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Shape factors for β-decay

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Expressions for the shape factors of the Lth degree of forbidden β-transition were given by Greuling (1942) for the pure interactions and by Pursey (1951) for different mixtures of the pure forms. The same results have been derived here by a method due to Spiers & Blinstoyle (1952) and formulated neatly in terms of three parameters, (*a*) ξ giving the spatial covariance, (*b*) η giving the spatial parity and (*c*) ζ giving the space-time parity. The results readily point out that the correct form of interaction in the β-processes is either a STP combination or a VA combination. It has been concluded that the proper way of setting up the β-interaction is to require that all the Dirac covariants, whose scalar products appear in the Hamiltonian, must behave in the same way under space-time reflexion.

A brief sketch of the principal mathematical tools required in the method of Spiers & Blinstoyle has also been given.

Introduction

Konopinski & Uhlenbeck (1941) were the first to calculate the shape factor for the forbidden β-transitions. In their method the Hamiltonian is expanded in a power series of x/r, y/r and z/r. The calculation of the shape factor is performed with terms of a selected order of magnitude. In the final result, the terms are rearranged and expressed as the sum of invariant scalar products of irreducible tensor operators in the Cartesian form. The absence of a convenient algebra for composition and reduction of these Cartesian forms limits the applicability of the K.-U. method to only the first few degrees of forbiddenness. Greuling (1942) was able to write down the shape factor for the five pure interactions for any degree of forbiddenness by inspection of the results of actual calculation for the first five degrees of forbidden transition. Pursey (1951) gave similar results for any mixture of the five pure interactions. Though he does not explain his method of obtaining the results, it appears to be similar to that adopted by Greuling. Spiers & Blinstoyle (1952) developed a very elegant method of calculation of the shape factor, based on the use of the properties of irreducible tensor operators in the solid harmonic form and the rules of Wigner-Racah algebra. They were able to calculate the shape factor for the Lth degree of forbiddenness and the results corroborated those given by Greuling. In this paper, the method of Spiers & Blinstoyle has been used to calculate the shape factor in a general degree of forbiddenness, assuming a linear combination of the five forms of β-interactions to be operative, and we have obtained agreement with the results of Pursey. It is well known that the experimental results demand that the proper combination in the Hamiltonian should be either of the STP type or of the VA type. We have found, as will be shown presently, that the STP combination contains the Dirac covariants which do not change sign to a space-time reflexion, whereas VA contains those covariants which change sign to space-time reflexion.

Our paper is divided into two sections. §1 is devoted to a sketch of the two principal mathematical methods needed in our work, namely, (A) the Wigner-Racah algebra and (B) the factorization of the Dirac matrices. §2 contains the method and the results of calculation of the shape factor.

1. Mathematical apparatus

(A) *Wigner-Racah algebra*

(i) *The irreducible tensor operator*

Whenever a problem requires the construction of rotationally invariant quantities, the irreducible tensor operators, defined by Racah (1942), are most helpful. The $2l+1$ components of an irreducible tensor U^l_m (l, m are integers and $m = l, l-1, \ldots, -l$) satisfy the following commutation rules with the components of the total angular momentum operator $\mathbf{J}$:

$$\left.\begin{aligned}[(J_x \pm \mathrm{i}J_y), U^l_m] &= \sqrt{\{(l \mp m)(l \pm m+1)\}}\, U^l_{m\pm1},\\ [J_z, U^l_m] &= mU^l_m.\end{aligned}\right\} \tag{1}$$

Here the bracket symbols $[A, B]$ stand for the commutator $AB-BA$ and J_x, etc., are the components of $\mathbf{J}$. It is well known that the solid harmonics $\mathbf{r}^l Y^l_m(\theta, \phi)$ satisfy (1). These will be called the irreducible tensor operators of the vector $\mathbf{r}$. In a similar fashion, one defines the irreducible tensor operator of a general vector $\mathbf{A}$ as $A^l Y^l_m(\theta_A, \phi_A)$, in which θ_A and ϕ_A are the polar angles of $\mathbf{A}$. We shall denote this solid harmonic of $\mathbf{A}$ by the symbol $(\mathbf{A})^l_m$.

(ii) *The laws of combination and reduction of tensors*

The advantage of casting the operators in the form of solid harmonics is that all the well-known properties of the latter can be made use of. Thus the vector addition theorem (Wigner 1931) can be used to construct an irreducible tensor out of two kinds of argument vectors. Let $(\mathbf{A})^l_m$ be an irreducible tensor operator of rank l formed out of l vectors $\mathbf{A}$ and $(\mathbf{B})^{l'}_{m'}$ be one of rank l', formed out of l' vectors $\mathbf{B}$. These two can be combined to give an irreducible tensor of rank

$$L = l+l', \quad l+l'-1, \quad \ldots, \quad |l-l'|, \tag{2a}$$

formed out of l vectors $\mathbf{A}$ and l' vectors $\mathbf{B}$. This tensor is denoted by the symbol $(\mathbf{A}^l, \mathbf{B}^{l'})^L_M$. The law of composition is

$$(\mathbf{A}^l, \mathbf{B}^{l'})^L_M = \sum_m \begin{pmatrix} L & l & l' \\ M & m & m' \end{pmatrix} (\mathbf{A})^l_m (\mathbf{B})^{l'}_{m'}, \tag{2}$$

