$$=\frac{\partial}{\partial x_\mu}\left[\hbar c\overline{G'_{\mu\nu}}\left(\frac{\partial}{\partial x_\rho}-\frac{ie}{\hbar c}\phi_\rho\right)U_\nu+\text{conj.}+\psi^+\gamma^\mu c\left(p_\rho-\frac{e}{c}\phi_\rho\tau_P\right)\psi+\delta_{\mu\rho}L\right]. \tag{24}$$

Further,

$$\frac{\partial}{\partial x_\mu}\left\{-\hbar c\overline{G'_{\mu\nu}}\left(\frac{\partial}{\partial x_\nu}-\frac{ie}{\hbar c}\phi_\nu\right)U_\rho+M_u^2c^4\overline{U_\mu}U_\rho\right\}+\text{conj.}$$
$$=\frac{\partial}{\partial x_\mu}\left\{-\hbar c\frac{\partial}{\partial x_\nu}(\overline{G'_{\mu\nu}}U_\rho)-\hbar c\left(\frac{\partial}{\partial x_\nu}+\frac{ie}{\hbar c}\phi_\nu\right)\overline{G'_{\mu\nu}}U_\rho+M_u^2c^4\overline{U_\mu}U_\rho\right\}+\text{conj.}$$
$$=\frac{\partial}{\partial x_\mu}\left\{-g_1\psi^+\gamma^\mu(U_\rho\tau_{PN}+\overline{U_\rho}\tau_{NP})\psi\right\}.$$

Adding this equation to (21) we get (22) with

$$T_{\mu\rho}=\left[\psi^+\gamma^\mu c\left(p_\rho-\frac{e}{c}\phi_\rho\tau_P\right)\psi+\hbar^2c^2\overline{G_{\mu\nu}}G_{\rho\nu}+\text{conj.}\right.$$
$$+M_u^2c^4(\overline{U_\mu}U_\rho+\overline{U_\rho}U_\mu)+\delta_{\mu\rho}L$$
$$+g_1\psi^+\gamma^\mu(U_\rho\tau_{PN}+\overline{U_\rho}\tau_{NP})\psi$$
$$\left.+g_2\psi^+\gamma^\mu\gamma^\nu(G_{\rho\nu}\tau_{PN}+\overline{G_{\rho\nu}}\tau_{NP})\psi\right]. \tag{25}$$

This is the energy-momentum tensor for the material part of the system. The energy is then just

$$T_{00}=-T_{44}=\psi^+\left\{\gamma^k c\left(p_k-\frac{e}{c}\phi_k\tau_P\right)-ic^2(M_N\tau_N+M_P\tau_P)\right\}\psi$$
$$+\{\hbar^2c^2(\overline{G_{0k}}G_{0k}+\tfrac{1}{2}\overline{G_{kl}}G_{kl})+M_u^2c^4(\overline{U_0}U_0+\overline{U_k}U_k)\}$$
$$+g_1\psi^+\gamma^k(U_k\tau_{PN}+\overline{U_k}\tau_{NP})\psi$$
$$+\tfrac{1}{2}g_2\psi^+\gamma^k\gamma^l(G_{kl}\tau_{PN}+\overline{G_{kl}}\tau_{NP})\psi. \tag{26}$$

The first term represents the energy of the proton-neutron, the second term that of the U-particles, and the last term the interaction energy. The

Hamiltonian for the free U-particles is a positive definite form. One should notice particularly that here, as in the electromagnetic field, the time components U_0 and G_{0k} are absent from the interaction.

2. Quantization of the equations of motion

To quantize these equations one has now only to find the right commutation rules for the G's and U's. The momenta conjugate to U_k and $\overline{U_k}$ are $\overline{G_k}$ and G_k respectively, given by

$$\overline{G_k} \equiv \frac{1}{\hbar}\frac{\partial L}{\partial\left(\frac{\partial U_k}{\partial t}\right)} = -\overline{G'_{0k}} = -\hbar c\,\overline{G_{0k}} + \frac{g_2}{\hbar c} i\bar{\psi}\beta\alpha^k\tau_{PN}\psi, \tag{27}$$

and

$$G_k \equiv \frac{1}{\hbar}\frac{\partial L}{\partial\left(\frac{\partial \overline{U_k}}{\partial t}\right)} = -G'_{0k} = -\hbar c\,G_{0k} + \frac{g_2}{\hbar c} i\bar{\psi}\beta\alpha^k\tau_{NP}\psi,$$

so that following Pauli and Weisskopf (1934) the commutation rules may be taken as

$$i[\overline{G_k}(x,t),\ U_l(x',t)] \equiv i\{\overline{G_k}U_l - U_l\overline{G_k}\} = \delta(x-x')\,\delta_{kl}$$

and

$$i[G_k(x,t),\ \overline{U_l}(x',t)] = \delta(x-x')\,\delta_{kl}, \tag{28}$$

with all other combinations commuting. The charge density given by (19) may be written

$$\sigma_0 = ie(\overline{U_k}G_k - U_k\overline{G_k}), \tag{29}$$

which in consequence of the commutation relations (28) has eigenvalues which are positive or negative integral multiples of e as has been shown by Pauli and Weisskopf.

From the equations (28) it may be deduced that

$$i[\overline{G_k}(x'),\ G_{lm}(x)] = \left\{\delta_{km}\left(\frac{\partial}{\partial x_l} - \frac{ie}{\hbar c}\phi_l\right) - \delta_{kl}\left(\frac{\partial}{\partial x_m} - \frac{ie}{\hbar c}\phi_m\right)\right\}\delta(x'-x), \tag{30a}$$

$$[G_k(x'),\ G_{lm}(x)] = 0, \tag{30b}$$

and the conjugate complex equations. The commutation rules for U_0 will be discussed below.

The commutation rules for ψ, $\bar{\psi}$ are as usual

$$\begin{aligned} \psi(x)\,.\,\bar{\psi}(x')+\bar{\psi}(x')\,.\,\psi(x) &= \delta(x-x')\,.\,1 \\ \psi(x)\,.\,\psi(x')+\psi(x')\,.\,\psi(x) &= 0 \\ \overline{\psi(x)}\,.\,\overline{\psi(x')}+\overline{\psi(x')}\,.\,\overline{\psi(x)} &= 0, \end{aligned} \tag{31}$$

where 1 is a unit matrix with eight rows and columns corresponding to the eight rows of ψ.

Finally, the commutation rules for the electromagnetic field may be written in the form

$$[E_k(x'), \phi_l(x)] = 4\pi ic\hbar\,\delta_{kl}\,\delta(x-x'), \tag{32a}$$

from which follow, using (8),

$$[E_k(x'), H_{lm}(x)] = 4\pi ic\hbar\left\{\delta_{kl}\frac{\partial}{\partial x_m}-\delta_{km}\frac{\partial}{\partial x_l}\right\}\delta(x-x'), \tag{32b}$$

and $$[\phi_k, H_{lm}] = 0.$$

The quantities describing either the electromagnetic field, or the proton-neutron, or the U-particles of course commute with those describing any other of the above fields.

The Hamiltonian for the whole system may then be written in the form

$$\mathscr{H} = \iiint dx_1\,dx_2\,dx_3(H^M+H^U+H^I+H^{\text{Max}}+H^0), \tag{33}$$

where, consistently with (26), H^M is the Hamiltonian for the heavy particles including their interaction with the electromagnetic field given by

$$H^M = \bar{\psi}\left[\alpha^k c\left(p_k-\frac{e}{c}\phi_k\tau_P\right)+\beta c^2(M_N\tau_N+M_P\tau_P)\right]\psi. \tag{34}$$

H^U is the Hamiltonian for the free U-particles together with their interaction with the electromagnetic field and a part of their interaction with the proton-neutron, being the term in curly brackets in (26), which by (27) becomes

$$\begin{aligned} H^U = \left(\overline{G_k}+\frac{g_2}{e\hbar c}\psi^+\gamma^4\gamma^k\tau_{PN}\psi\right)\left(G_k+\frac{g_2}{e\hbar c}\psi^+\gamma^4\gamma^k\tau_{NP}\psi\right) \\ +\tfrac{1}{2}\hbar^2c^2\,\overline{G_{kl}}G_{kl}+M_u^2c^4(\overline{U_k}\,U_k+\overline{U_0}\,U_0). \end{aligned} \tag{35}$$

I is the Hamiltonian for the interaction of the U-particles with the proton-neutron given by

$$I = g_1\psi^+\gamma^k(U_k\tau_{PN}+\overline{U_k}\tau_{NP})\psi+\tfrac{1}{2}g_2\psi^+\gamma^k\gamma^l(G_{kl}\tau_{PN}+\overline{G_{kl}}\tau_{NP}\psi). \tag{36}$$

$H^{\max}$ is the usual Hamiltonian for the electromagnetic field*

$$H^{\mathrm{Max}} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{H}^2). \tag{37}$$

Without going into the well-known complications which arise in the quantization of fields, I would only remark that H^0 is just the vanishing term which must be added to the Hamiltonian in order that the right equations of motion should result by an application of the rule

$$i\hbar\frac{\partial f}{dt} = [f, \mathscr{H}], \tag{38}$$

where f is any operator not explicitly involving the time. Here H^0 contains terms in G_k and U_0 similar to those in the electromagnetic field. We have

$$H^0 = \phi_0\{e\overline{\psi}\tau_P\psi + ie(\overline{U_k}G_k - U_k\overline{G_k}) - \tfrac{1}{4}\pi\,\mathrm{div}.\mathbf{E}\}$$
$$+\overline{U_0}\left\{\hbar c\left(\frac{\partial}{\partial x_k} - \frac{ie}{\hbar c}\phi_k\right)G_k - M_u^2c^4U_0 - g_1\overline{\psi}\tau_{NP}\psi\right\} + \mathrm{conj}. \tag{39}$$

The last terms also vanish on account of (15) and (27).

The commutation rules (28), (30), (31) and (32) suffice to derive the equations of motion of the whole system completely by an application of (38) to the various quantities concerned, treating U_0 and $\overline{U_0}$ as if they commuted with all quantities. The reason why U_0 and $\overline{U_0}$ behave *as if* they commuted with all quantities concerned will be discussed below.

Writing ψ, $\overline{\psi}$ in place of f we get just the equations (10) and (11) respectively.

Similarly, an application of (38) to ϕ_k and E_k leads just to the equations (8) and (16). The equation (17*b*) has as usual to be imposed as an initial condition at time $t = 0$.

Finally, applying the rule (38) to U_k,

$$i\hbar\dot{U}_k = i\left\{G_k + \frac{g_2}{e\hbar c}\psi^+\gamma^4\gamma^k\tau_{NP}\psi\right\} + e\phi_0 U_k - i\hbar c\left(\frac{\partial}{\partial x_k} - \frac{ie}{\hbar c}\phi_k\right)U_0,$$

which, remembering (27), are just three of the equations (14). Similarly, writing G_k for f in (38),

$$-i\hbar\dot{G}_k = -\hbar c\left(\frac{\partial}{\partial x_l} - \frac{ie}{\hbar c}\phi_l\right)G'_{lk} + M_u^2c^4U_k + g_1\psi^+\gamma^k\tau_{NP}\psi + ie\phi_0G_k,$$

* W. Heisenberg and W. Pauli (1929*a*, *b*). Cf. also W. Pauli (1933).

which are in fact just three of the equations (15) putting $\nu = k$. The fourth equation of the set (15a) got by putting $\nu = 4$, namely

$$\hbar c\left(\frac{\partial}{\partial x_k} - \frac{ie}{\hbar c}\phi_k\right) G_k - M_u^2 c^4 U_0 - g_1 \overline{\psi} \tau_{NP} \psi = 0 \tag{40}$$

and its conjugate equation in (15b) cannot be derived as equations of motion. This is obvious since they contain the operators G_k and U_0 but no time derivatives, so that they must be fulfilled at any given instant of time. They have to be imposed as initial conditions at $t = 0$, as, indeed, is also the case with equation (17b).