where $M = m+m'$ and L has one of the values specified in (2a). $\begin{pmatrix} L & l & l' \\ M & m & m' \end{pmatrix}$ stands for the Clebsch-Gordon vector addition coefficient $\langle ll'LM \mid ll'mm' \rangle$ used by Condon & Shortley (1935). Although $(\mathbf{A})^l_m$ and $(\mathbf{B})^{l'}_{m'}$ are irreducible tensors, their product $(\mathbf{A})^l_m (\mathbf{B})^{l'}_{m'}$ is, in general, not so. However, it is possible to express the product as a linear sum of irreducible tensors of rank ranging from $|l-l'|$ to $l+l'$. This process of reduction is the reverse of (2) and is obtained from (2) by means of the orthogonality relations between the Clebsch-Gordon coefficients, given in (4). We have

$$(\mathbf{A})^l_m (\mathbf{B})^{l'}_{m'} = \sum_{L=|l-l'|}^{l+l'} \begin{pmatrix} L & l & l' \\ M & m & m' \end{pmatrix} (\mathbf{A}^l, \mathbf{B}^{l'})^L_M. \tag{3a}$$

In the special case $\mathbf{A} = \mathbf{B}$ the equation ($3a$) is modified as follows:

$$(\mathbf{A})^l_m\,(\mathbf{A})^{l'}_{m'} = \sum_{L=|l-l'|}^{l+l'} \left[\frac{(2l+1)\,(2l'+1)}{4\pi(2L+1)}\right]^{\frac{1}{2}} \begin{pmatrix} L & l & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & l & l' \\ M & m & m' \end{pmatrix} (\mathbf{A}^{l+l'})^L_M. \tag{36}$$

(iii) *Properties of the Clebsch-Gordon coefficient*

The following relations are useful:

$$\left.\begin{aligned} \sum_{m,m'} \begin{pmatrix} L & l & l' \\ M & m & m' \end{pmatrix} \begin{pmatrix} L' & l & l' \\ M' & m & m' \end{pmatrix} &= \delta_{LL'}\delta_{MM'} \\ \sum_{LM} \begin{pmatrix} L & l & l' \\ M & m & m' \end{pmatrix} \begin{pmatrix} L & l & l' \\ M & n & n' \end{pmatrix} &= \delta_{mn}\delta_{m'n'} \end{aligned}\right\} \text{(orthogonality relations)}, \tag{4}$$

$$\left.\begin{aligned} \begin{pmatrix} L & l & l' \\ M & m & m' \end{pmatrix} &= (-)^{l+l'-L} \begin{pmatrix} L & l & l' \\ -M & -m & -m' \end{pmatrix} \\ &= (-)^{l+L+m'} \left[\frac{2L+1}{2l+1}\right]^{\frac{1}{2}} \begin{pmatrix} l & l' & L \\ m & -m' & M \end{pmatrix} \\ &= (-)^{l-m} \left[\frac{2L+1}{2l'+1}\right]^{\frac{1}{2}} \begin{pmatrix} l' & L & l \\ m' & M & -m \end{pmatrix} \\ &= (-)^{l+l'-L} \begin{pmatrix} L & l' & l \\ M & m' & m \end{pmatrix} \end{aligned}\right\} \text{(symmetry properties)}. \tag{5}$$

The conditions for non-vanishing of the vector addition coefficients are that the triangular inequality will be satisfied, namely,

$$L+l-l' \geqslant 0, \quad l+l'-L \geqslant 0, \quad l'+L-l \geqslant 0. \tag{6}$$

The six numbers $L+l-l'$, $l+l'-L$, $l'+L-l$, $l-m$, $l'-m'$ and $L-M$ must be integers:

$$\begin{pmatrix} L & l & l' \\ 0 & 0 & 0 \end{pmatrix} = 0, \quad \text{if } L+l+l' \text{ is odd}, \tag{7a}$$

$$= \left[\frac{(2l)!\,(2l')!}{(2L)!}\right]^{\frac{1}{2}} \frac{L!}{l!\,l'!}, \quad \text{when } L = l+l'. \tag{76}$$

(iv) *Racah coefficients*

Consider three irreducible tensors $(\mathbf{A})^{l_1}_{m_1}$, $(\mathbf{B})^{l_2}_{m_2}$ and $(\mathbf{C})^{l_3}_{m_3}$. The three can be combined in two ways:

$$\begin{aligned} (\mathbf{A})^{l_1}_{m_1}(\mathbf{B})^{l_2}_{m_2}(\mathbf{C})^{l_3}_{m_3} &= \sum_{L,l_{12}} \begin{pmatrix} L & l_{12} & l_3 \\ M & m_1+m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_{12} & l_1 & l_2 \\ m_1+m_2 & m_1 & m_2 \end{pmatrix} ((\mathbf{A}^{l_1}, \mathbf{B}^{l_2})^{l_{12}}, \mathbf{C}^{l_3})^L_M \\ &= \sum_{L,l_{23}} \begin{pmatrix} L & l_1 & l_{23} \\ M & m_1 & m_2+m_3 \end{pmatrix} \begin{pmatrix} l_{23} & l_2 & l_3 \\ m_2+m_3 & m_2 & m_3 \end{pmatrix} (\mathbf{A}^{l_1}, (\mathbf{B}^{l_2}, \mathbf{C}^{l_3})^{l_{23}})^L_M. \end{aligned} \tag{8}$$

$((\mathbf{A}^{l_1}, \mathbf{B}^{l_2})^{l_{12}}, \mathbf{C}^{l_3})^L_M$ is a tensor formed out of the tensors $(\mathbf{A}^{l_1}, \mathbf{B}^{l_2})^{l_{12}}_{m_1+m_2}$ and $(\mathbf{C})^{l_3}_{m_3}$, and similarly $(\mathbf{A}^{l_1}, (\mathbf{B}^{l_2}, \mathbf{C}^{l_3})^{l_{23}})^L_M$ from the tensors $(\mathbf{A})^{l_1}_{m_1}$ and $(\mathbf{B}^{l_2}, \mathbf{C}^{l_3})^{l_{23}}_{m_2+m_3}$. l_{12} is the vector sum of l_1 and l_2 and l_{23} that of l_2 and l_3. Using (4), one obtains

$$(\mathbf{A}^{l_1}, (\mathbf{B}^{l_2}, \mathbf{C}^{l_3})^{l_{23}})^L_M = \sum_{l_{12}} (2l_{12}+1)^{\frac{1}{2}}\,(2l_{23}+1)^{\frac{1}{2}} \begin{bmatrix} l_1 & l_2 & \\ L & l_3 & l_{23} \\ & l_{12} & \end{bmatrix} ((\mathbf{A}^{l_1}, \mathbf{B}^{l_2})^{l_{12}}, \mathbf{C}^{l_3})^L_M, \tag{9}$$

where

$$(2l_{12}+1)^{\frac{1}{2}}(2l_{23}+1)^{\frac{1}{2}}\begin{bmatrix} l_1 & l_2 & \\ L & l_3 & l_{23} \\ & l_{12} & \end{bmatrix} = \sum_{m_1 m_2 m_3} \begin{pmatrix} L & l_1 & l_{23} \\ M & m_1 & m_2+m_3 \end{pmatrix}\begin{pmatrix} L & l_{12} & l_3 \\ M & m_1+m_2 & m_3 \end{pmatrix} \times \begin{pmatrix} l_{12} & l_1 & l_2 \\ m_1+m_2 & m_1 & m_2 \end{pmatrix}\begin{pmatrix} l_{23} & l_2 & l_3 \\ m_2+m_3 & m_2 & m_3 \end{pmatrix}. \quad (10)$$

Here $\begin{bmatrix} a & b & \\ c & d & f \\ & e & \end{bmatrix}$ stands for $W(abcd;\, ef)$ defined by Racah.

The advantage of using this form is that the symmetry properties are easily recalled, namely,

$$\begin{bmatrix} a & b & \\ c & d & f \\ & e & \end{bmatrix} = \begin{bmatrix} b & d & \\ a & c & e \\ & f & \end{bmatrix} = \begin{bmatrix} d & c & \\ b & a & f \\ & e & \end{bmatrix} = \begin{bmatrix} c & a & \\ d & b & e \\ & f & \end{bmatrix}$$
$$= (-)^{e+f-a-d}\begin{bmatrix} e & b & \\ c & f & d \\ & a & \end{bmatrix} = (-)^{e+f-b-c}\begin{bmatrix} a & e & \\ f & d & c \\ & b & \end{bmatrix}. \quad (10)$$

The orthogonality relations for the Racah coefficients are

$$\sum_e (2e+1)\begin{bmatrix} a & e & \\ f & d & c \\ & b & \end{bmatrix}\begin{bmatrix} a & e & \\ g & d & c \\ & b & \end{bmatrix} = \frac{\delta_{f,g}}{2f+1}, \quad (10a)$$

$$\sum_e (-)^{a+b+c+d+e+f+g}(2e+1)\begin{bmatrix} a & c & \\ b & d & e \\ & f & \end{bmatrix}\begin{bmatrix} a & b & \\ d & c & g \\ & e & \end{bmatrix} = \begin{bmatrix} a & c & \\ d & b & g \\ & f & \end{bmatrix}. \quad (10b)$$

(v) *LS-jj recoupling coefficient*

Of particular importance is the transformation coefficient between the j-j coupling scheme and the L-S coupling scheme.

We have

$$((\mathbf{A}^{l_1}, \mathbf{B}^{s_1})^{j_1}, (\mathbf{C}^{l_2}, \mathbf{D}^{s_2})^{j_2})^J_M = \sum_{m\mu\sigma}\begin{pmatrix} J & j_1 & j_2 \\ M & m_1 & m_2 \end{pmatrix}\begin{pmatrix} j_1 & l_1 & s_1 \\ m_1 & \mu_1 & \sigma_1 \end{pmatrix}\begin{pmatrix} j_2 & l_2 & s_2 \\ m_2 & \mu_2 & \sigma_2 \end{pmatrix}(\mathbf{A})^{l_1}_{\mu_1}(\mathbf{B})^{s_1}_{\sigma_1}(\mathbf{C})^{l_2}_{\mu_2}(\mathbf{D})^{s_2}_{\sigma_2},$$