It is now possible to discuss the commutation rules for U_0 and $\overline{U}_0$. These occur in the Hamiltonian (33) only in the expression

$$M_u^2 c^4 \overline{U_0} U_0 + \overline{U_0}\left\{\hbar c\left(\frac{\partial}{\partial x_k} - \frac{ie}{\hbar c}\phi_k\right) G_k - M_u^2 c^4 U_0 - g_1 \overline{\psi} \tau_{NP} \psi\right\} + \text{conj.} \tag{41}$$

The result, therefore, of some operator, say f, not commuting with U_0 or $\overline{U_0}$ is only to add to the expression for $[f, \mathscr{H}]$ terms containing the expression in curly brackets in (41) or its conjugate as factors on the right, which on account of the initial condition (40) vanish. This is the basis of what has been said before, namely that the commutation properties of U_0 and $\overline{U_0}$ are irrelevant in deriving the equations of motion, and these may be taken as commuting with all other quantities for this purpose.

It is clear, however, that U_0 and $\overline{U_0}$ cannot really commute with all quantities in $\mathscr{H}$, since this would mean by (38) that $\dot{U}_0$ and $\dot{\overline{U}}_0$ were zero. The commutation properties of U_0 and $\overline{U_0}$ can be derived from the fact that if the equations (40) and its conjugate are imposed at the time $t = 0$, they must be preserved for all time by the equations of motion. In other words, the expression on the left-hand side of (40) must commute with $\mathscr{H}$, and it must also commute with the corresponding conjugate expression. Denote by χ the expression on the left-hand side of (40) except the term $M_u^2 c^4 U_0$. Then

$$[\chi - M_u^2 c^4 U_0, \mathscr{H}] = 0, \tag{42}$$

with its conjugate equation. Indeed (42) demands that

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}(M_u^2 c^4 U_0) &= [M_u^2 c^4 U_0, \mathscr{H}] = [\chi, \mathscr{H}] \\ &= g_1 \psi^+[ic^2(M_P - M_N)\tau_{NP} + (g_1 U_\mu \gamma^\mu + \tfrac{1}{2} g_2 \gamma^\mu \gamma^\nu G_{\mu\nu})(\tau_P - \tau_N)]\psi \\ &\qquad - e\hbar c(\tfrac{1}{2} F_{kl} G'_{kl} + F_{0k} G_k) - M_u^2 c^4\, i\hbar c\left(\frac{\partial}{\partial x_k} - \frac{ie}{\hbar c}\phi_k\right) U_k \\ &\qquad + e M_u^2 c^4 \phi_0 U_0. \end{aligned} \tag{43}$$

This equation is, of course, just what could be derived from the equations (15). Applying the operator $\left(\frac{\partial}{\partial x_\nu} - \frac{ie}{\hbar c}\phi_\nu\right)$ to the equations (15) and summing over ν,

$$M_u^2 c^4 \left(\frac{\partial}{\partial x_\nu} - \frac{ie}{\hbar c}\phi_\nu\right) U_\nu = \frac{ie}{2} F_{\mu\nu} G'_{\mu\nu} - \left(\frac{\partial}{\partial x_\nu} - \frac{ie}{\hbar c}\phi_\nu\right)(g_1 \psi^+ \gamma^\nu \tau_{NP} \psi), \quad (43a)$$

which by using the equations (10) can be reduced to (43). Since the equations of motion do not depend on the commutation rules of U_0 and $\overline{U}_0$, the latter may clearly be chosen to satisfy (42), and its conjugate equation.

We therefore assume that the commutation rules of $M_u^2 c^4 U_0$ with all other quantities are exactly those of χ, so that (42) is satisfied identically. Indeed, this procedure is equivalent to treating (40) and its conjugate as the definition of U_0 and $\overline{U}_0$. The introduction of these quantities is, however, convenient for the general theory because it makes the relativistic invariance of the equations obvious.

Using (40) and its conjugate, H^U becomes

$$\begin{aligned} H^U = {} & \left(\overline{G}_k + \frac{g_2}{\epsilon\hbar c}\psi^+\gamma^4\gamma^k\tau_{PN}\psi\right)\left(G_k + \frac{g_2}{\epsilon\hbar c}\psi^+\gamma^4\gamma^k\tau_{NP}\psi\right) \\ & + \tfrac{1}{2}\hbar^2 c^2 \overline{G}_{kl} G_{kl} + M_u^2 c^4 \overline{U}_k U_k \\ & + \frac{\hbar^2 c^2}{M_u^2 c^4}\left\{\left(\frac{\partial}{\partial x_k} + \frac{ie}{\hbar c}\phi_k\right)\overline{G}_k - \frac{g_1}{\epsilon\hbar c}\psi^+\gamma^4\tau_{PN}\psi\right\} \\ & \times \left\{\left(\frac{\partial}{\partial x_k} - \frac{ie}{\hbar c}\phi_k\right) G_k - \frac{g_1}{\epsilon\hbar c}\psi^+\gamma^4\tau_{NP}\psi\right\}. \end{aligned} \quad (35a)$$

Now one may go over by a well-known procedure* to a state where there are only $1 \ldots s \ldots n$ proton-neutrons in the field. In this case in virtue of the commutation rules (31),

$$\overline{\psi}\gamma\tau\psi = \sum_{(s)} \gamma^{(s)} \tau^{(s)} \delta(x - X^{(s)}), \quad (44a)$$

where $\gamma^{(s)}, \tau^{(s)}$ and $X^{(s)}$ are the corresponding matrices and co-ordinates which refer to the sth particle. It is then obvious that each of those terms in (35a) which is a product of the type $(\overline{\psi}\gamma\tau\psi)(\overline{\psi}\gamma'\tau'\psi)$ gives rise to n infinities in the Hamiltonian (33) when (35a) is integrated over the whole of space, each infinity resulting according to (44) from cross terms in which the same suffix s appears twice. These terms represent just the infinite self energy of the proton-neutrons due to their interaction with the U-particles, and

* Cf. V. Fock (1932).

have to be neglected as must the electromagnetic self-energies. In particular, these terms are

$$-\frac{g_1^2}{M_u^2c^4}(\psi^+\gamma^4\tau_{PN}\psi)(\psi^+\gamma^4\tau_{NP}\psi)-\frac{g_2^2}{\hbar^2c^2}(\psi^+\gamma^4\gamma^k\tau_{PN}\psi)(\psi^+\gamma^4\gamma^k\tau_{NP}\psi)$$
$$=\sum_{s\neq t}\tau_{PN}^{(s)}\tau_{NP}^{(t)}\left(\frac{g_1^2}{M_u^2c^4}+\frac{g_2^2}{\hbar^2c^2}\gamma^{k(s)}\gamma^{k(t)}\right)\delta(\mathbf{X}^{(s)}-\mathbf{X}^{(t)}). \qquad (44b)$$
$$+\text{self-energy terms.}$$