$$((\mathbf{A}^{l_1}, \mathbf{C}^{l_2})^{L}, (\mathbf{B}^{s_1}, \mathbf{D}^{s_2})^{S})^J_M = \sum\begin{pmatrix} J & L & S \\ M & \mu & \sigma \end{pmatrix}\begin{pmatrix} L & l_1 & l_2 \\ \mu & \mu_1 & \mu_2 \end{pmatrix}\begin{pmatrix} S & s_1 & s_2 \\ \sigma & \sigma_1 & \sigma_2 \end{pmatrix}(\mathbf{A})^{l_1}_{\mu_1}(\mathbf{B})^{s_1}_{\sigma_1}(\mathbf{C})^{l_2}_{\mu_2}(\mathbf{D})^{s_2}_{\sigma_2},$$

where $((\mathbf{A}^{l_1}, \mathbf{B}^{s_1})^{j_1}, (\mathbf{C}^{l_2}, \mathbf{D}^{s_2})^{j_2})^J_M$ is a tensor composed of the tensors $(\mathbf{A}^{l_1}, \mathbf{B}^{s_1})^{j_1}_{m_1}$ and $(\mathbf{C}^{l_2}, \mathbf{D}^{s_2})^{j_2}_{m_2}$ and similarly $((\mathbf{A}^{l_1}, \mathbf{C}^{l_2})^{L}, (\mathbf{B}^{s_1}, \mathbf{D}^{s_2})^{S})^J_M$ is a tensor composed of the tensors $(\mathbf{A}^{l_1}, \mathbf{C}^{l_2})^L_\mu$ and $(\mathbf{B}^{s_1}, \mathbf{D}^{s_2})^S_\sigma$. From these two equations one finds

$$((\mathbf{A}^{l_1}, \mathbf{B}^{s_1})^{j_1}, (\mathbf{C}^{l_2}, \mathbf{D}^{s_2})^{j_2})^J_M = \sum_{L,S}\begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix}((\mathbf{A}^{l_1}, \mathbf{C}^{l_2})^{L}, (\mathbf{B}^{s_1}, \mathbf{D}^{s_2})^{S})^J_M, \quad (11)$$

where, as can be easily shown with the help of (4) and (10),

$$\begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} = (2L+1)^{\frac{1}{2}}(2S+1)^{\frac{1}{2}}(2j_1+1)^{\frac{1}{2}}(2j_2+1)^{\frac{1}{2}} \times \sum_{\lambda}(-)^{2\lambda}(2\lambda+1)\begin{bmatrix} l_2 & j_2 & \lambda \\ s & s_1 & \\ & s_2 & \end{bmatrix}\begin{bmatrix} l_1 & l_2 & \lambda \\ J & S & \\ & L & \end{bmatrix}\begin{bmatrix} l_1 & s_1 & \lambda \\ J & j_2 & \\ & j_1 & \end{bmatrix}. \tag{12}$$

There are other perfectly equivalent expressions for the right-hand side.

The values of $\begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix}$ for the cases $s_1 = s_2 = \frac{1}{2}$, $S = 0, 1$, $L = l_1+l_2$ and $J = L, L\pm 1$ are listed below, as they will be useful for our work.

$S = 0, J = L = l_1+l_2$:

$$\begin{aligned} &\begin{bmatrix} l_1 & \frac{1}{2} & l_1+\frac{1}{2} \\ l_2 & \frac{1}{2} & l_2+\frac{1}{2} \\ L & 0 & L \end{bmatrix} = \left[\frac{L+1}{(2l_1+1)(2l_2+1)}\right]^{\frac{1}{2}}, \quad \begin{bmatrix} l_1 & \frac{1}{2} & l_1-\frac{1}{2} \\ l_2 & \frac{1}{2} & l_2+\frac{1}{2} \\ L & 0 & L \end{bmatrix} = -\left[\frac{l_1}{2l_1+1}\right]^{\frac{1}{2}}, \\ &\begin{bmatrix} l_1 & \frac{1}{2} & l_1+\frac{1}{2} \\ l_2 & \frac{1}{2} & l_2-\frac{1}{2} \\ L & 0 & L \end{bmatrix} = \left[\frac{l_2}{2l_2+1}\right]^{\frac{1}{2}}, \quad \begin{bmatrix} l_1 & \frac{1}{2} & l_1-\frac{1}{2} \\ l_2 & \frac{1}{2} & l_2-\frac{1}{2} \\ L & 0 & L \end{bmatrix} = 0. \end{aligned} \tag{13}$$

$S = 1, J = L+1 = l_1+l_2+1$:

$$\begin{bmatrix} l_1 & \frac{1}{2} & l_1+\frac{1}{2} \\ l_2 & \frac{1}{2} & l_2+\frac{1}{2} \\ L & 1 & L+1 \end{bmatrix} = 1. \text{ All others vanish.} \tag{14}$$

$S = 1, J = L = l_1+l_2$:

$$\begin{aligned} &\begin{bmatrix} l_1 & \frac{1}{2} & l_1+\frac{1}{2} \\ l_2 & \frac{1}{2} & l_2+\frac{1}{2} \\ L & 1 & L \end{bmatrix} = \frac{l_1-l_2}{\sqrt{\{(2l_1+1)(2l_2+1)\}}}, \quad \begin{bmatrix} l_1 & \frac{1}{2} & l_1-\frac{1}{2} \\ l_2 & \frac{1}{2} & l_2+\frac{1}{2} \\ L & 1 & L \end{bmatrix} = \left[\frac{l_1(L+1)}{L(2l_1+1)}\right]^{\frac{1}{2}}, \\ &\begin{bmatrix} l_1 & \frac{1}{2} & l_1+\frac{1}{2} \\ l_2 & \frac{1}{2} & l_2-\frac{1}{2} \\ L & 1 & L \end{bmatrix} = \left[\frac{l_2(L+1)}{L(2l_2+1)}\right]^{\frac{1}{2}}, \quad \begin{bmatrix} l_1 & \frac{1}{2} & l_1-\frac{1}{2} \\ l_2 & \frac{1}{2} & l_2-\frac{1}{2} \\ L & 1 & L \end{bmatrix} = 0. \end{aligned} \tag{15}$$

$S = 1, J = L-1 = l_1+l_2-1$:

$$\begin{bmatrix} l_1 & \frac{1}{2} & l_1+\frac{1}{2} \\ l_2 & \frac{1}{2} & l_2+\frac{1}{2} \\ L & 1 & L-1 \end{bmatrix} = -\left[\frac{1}{2(2L-1)}\frac{4l_1 l_2}{(2l_1+1)(2l_2+1)}\right],$$

$$\begin{bmatrix} l_1 & \frac{1}{2} & l_1-\frac{1}{2} \\ l_2 & \frac{1}{2} & l_2+\frac{1}{2} \\ L & 1 & L-1 \end{bmatrix} = \left[\frac{2L+1}{2L(2L-1)}\frac{2l_1(2l_1-1)}{(2l_1+1)(2l_2+1)}\right]^{\frac{1}{2}},$$

$$\begin{bmatrix} l_1 & \frac{1}{2} & l_1+\frac{1}{2} \\ l_2 & \frac{1}{2} & l_2-\frac{1}{2} \\ L & 1 & L-1 \end{bmatrix} = -\left[\frac{2L+1}{2L(2L-1)}\frac{2l_2(2l_2-1)}{(2l_1+1)(2l_2+1)}\right]^{\frac{1}{2}},$$

$$\begin{bmatrix} l_1 & \frac{1}{2} & l_1-\frac{1}{2} \\ l_2 & \frac{1}{2} & l_2-\frac{1}{2} \\ L & 1 & L-1 \end{bmatrix} = \left[\frac{2L+1}{2L-1}\frac{4l_1 l_2}{(2l_1+1)(2l_2+1)}\right]^{\frac{1}{2}}.$$

These results are given by Spiers & Blinstoyle whose

$$B^{l_1 l_2}_{j_1 j_2}(JLS) = \begin{bmatrix} l_1 & \frac{1}{2} & j_1 \\ l_2 & \frac{1}{2} & j_2 \\ L & S & J \end{bmatrix}.$$

An immediate corollary of (11), (11′) and (11″) is

$$\begin{pmatrix} J & j_1 & j_2 \\ M & m_1 & m_2 \end{pmatrix} \begin{pmatrix} j_1 & l_1 & s_1 \\ m_1 & \mu_1 & \sigma_1 \end{pmatrix} \begin{pmatrix} j_2 & l_2 & s_2 \\ m_2 & \mu_2 & \sigma_2 \end{pmatrix}$$
$$= \sum_{LS} \begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} \begin{pmatrix} J & L & S \\ M & \mu & \sigma \end{pmatrix} \begin{pmatrix} L & l_1 & l_2 \\ \mu & \mu_1 & \mu_2 \end{pmatrix} \begin{pmatrix} S & s_1 & s_2 \\ \sigma & \sigma_1 & \sigma_2 \end{pmatrix}. \quad (16)$$

For convenience in subsequent work, we define a quantity

$$\begin{Bmatrix} l_1 & \frac{1}{2} & j_1 \\ l_2 & \frac{1}{2} & j_2 \\ L & S & J \end{Bmatrix} = \begin{bmatrix} l_1 & \frac{1}{2} & j_1 \\ l_2 & \frac{1}{2} & j_2 \\ L & S & J \end{bmatrix} \left[\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}\right]^{\frac{1}{2}} \begin{pmatrix} L & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{(2l_2+1)!!}$$
$$= \frac{1}{4\pi} \begin{bmatrix} l_1 & \frac{1}{2} & j_1 \\ l_2 & \frac{1}{2} & j_2 \\ L & S & J \end{bmatrix} N(L) \left[\frac{2^{L-2l_1}(2l_1+1)!}{(2L-2l_1+1)!\,(l_1!)^2}\right]^{\frac{1}{2}}, \quad (12)$$

where
$$N(L) = \left[\frac{4\pi L!}{(2L+1)!!}\right]^{\frac{1}{2}}. \quad (18)$$

(vii) *Scalar product of irreducible tensors*

The Hermitian conjugate of equation (1) is

$$\left.\begin{aligned} [(J_x \pm \mathrm{i}J_y), U^{L\dagger}_M] &= -\sqrt{\{(L \pm M)(L \mp M + 1)\}}\, U^{l\dagger}_{m\mp 1}, \\ [J_z, U^{L\dagger}_M] &= -MU^{L\dagger}_M. \end{aligned}\right\} \quad (1')$$

Now it is trivial to prove with the help of (1) and (1′) that the commutators of $\sum_{M=-L}^{L} U^{L\dagger}_M V^L_M$ with $J_x \pm \mathrm{i}J_y$ and J_z vanish, showing that the quantity is invariant under rotation. $\sum_{M=-L}^{L} U^{L\dagger}_M V^L_M$ is called the scalar product of the two tensors. The invariant nature is independent of the composition of the two tensors.