There is, in other words, an exchange interaction between different particles of the form of δ functions. The term in g_1^2 *exactly compensates* a δ function which appears in the second order interaction given in (67). These δ function interactions are however to some extent arbitrary. One could, for example, have started equally well from a Lagrange function in which the $\hbar c G_{\mu\nu}$ and $\hbar c \overline{G_{\mu\nu}}$ of (2) were replaced by $G'_{\mu\nu}$ and $\overline{G'_{\mu\nu}}$ of (14), at the same time omitting the g_2 term in (6). This is equivalent to adding

$$-\frac{1}{2}\frac{g_2^2}{\hbar^2c^2}(\psi^+\gamma^\mu\gamma^\nu\tau_{PN}\psi)(\psi^+\gamma^\mu\gamma^\nu\tau_{NP}\psi)$$

to the total Lagrangian L^U+L^I, or just minus this expression to the Hamiltonian (26). Such an addition leaves all the previous equations unchanged, and merely adds additional terms to the equations (10). Their effect is however to change the terms in g_2^2 in (44b) so that these now *exactly* cancel the corresponding δ function terms in (67). The reason why this is possible is essentially due to the fact that a δ-function is a relativistically invariant interaction and can therefore be added on at will. That, however, exactly the same spin dependence results as in (67) is not trivial.

For the purposes of calculation it is convenient to make a Fourier analysis of the U's and G's. It will be assumed as usual that all the functions are periodic in some very large volume V. Write

$$U_k=\frac{1}{\sqrt{V}}\sum_p u_{pk}\,e^{\frac{i}{\hbar}(\mathbf{p},\mathbf{x})} \qquad \overline{U_k}=\frac{1}{\sqrt{V}}\sum_p \overline{u_{pk}}\,e^{-\frac{i}{\hbar}(\mathbf{p},\mathbf{x})}$$
$$G_k=\frac{1}{\sqrt{V}}\sum_p g_{pk}\,e^{\frac{i}{\hbar}(\mathbf{p},\mathbf{x})} \qquad \overline{G_k}=\frac{1}{\sqrt{V}}\sum_p \overline{g_{pk}}\,e^{-\frac{i}{\hbar}(\mathbf{p},\mathbf{x})} \qquad (45)$$

It then follows from (28) that

$$\left.\begin{aligned} &i[\overline{g_{p'l}},u_{pk}]=i[g_{p'l},\overline{u_{pk}}]=\delta_{kl}\delta_{pp'},\\ \text{and}\quad &[\overline{g},\overline{u}]=[g,u]=[\overline{g},g]=[\overline{u},u]=0.\end{aligned}\right\} \qquad (46)$$

Introducing three real unit vectors ϵ_{1p}, ϵ_{2p}, ϵ_{3p} for each Fourier component, such that ϵ_{3p} is in the direction of p and the other two are perpendicular to it,

$$(\boldsymbol{\epsilon}_{3p}, \mathbf{p}) = |p|; \ (\boldsymbol{\epsilon}_{1p}, \boldsymbol{\epsilon}_{2p}) = (\boldsymbol{\epsilon}_{1p}, \boldsymbol{\epsilon}_{3p}) = 0$$
$$(\boldsymbol{\epsilon}_{1p}, \boldsymbol{\epsilon}_{1p}) = (\boldsymbol{\epsilon}_{2p}, \boldsymbol{\epsilon}_{2p}) = (\boldsymbol{\epsilon}_{3p}, \boldsymbol{\epsilon}_{3p}) = 1. \tag{47}$$

One may then resolve any vector v_k along the three directions $\epsilon_{1,2,3}$ thus

$$v_k = \sum_{r=1,2,3} \epsilon_{rpk} v_{rp},$$

so that

$$v_{rp} = \sum_k \epsilon_{rpk} v_k. \tag{48}$$

In particular, the u_{pk} and g_{pk} of (45) may be resolved into three components in the same way. The total Hamiltonian for the U-particles may then be written in the form

$$\mathscr{H}^U + \mathscr{I} \equiv \iiint dx_1 dx_2 dx_3 (H^U + I) = \mathscr{H}_0^U + \mathscr{H}_1^U + \mathscr{I}_0 + \mathscr{I}_1, \tag{49}$$

where $\mathscr{H}_0^U$ is the Hamiltonian for the free U-particles without any interaction with the proton-neutron or the electromagnetic field. Hence

$$\mathscr{H}_0^U = \sum_p \left\{ \sum_{r=1,2} (\overline{g_{rp}} g_{rp} + E^2 \overline{u_{rp}} u_{rp}) + \frac{E^2}{M_u^2 c^4} \overline{g_{3p}} g_{3p} + M_u^2 c^4 \overline{u_{3p}} u_{3p} \right\}, \tag{50}$$

where

$$E = +c(p^2 + M_u^2 c^2)^{\frac{1}{2}}. \tag{51}$$

$\mathscr{H}_1^U$, $\mathscr{I}_0$, $\mathscr{I}_1$, will be given below. Following Pauli and Weisskopf and introducing quantities a, b and their conjugates defined by

$$g_{1p} = \sqrt{\frac{E}{2}} (a_{1p} + \overline{b_{1p}}), \quad u_{1p} = \frac{i}{\sqrt{(2E)}} (a_{2p} - \overline{b_{1p}}),$$
$$\overline{g_{1p}} = \sqrt{\frac{E}{2}} (\overline{a_{1p}} + b_{1p}), \quad \overline{u_{1p}} = -\frac{i}{\sqrt{(2E)}} (a_{1p} - \overline{b_{1p}}), \tag{52}$$

it follows from (46) that

$$[a_{1p'}, \overline{a_{1p}}] = [b_{1p'}, \overline{b_{1p}}] = \delta_{pp'}$$
$$[a_{1p'}, a_{1p}] = [b_{1p'}, b_{1p}] = [a_{1p'}, b_{1p}] = [a_{1p'}, \overline{b_{1p}}] = 0. \tag{53}$$

Similar equations hold for the a's and b's with the suffix 2, while for the longitudinal waves it is more convenient to introduce a_{3p}, b_{3p} by

$$g_{3p} = \frac{M_u c^2}{\sqrt{(2E)}} (a_{3p} + \overline{b_{3p}}); \quad u_{3p} = \frac{i}{M_u c^2} \sqrt{\frac{E}{2}} (a_{3p} - \overline{b_{3p}}),$$
$$\overline{g_{3p}} = \frac{M_u c^2}{\sqrt{(2E)}} (\overline{a_{3p}} + b_{3p}); \quad \overline{u_{3p}} = -\frac{i}{M_u c^2} \sqrt{\frac{E}{2}} (\overline{a_{3p}} - b_{3p}), \tag{54}$$

so that (50) reduces to*

$$\mathscr{H}_0^U = \sum_p E\{\overline{a_{rp}}\, a_{rp} + \overline{b_{rp}}\, b_{rp} + 3\}.$$

a_{3p} and b_{3p} satisfy the same commutation relations as (53), and all quantities with different suffices always commute.

After some easy calculation it is found that

$$\mathscr{H}_1^U = \sum_{p,p'} [-ec\{\phi_k^{p-p}(p_k+p_k')(\overline{u_{p'l}u_{pl}}) - (\phi_k^{p-p'}u_{pk})(p_l\overline{u_{p'l}})$$
$$-(\phi_k^{p-p'}\overline{u_{p'l}})(p_l'u_{pl})\} - \frac{e}{M_u^2c^3}\{(\phi_k^{p-p'}\overline{g_{p'k}})pg_{3p} + (\phi_k^{p-p'}g_{pk})p'\overline{q_{3p'}}\}$$
$$+e^2\{(\phi_k\phi_k)^{p-p'}(\overline{u_{p'l}}u_{pl}) - (\phi_k\phi_l)^{p-p'}\overline{u_{p'l}}u_{pk}\}$$
$$+\frac{e^2}{M_u^2c^4}\{(\phi_k\phi_l)^{p-p'}\overline{g_{p'k}}g_{pl}\}], \tag{55}$$

where
$$\phi_k^{p-p'} \equiv \frac{1}{V}\iiint_V \mathbf{dx}\,\phi_k\, e^{\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}',\,\mathbf{x})} \tag{56}$$

It is convenient for the purposes of calculation to introduce a new constant g_2' defined by

$$g_2' = g_2\frac{M_u c}{\hbar}, \tag{57}$$

so that g_2' now has the same dimensions as g_1, namely $(\hbar c)^{\frac{1}{2}}$. The interaction of the U-particles with the proton-neutron is then given by

$$\mathscr{I}_0 = \sum_s\sum_p \frac{1}{\sqrt{V}\sqrt{(2E)}}\Bigg\{\Bigg[-ig_1\Bigg\{\sum_{r=1,2}\alpha_{rp}^{(s)}(\overline{a_{rp}}-b_{rp}) - \frac{p}{M_uc}(\overline{a_{3p}}+b_{3p})$$
$$+\alpha_{3p}^{(s)}\frac{E}{M_uc^2}(\overline{a_{3p}}-b_{3p})\Bigg\} - ig_2'\beta^{(s)}\Bigg\{\sum_{r=1,2}\alpha_{rp}^{(s)}\frac{E}{M_uc^2}(\overline{a_{rp}}+b_{rp})$$
$$+\sum_{r=1,2}\alpha_{3p}^{(s)}\alpha_{rp}^{(s)}\frac{p}{M_uc}(\overline{a_{rp}}-b_{rp}) + \alpha_{3p}^{(s)}(\overline{a_{3p}}+b_{3p})\Bigg\}\Bigg]\tau_{NP}^{(s)}e^{-\frac{i}{\hbar}(\mathbf{p},\,\mathbf{X}^{(s)})}$$
$$+\Bigg[ig_1\Bigg\{\sum_{r=1,2}\alpha_{rp}^{(s)}(a_{rp}-\overline{b_{rp}}) - \frac{p}{M_uc}(a_{3p}+\overline{b_{3p}}) + \alpha_{3p}^{(s)}\frac{E}{M_uc^2}(a_{3p}-\overline{b_{3p}})\Bigg\}$$
$$-ig_2'\beta^{(s)}\Bigg\{\sum_{r=1,2}\alpha_{rp}^{(s)}\frac{E}{M_uc^2}(a_{rp}+\overline{b_{rp}}) + \sum_{r=1,2}\alpha_{3p}^{(s)}\alpha_{rp}^{(s)}\frac{p}{M_uc}(a_{rp}-\overline{b_{rp}})$$
$$+\alpha_{3p}^{(s)}(a_{3p}+\overline{b_{3p}})\Bigg\}\Bigg]\tau_{PN}^{(s)}e^{\frac{i}{\hbar}(\mathbf{p},\,\mathbf{X}^{(s)})}\Bigg\}, \tag{58a}$$

* A summation from 1 to 3 is always implied over all repeated Latin suffices.

with

$$\mathscr{I}_1 = \sum_s \sum_p \frac{e}{\sqrt{V}} \left[\left\{ \frac{ig_1}{M_u^2 c^4} \phi_l(X^{(s)})\, g_{pl} + \sum_{k \neq l} \frac{g_2'}{M_u c^2} (\beta\alpha^k \alpha^l)^{(s)} \phi_k(X^{(s)})\, u_{pl} \right\} \tau_{PN}^{(s)} e^{\frac{i}{\hbar}(\mathbf{p},\, \mathbf{X}^{(s)})} + \text{conj.} \right]. \quad (58b)$$

The expression (58b) is remarkable in that it contains a product of the variables describing the U-particles, the proton-neutron, and the electromagnetic field.

It is important to notice that both (35a) and (36) contain interaction terms in which differentials of the U's and G's appear, and which consequently contain additional constants of the dimensions of a length. This would mean that on this theory there would also be Heisenberg showers in which several particles are created in one elementary process, but with this difference, that whereas on Heisenberg's original theory these showers would have consisted mostly of electrons, on the present theory they would consist mainly of U-particles, a few electrons, and some protons or neutrons. This would also seem to agree with what little experimental evidence there is on the subject. The existence of such processes can be seen at once from the fact that in the interaction (58a) there are terms which increase proportionally to p or E, and, indeed, such terms are connected with the longitudinal U-waves alone due to the vector interaction in (36), and to the transverse U-waves alone due to the tensor interaction in (36).