(viii) *Selection rule of the tensor operator* $(\mathbf{A})^J_M$

Racah (1942) has proved that

$$\langle \alpha_f J_f M_f \,|\, (\mathbf{A})^J_M \,|\, \alpha_i J_i M_i \rangle = \begin{pmatrix} J_i & J_f & J \\ M_i & M_f & M \end{pmatrix} \langle \alpha_f J_f \,\|\, (\mathbf{A})^J \,\|\, \alpha_i J_i \rangle, \quad (19)$$

where J_i and J_f are the total angular momenta of the states between which the matrix element is taken and M_i, M_f are the corresponding magnetic quantum numbers and α's denotes the other quantum numbers necessary for a complete description of the states. It is well known that the second factor in (19) is independent of the magnetic quantum numbers. The selection rules on ΔJ and ΔM follow at once from the conditions of non-vanishing of the vector addition coefficient. These are

$$\left.\begin{aligned} &\Delta J = J_i - J_f = \pm J,\ \pm(J-1), \ldots, 0, \\ &\text{provided} \quad J_i + J_f \geqslant J, \end{aligned}\right\} \quad (20)$$

$$\Delta M = M_i - M_f = M. \quad (21)$$

(B) *The factorization of the Dirac matrices*

The contents of this section consist essentially of a proof of equation (8) of Spiers & Blinstoyle. We have given the proof in some detail to bring out certain symmetry properties which lead to a deeper insight in the β-decay problem. In the usual representation of the Dirac matrices β and σ_z are diagonal and the four components of the wave function may be labelled by the eigenvalues (± 1) of β and σ_z, namely, $\psi_{\beta\sigma}$. Thus, $\psi_1 = \psi_{1,1}$, $\psi_2 = \psi_{1,-1}$, $\psi_3 = \psi_{-1,1}$ and $\psi_4 = \psi_{-1,-1}$. In the formalism of Spiers & Blinstoyle use is made of the fact that each Dirac matrix can be represented as the Kronecker product of two 2×2 matrices, namely,

$$(\beta\sigma \mid A \mid \beta'\sigma') = (\beta \mid B \mid \beta')(\sigma \mid C \mid \sigma'), \tag{22}$$

where A is a 4×4 matrix and B and C are two 2×2 matrices. Using the above correspondence between ψ_i and $\psi_{\beta\sigma}$, the Kronecker product rule is

$$A = \begin{pmatrix} B_{11} & B_{1-1} \\ B_{-11} & B_{-1-1} \end{pmatrix} \times \begin{pmatrix} C_{11} & C_{1-1} \\ C_{-11} & C_{-1-1} \end{pmatrix} = \begin{pmatrix} B_{11}\begin{pmatrix} C_{11} & C_{1-1} \\ C_{-11} & C_{-1-1} \end{pmatrix} & B_{1-1}\begin{pmatrix} C_{11} & C_{1-1} \\ C_{-11} & C_{-1-1} \end{pmatrix} \\ B_{-11}\begin{pmatrix} C_{11} & C_{1-1} \\ C_{-11} & C_{-1-1} \end{pmatrix} & B_{-1-1}\begin{pmatrix} C_{11} & C_{1-1} \\ C_{-11} & C_{-1-1} \end{pmatrix} \end{pmatrix}. \tag{23}$$

The Kronecker product will be indicated by a cross. The matrix multiplication rule is as follows: if there are two 4×4 matrices A and B and if $A = a\times\alpha$ and $B = b\times\beta$, where a, b, α and β are 2×2 matrices, then

$$AB = ab\times\alpha\beta. \tag{24}$$

The three Pauli spin matrices and the unit matrix δ,

$$S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{25}$$

form a complete set of 2×2 matrices. Hence a complete set of independent 4×4 matrices can be constructed by forming Kronecker products of these matrices. Suppose now that the space-time axes are subjected to an extended Lorentz transformation (i.e. including reflexions), and suppose also that under the influence of this transformation, a matrix element $\psi^\dagger A\phi$ (where $\psi^\dagger = \psi^*\beta$, A any 4×4 matrix) be transformed into $P_\alpha\psi^\dagger A\phi$, then it is well known that there exists a non-singular 4×4 matrix Λ, such that

$$P_\alpha\psi^\dagger A\phi = \psi^\dagger\Lambda^{-1}A\Lambda\phi. \tag{26}$$

Table 1 gives the factorized representations of Λ and Λ^{-1} associated with four types of extended Lorentz transformations that are useful for our purpose.