Finally the expression for the current (19) in terms of the a's and b's for the case of free U-particles with no interaction is

$$\sigma_k = e\frac{cp_k}{E}\{\overline{a_{rp}}\, a_{rp} + \overline{b_{rp}}\, b_{rp} - \overline{a_{rp}}\,\overline{b_{rp}} - a_{rp}\, b_{rp} + 3\}$$
$$+ \frac{ep}{M_u c}\{\overline{a_{3p}}(\epsilon_{rpk}\overline{b_{rp}}) + (\epsilon_{rpk}\overline{a_{rp}})\,\overline{b_{3p}} + b_{3p}(\epsilon_{rpk} a_{rp}) + a_{3p}(\epsilon_{rpk} b_{rp})\} \quad (59)$$

The current therefore contains terms which give rise to a rapid oscillatory motion as in the Dirac and Pauli-Weisskopf theory. The last term in (59) expresses the fact that there is a current due to the U-particles having a spin, which is also in the nature of a rapid oscillatory motion.

3. The interaction of neutrons with protons

We now proceed to calculate the interaction between a neutron and a proton which results from the interaction of the heavy particles with the field of the U-particles. The method here employed is the usual method of quantum mechanics, consisting in essence of a development in powers of the constants $(g_1^2/\hbar^3c^3)$ and $(g_2'^2/\hbar^3c^3)$ which correspond in our theory to the fine structure constant $e^2/\hbar c$ of electrodynamics. This interaction appears already in the second approximation, and leads to a Heisenberg plus a Majorana force between a proton and a neutron, as is found from experiment, to fit which we have then to assume that the above constants are of the order of a tenth. Now this would make our method of calculation very bad, were it not for the fact that the next contribution to the proton-neutron force comes from a process of the sixth order, being thus $(g_1^2/\hbar^3c^3)^2$ times smaller than the one we have calculated. In the extreme relativistic case, however, the effect of the higher order processes cannot be considered as small, since the collision cross-section, for example, increases with the energy, due to the fact pointed out in the previous sections, that our theory leads to showers of Heisenberg's type. The expression for the second order process in the extreme relativistic case is however independent of the actual magnitude of the constants g_1 and g_2, so that if they were sufficiently small, there would clearly be some *relativistic* region where the interaction would be given to a good approximation by the second order process alone, with neglect of the processes of higher order. Now this case, although it may not be of direct practical importance, is of interest inasmuch as it gives us the mathematical generalizations of the Heisenberg and Majorana exchange force for the relativistic case.

As is usual in problems of this sort, it is convenient to carry out the calculation in the system of co-ordinates in which the centre of gravity of the two particles is at rest. There are therefore initially two heavy particles, one in a proton state, and the other in a neutron state, moving with equal and opposite momenta p_0, and we wish to calculate the differential effective cross-section dQ for their being scattered through the angle θ, the proton becoming a neutron and vice versa. The scattering takes place through the following intermediate states.

$$P(\mathbf{p}_0)+N(-\mathbf{p}_0)\rightarrow\left\{\begin{matrix}N(\mathbf{p}_0')+U^+(\mathbf{p}_0-\mathbf{p}_0')+N(-\mathbf{p}_0)\\ P(\mathbf{p}_0)+U^-(\mathbf{p}_0'-\mathbf{p}_0)+P(-\mathbf{p}_0')\end{matrix}\right\}\rightarrow N(\mathbf{p}_0')+P(-\mathbf{p}_0').$$

The quantities in brackets denote the momenta of the proton, neutron and U-particles in the corresponding state.

Then, as usual,

$$dQ=\frac{1}{16\pi^2}\frac{E_0^2V^2}{(\hbar c)^4}\sum_m\left|\frac{(f\,|\,\mathscr{I}_0\,|\,m)\,(m\,|\,\mathscr{I}_0\,|\,i)}{E_i-E_m}\right|^2 d\Omega. \tag{60}$$

Here $d\Omega$ denotes the element of solid angle into which the scattering takes place, $E_0 \equiv c(p_0^2 + M^2c^2)^{\frac{1}{2}}$ is the initial energy of either of the heavy particles, and we take the masses of the proton and neutron to be equal, say M. The letters i, m and f denote the initial intermediate and final states of the whole system. The momenta of the U-particles in the intermediate states is $\pm p$ where

$$p = |\mathbf{p}_0 - \mathbf{p}_0'| = 2p_0 \sin \tfrac{1}{2}\theta, \tag{61}$$

so that
$$E_i - E_m = E = c(p^2 + M_u^2 c^2)^{\frac{1}{2}}. \tag{62}$$

Using (58a), one finds after some calculation that the expression within the modulus in (60) can be written in the form of a matrix element $(f|\,W\,|i)$ with

$$\begin{aligned} W = -\tau_{NP}^{(1)}\tau_{PN}^{(2)} \frac{e^{\frac{i}{\hbar}(\mathbf{p},\, \mathbf{x}_2 - \mathbf{x}_1)}}{VE^2} \Big[&\{g_1^2 - g_2'^2 \beta^{(1)}\alpha_{3p}^{(1)}\alpha_{sp}^{(1)}\beta^{(2)}\alpha_{3p}^{(2)}\alpha_{sp}^{(2)}\} \\ &\times \left\{\alpha_{sp}^{(1)}\alpha_{sp}^{(2)} + \frac{p^2}{M_u^2c^2} + \frac{E^2}{M_u^2c^4}\alpha_{3p}^{(1)}\alpha_{3p}^{(2)}\right\} \\ &+ g_1 g_2' \frac{p}{M_u c}\{(1 - \alpha_{sp}^{(1)}\alpha_{sp}^{(2)})(\beta^{(2)}\alpha_{3p}^{(2)} - \beta^{(1)}\alpha_{3p}^{(1)})\}\Big] \end{aligned}$$

$$-\,[\text{Same expression with indices (1) and (2) interchanged}] \tag{63}$$

Here the suffix s denotes a summation over the values 1 and 2, i.e. over the two transverse U-particle states for the given momentum p. The last term in (63) results from the fact that in accordance with (31) the wave-function for the two heavy particles (1) and (2) has to be antisymmetrical in the initial and final states.

First consider the non-relativistic form of (63). In this case $|E_i - E_m| = E$ exactly, since in the system chosen the centre of gravity is at rest. But in the non-relativistic case, i.e. when the heavy particles are moving with small velocity, this is still approximately true in any other system, since the change in energy of the heavy particles is small compared with E. Hence the most general non-relativistic interaction between the two heavy particles is just derived by summing (63) over all intermediate momenta p. We now introduce the matrices σ defined by

$$i\sigma_k = \alpha_l \alpha_m, \tag{64}$$

the suffices being interchanged cyclically, so that the σ's correspond to the well-known Pauli spin matrices. In the non-relativistic approximation one may then leave out all the terms containing α's in (63) excepting those products giving σ, and put $\beta = -1$. (63) then reduces to

$$W = -(\tau_{NP}^{(1)}\tau_{PN}^{(2)}+\tau_{PN}^{(1)}\tau_{NP}^{(2)})\sum_p \frac{e^{\frac{i}{\hbar}(\mathbf{p},\, \mathbf{X}_2-\mathbf{X}_1)}}{V}\frac{p^2}{E^2M_u^2c^2}[g_1^2+g_2'^2(\sigma_{1p}^{(1)}\sigma_{1p}^{(2)}+\sigma_{2p}^{(1)}\sigma_{2p}^{(2)})]$$

$$= -(\tau_{NP}^{(1)}\tau_{PN}^{(2)}+\tau_{PN}^{(1)}\tau_{NP}^{(2)})\sum_p\Big[g_1^2+g_2'^2(\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)}) + \frac{g_2'^2\hbar^2}{p^2}(\boldsymbol{\sigma}^{(1)}\,\mathrm{grad})(\boldsymbol{\sigma}^{(2)}\,\mathrm{grad})\Big]\frac{1}{V}\frac{p^2}{E^2M_u^2c^2}e^{\frac{i}{\hbar}(\mathbf{p},\,\mathbf{X}_2-\mathbf{X}_1)}. \tag{65}$$

Now

$$\sum_p e^{\frac{i}{\hbar}(\mathbf{p},\,\mathbf{X}_2-\mathbf{X}_1)}\frac{c^2p^2}{VE^2} = \iiint\frac{d\mathbf{p}}{p^3}\left(1-\frac{M_u^2c^4}{E^2}\right)e^{\frac{i}{\hbar}(\mathbf{p},\,\mathbf{X}_2-\mathbf{X}_1)}$$

$$= \delta(\mathbf{X}_2-\mathbf{X}_1)-\frac{M_u^2c^4}{4\pi\hbar^2}\frac{e^{-\lambda|\mathbf{X}_2-\mathbf{X}_1|}}{|\mathbf{X}_2-\mathbf{X}_1|}, \tag{66}$$

with $\lambda \equiv M_u c/\hbar$. Therefore

$$W = (\tau_{NP}^{(1)}\tau_{PN}^{(2)}+\tau_{PN}^{(1)}\tau_{NP}^{(2)})\Bigg\{\Big[\frac{g_1^2}{\hbar^2c^2}+\frac{g_2'^2}{\hbar^2c^2}(\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)}) - \frac{g_2'^2}{M_u^2c^4}(\boldsymbol{\sigma}^{(1)},\mathrm{grad})(\boldsymbol{\sigma}^{(2)},\mathrm{grad})\Big]\frac{1}{4\pi}\frac{e^{-\lambda|\mathbf{X}_2-\mathbf{X}_1|}}{|\mathbf{X}_2-\mathbf{X}_1|} - \Big[\frac{g_1^2}{M_u^2c^4}+\frac{g_2'^2}{M_u^2c^4}(\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)})\Big]\delta(\mathbf{X}_2-\mathbf{X}_1)\Bigg\}. \tag{67}$$

The *interaction is therefore just of the required form consisting of Heisenberg and Majorana forces of the right sign so as to allow one to make the triplet state of the deuteron the lowest stable state. We would emphasize the fact that since only the squares of g_1 and g_2' enter into this expression, the sign of the Majorana force is beyond our control, and it is to be looked upon as a strong argument in favour of this theory that it allows only that sign of the force which actually occurs in nature.* There are additional terms consisting of δ functions which have already been discussed above. Finally, there is a term which depends on the spins of the heavy particles in the direction of their mutual separation. The effect of this term on the binding energy of the deuteron and on the scattering of neutrons has not yet been investigated.

To evaluate the relativistic scattering of protons by neutrons, it is necessary to evaluate the matrix element (63), summing over the two possible states of spin of both heavy particles in their initial and final states. This summation can be carried out by the usual methods, and leads to a complicated expression for the cross-section (60) containing terms in

g_1^4, $g_1^3 g_2'$, $g_1^2 g_2'^2$, $g_1 g_2'^3$ and $g_2'^4$. Therefore only the expression to which this reduces when g_2' is put equal to zero is given, since this is then the relativistic generalization of a pure Heisenberg force. Thus,

$$dQ = \frac{1}{8\pi}\frac{g_1^4}{\hbar^4 c^4}\frac{1}{E_0^2 E^4}\left[c^2 p_0^4 + \frac{c^4 p_0^4}{4}(1-\cos\theta)^2 - \frac{2(E_0^2 - E^2)\,p_0^4}{M_u^2}\sin^2\theta \right.$$
$$\left. + \{2M^2c^4 + c^2p_0^2(1+\cos\theta)\}^2 \frac{p_0^4}{M_u^4 c^4}(1-\cos\theta)^2\right] d\cos\theta. \quad (68)$$

In the general extreme relativistic case

$$cp_0 \gg M, \quad (69)$$

one gets
$$dQ = \frac{1}{32\pi}\frac{(g_1^4 + 2g_2'^4)}{\hbar^4 c^4}\frac{p_0^2}{M_u^4 c^6}(1+\cos\theta)^2\, d\cos\theta. \quad (70)$$

Putting g_2 equal to zero this goes over into the corresponding form of (68) for extremely high energies. Now the spin dependence of a pure Majorana force is given by the operator $\frac{1}{2}\{1 + (\sigma^{(1)}, \sigma^{(2)})\}$ so that (67) would reduce to a pure Majorana force if we put $g_1 = g_2'$. Thus (70) shows that a pure Majorana force scatters in the extreme relativistic case $\frac{3}{4}$ times as much as a pure Heisenberg force of the same magnitude, but the angular dependence of the scattering is identical, as might have been expected, for then neither particle conserves its spin in the collision, whatever its interaction.