TABLE 1

Lorentz transformation	Λ	Λ^{-1}
rotation about x_i axes	$\delta\times e^{\frac{1}{2}\mathrm{i}\theta S_i}$	$\delta\times e^{-\frac{1}{2}\mathrm{i}\theta S_i}$
reflexion of all the space co-ordinates	$\mathrm{i}S_z\times\delta$	$-\mathrm{i}S_z\times\delta$
reflexion of time co-ordinate	$-\mathrm{i}S_y\times\delta$	$\mathrm{i}S_y\times\delta$
total reflexion	$\mathrm{i}S_x\times\delta$	$-\mathrm{i}S_x\times\delta$

We have further $$\Lambda^{-1}A\Lambda = (a^{-1}ca)\times(b^{-1}db), \tag{27}$$
where $\Lambda = a\times b$ and $A = c\times d$, a, b, c and d being 2×2 matrices. From table 1 it can be seen that the behaviour under three-dimensional rotation is determined solely by the second-factor matrix and that under the reflexions by the first-factor matrix. In particular, a 4×4 matrix, which is a scalar, as far as behaviour under three-dimensional rotations is concerned, would contain δ as the second-factor matrix. The components of a vector 4×4 matrix will have as the second-factor matrix the respective component of S. The first-factor matrix which is determined by the behaviour of the 4×4 matrix under reflexion can be found for the different cases from table 2. The top row gives the change of sign under reflexion of space co-ordinates and the first column that under total reflexion. A plus sign denotes that there is no change of sign, while a minus denotes change of sign.

TABLE 2

total reflexion \ reflexion of XYZ	+	−
+	δ	S_x
−	S_z	S_y

In subsequent calculations we shall be required to factorize $(KA^*)^* = Q = K^*A$, where K is one of the sixteen Dirac covariants, $A = -\mathrm{i}\alpha_y\beta$ and the star denotes complex conjugate. When $\mathbf{K}$ is a 3-vector, then out of the irreducible components of $\mathbf{K}$ we get the three functions Q_λ as follows:

$$Q_{\pm 1} = \mp\frac{K_x^*\pm \mathrm{i}K_y^*}{\sqrt{2}}A, \quad Q_0 = K_z^*A. \tag{28}$$

Denoting the three components by a single symbol $\mathbf{K}$, we have altogether eight Dirac covariants which are characterized by their behaviour under spatial rotation, space reflexion and total reflexion. There are two possibilities for each operation which are denoted by the values 0 and 1 for three parameters ξ, η and ζ as defined in table 3. Thus, for example, if $\mathbf{K} = \beta\boldsymbol{\alpha}$, $\xi = 1$, $\eta = 1$, $\zeta = 0$.

TABLE 3

transformation	parameter	0	1
spatial rotation	ξ	invariant (scalar)	transforms like a vector
space reflexion	η	invariant (even parity)	changes sign (odd parity)
total reflexion	ζ	invariant	changes sign

The two-factor matrices are completely determined by the three-parameter. Let $Q = a\times b$, then
$$(\beta'\,|\,a\,|\,\beta'') = \mathrm{i}(-)^{\xi+1}(\beta')^{\zeta}\,\delta_{\beta',(2\eta-1)\beta''}, \tag{29}$$
If $\mathbf{K}$ is a 3-vector, then $Q_\lambda = a\times\omega_\lambda$ and

$$\omega_1 = -\mathrm{i}\sqrt{2}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \quad \omega_{-1} = -\mathrm{i}\sqrt{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \quad \text{and} \quad \omega_0 = \mathrm{i}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

If K is a scalar, then $Q = a\times\sigma_y$.

2. Calculation of shape factor

The general expression for the shape factor of the β-energy spectrum is

$$C = \frac{2\pi^3}{Wpq^2F_0}P(W), \tag{30}$$

where W is the total energy of the electron, p and q are the momenta of the electron and neutrino, respectively. $P(W)\,\mathrm{d}W$ is the probability of the electron being emitted with energy between W and $W+\mathrm{d}W$. Our problem is to find C_L, the shape factor for the Lth forbidden transition and to express C as $\sum_{L=0}^{\infty} C_L$. C_L will be characterized by the distinctive features of the nuclear operators inducing the Lth forbidden transition. Thus C_L will satisfy the following criteria:

(*a*) C_L will contain operators whose order of magnitudes are nearly the same, being $\sim R^L$, where R is the nuclear radius.

(*b*) The nuclear operators in C_L shall have the same parity. The parity will be odd or even according as L is odd or even.

(*c*) We shall omit those operators which satisfy the first two criteria, yet give selection rules on angular momentum which are merely repetitions of those occurring in the lower degrees of forbiddenness.