Thus the relativistic calculation justified the result anticipated in a previous note (Bhabha 1934) that there would be a proton-neutron chain on the passage of a fast heavy particle through matter, though the angular distribution given by (70) is not nearly as asymmetrical as the incomplete treatment had led one to expect. If ϵ be the fraction of the kinetic energy of the moving particle communicated to the other particle in the system in which it was at rest, then

$$\epsilon = \tfrac{1}{2}(1+\cos\theta), \quad (71)$$

and the distribution (70) is proportional to $\epsilon^2 d\epsilon$, so that a rapid degradation of the energy follows resulting in a type of cascade shower in heavy particles alone. The total cross-section increases as p_0^2, which is connected, as already mentioned, with the phenomenon of the Heisenberg showers.

4. Scattering of U-particles by proton-neutrons

We now proceed to apply the theory to a process which is of interest in connexion with the collision of heavy electrons with nuclei, namely their scattering by heavy particles. This process is exactly analogous to the

Compton scattering of light by electrons. Calculating again in the system in which the centre of gravity is at rest, the process is represented schematically below.

$$U^{+}(\mathbf{p}_0)+N(-\mathbf{p}_0)+[P(0)]\to\left\{\begin{matrix} P(0)+[P(0)] \\ \\ U^{+}(\mathbf{p}_0)+N(-\mathbf{p}_0) \\ +U^{+}(\mathbf{p}_0')+N(-\mathbf{p}_0') \end{matrix}\right\}\to U^{+}(\mathbf{p}')+N(-\mathbf{p}')+[P(0)]$$

$$U^{-}(\mathbf{p}_0)+P(-\mathbf{p}_0)+[N(0)]\to\left\{\begin{matrix} N(0)+[N(0)] \\ \\ U^{-}(\mathbf{p}_0)+P(-\mathbf{p}_0) \\ +U^{-}(\mathbf{p}_0')+P(-\mathbf{p}_0') \end{matrix}\right\}\to U^{-}(\mathbf{p}_0')+P(-\mathbf{p}_0')+[N(0)].$$

Here $U^{+}(\mathbf{p}_0)$ denotes a positive U-particle with momentum $\mathbf{p}_0$. The square brackets are meant to indicate that the particle is in a state of negative energy corresponding to the momentum indicated.

The calculation is straightforward but complicated. We shall give only the result here. Summing or averaging over the two possible spin states of the heavy particles and over the three possible spin states of the U-particles, one gets for dq, the cross-section for the scattering of the U-particle through an angle θ in the system in which the centre of gravity is at rest, the expression

$$dq=\frac{1}{24\pi\hbar^4c^4}\frac{1}{E_i^2(E_i^2-M^2c^4)^2}[(E_i^2+M^2c^4)(A^2+B^2)+4E_iMc^2AB \\ +(E_i^2-M^2c^4)C^2\cos\theta]\,d\cos\theta, \qquad (72)$$

where

$$\left.\begin{aligned} A&=g_1^2\left\{2(E+E_N)\frac{E^2}{M_u^2c^4}+(E_N-2E)\right\}+6g_1g_2'E\frac{M}{M_u} \\ &\qquad+g_2'^2\left\{4(E_N+E)\frac{p^2}{M_u^2c^2}+3E_N\right\}, \\ B&=-3g_1^2Mc^2-6g_1g_2'\frac{c^2p^2+E_NE}{M_uc^2}-3g_2'^2Mc^2, \\ C&=g_1^2\left(cp-2E_N\frac{Ep}{M_u^2c^3}-2\frac{p^3}{M_u^2c}\right)-6g_1g_2'cp\frac{M}{M_u} \\ &\qquad-g_2'^2\left(4E_N\frac{Ep}{M_u^2c^3}+cp+4\frac{p^3}{M_u^2c}\right). \end{aligned}\right\} \qquad (73)$$

Here $|p|$ denotes the momentum of the U-particle or heavy particle, $E = c(p^2+M_u^2c^2)^{\frac{1}{2}}$ and $E_N = c(p^2+M^2c^2)^{\frac{1}{2}}$ their respective energies, and $E_i \equiv E+E_N$.

In the extreme relativistic case $p \gg Mc$ this reduces to

$$dq = \frac{1}{24\pi}\frac{(g_1^2+2g_2'^2)^2}{\hbar^4c^4}\frac{p^2}{M_u^4c^6}(1+\cos\theta)\,d\cos\theta, \tag{74}$$

and in the non-relativistic case $p \ll Mc$, M_uc, to

$$dq = \frac{3}{4\pi}\frac{(g_1+g_2')^4}{\hbar^4c^4}\frac{M^2}{(M+M_u)^2(M_u+2M)^2c^4}d\cos\theta. \tag{75}$$

It is interesting to note that whereas (67) does not depend on the sign of g_1 relative to g_2', a comparison of (75) with experiment would allow one to determine the relative sign. The cross section is however very small, being of the order 10^{-28} cm.^{2}. The application of these results to cosmic radiation and the calculation of some of the other processes mentioned in a previous note (Bhabha 1938 *b*) will be carried out in another paper.

Summary

A theory is developed based on the idea that the proton and neutron are two states of the same particle, which can go over from one state to the other by the emission of a charged particle of mass intermediate between those of the proton and electron, as originally suggested by Yukawa. These "U-particles" are described by four wave-functions. Quantization of the theory leads as usual to positive and negative U-particles with a spin of one unit. The U-particles are identified with the heavy electrons of cosmic radiation. The theory leads uniquely to short range forces of the Heisenberg and Majorana type of such a sort as to allow one to make the ground state of the deuteron the triplet state, *the sign of the Majorana force being not at our choice*. The range of the forces is connected with the mass of the U-particles as before, and demands a mass of some two hundred times the electron mass. The relativistic generalizations of a pure Heisenberg and pure Majorana force are given. The relativistic scattering of U-particles by protons and neutrons is calculated. The theory also leads to showers of Heisenberg's type, but consisting mainly of heavy electrons, a few electrons, and some heavy particles.

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