The nuclear operators are composed of $\mathbf{r}$, the nuclear co-ordinate, and K, one of the Dirac covariants. An odd-parity Dirac covariant ($\eta_K = 1$) mixes the small components of the wave function with the large components. So its presence reduces the matrix element by a factor $\sim v/c \sim R$. But an even-parity Dirac covariant ($\eta_K = 0$) mixes small components with small components and large with large. So it leaves the order of magnitude of the nuclear operator practically unaltered. This is because the Dirac matrices which have even parity with respect to space reflexion must have the structure $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, whereas those with odd parity must have the structure $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$. Therefore a nuclear operator of the order of magnitude $\sim R^L$ must be of degree $L-\eta_K$ in $\mathbf{r}$, when η_K determines the parity of the Dirac covariant contained in it. Thus the total parity is determined solely by L and is odd or even according as L is odd or even. So the two criteria (*a*) and (*b*) can be consistently fulfilled. An advantage of casting the nuclear operators in the solid harmonic form is that the selection rules on angular momentum come out automatically. The nuclear operators in C_L will be of the general form

$$(\mathbf{r}^{L-\eta_K}, K)^J_M, \quad \text{where} \quad J = L-\eta_K+\xi_K, \quad L-\eta_K+\xi_K-1, \ldots, \quad |L-\eta_K-\xi_K|.$$

The selection rule on angular momentum is given by (20) and (21). In most cases all the values of J, except the largest two, can be omitted on the basis of criterion (*c*).

In order to construct the nuclear tensors in the solid harmonic form, it is convenient to use the light particle wave function of an electron in positive-energy state in the nuclear Coulomb field as has been expressed by Spiers & Blinstoyle, which is

$$\psi_{j_1, m_1 a_1}(\mathbf{r}, \sigma_1, \beta_1) = \begin{pmatrix} j_1 & l_1 & \frac{1}{2} \\ m_1 & m_1-\sigma_1 & \sigma_1 \end{pmatrix} F^{\beta_1 a_1}_{l_1}(\Upsilon)^{l_1}_{m_1-\sigma_1}, \tag{31}$$

where $l_1 = j_1 + \frac{1}{2}a_1\beta_1$, $a_1 = \pm 1$ for wave functions of types A and B of Rose (1937) and σ and β have the significances mentioned in §1B. The connexion between f and g of Rose and the F's used here are

$$F_l^{1\,1} = \mathrm{i}\frac{f_{l-1}}{r^l}, \quad F_l^{-1\,1} = \frac{g_l}{r^l}, \quad F_l^{1\,-1} = \mathrm{i}\frac{f_{-l-z}}{r^l}, \quad F_l^{-1\,-1} = \frac{g_{-l-1}}{r^l}. \tag{32}$$

For the neutrino we can use a similar set where G is obtained from F by putting $Z = 0$. The neutrino wave functions for the positive-energy states are

$$\phi_{j_2 m_2 a_2}(\mathbf{r}, \sigma_2, \beta_2) = \begin{pmatrix} j_2 & l_2 & \frac{1}{2} \\ m_2 & m_2 - \sigma_2 & \sigma_2 \end{pmatrix} G_{l_2}^{\beta_2 a_2}(\Upsilon)_{m_2-\sigma_2}^{l_2}, \tag{33}$$

where $l_2 = j_2 + \frac{1}{2}a_2\beta_2$, $\quad G^{\beta_2 a_2}_{j_2+\frac{1}{2}a_2\beta_2} = G^{\beta_2 a_2}_{l_2} = \dfrac{q^{l_2+1}}{(2l_2+1)!!} G^{\beta_2 a_2}$, (34)

and $\quad G^{1\pm1} = -\mathrm{i} \quad \text{and} \quad G^{-1\pm1} = \pm 1.$ (35)

The matrix element of β-decay is of the form

$$H = \int (\Psi_f^* K \Psi_i)(\psi^* K \phi')\,\mathrm{d}\tau, \tag{36}$$

where Ψ_f and Ψ_i are the final and initial nuclear wave functions. ψ is the wave function for the electron in positive- energy state and ϕ' that for the neutrino in the negative-energy state. K is one of the Dirac covariants. In order to use (33), the following procedure due to Furry (1938) is adopted. Using the operator $A = \mathrm{i}\alpha_y\beta$, one gets

$$\phi' = (A\phi)^*, \tag{37}$$

where ϕ is the neutrino wave function for the positive-energy states, and the star signifies the procedure of taking complex conjugate. Hence H can be written as

$$H = \int (\Psi_f^* K \Psi_i)(\psi Q \phi)^*\,\mathrm{d}\tau, \tag{38}$$

where

$$Q = (KA^*)^* = K^* A. \tag{39}$$

The advantage of this procedure is that it avoids the use of projection operators in the summation over the negative-energy states after squaring of H. This permits the use of Racah algebra in forming the nuclear tensors from the very outset. The application of Wigner-Racah algebra after squaring of H and introduction of the projection operators is rather clumsy.

Following the procedure of Spiers & Blinstoyle we construct the Hamiltonian in a form different from, but perfectly equivalent to, the form given by Konopinski & Uhlenbeck. We specify that the electron-neutrino system carries away an angular momentum J, and that the individual total angular momenta are j_1 and j_2 for the electron and neutrino respectively. There is the further condition that j_1 and j_2 are such that $l_1 + l_2 = L - \eta_K$. This will ensure that the nuclear operator has the order of magnitude $\sim R^L$ (cf. criterion (*a*)). The orbital angular momentum of the electron-neutrino system will have values $l_1 + l_2$, $l_1 + l_2 - 2$, $l_1 + l_2 - 4, \ldots, |l_1 - l_2|$. We retain only the highest value $L - \eta_K$ on the basis of the criterion (*c*). The required matrix element is

$$^{L}H_{M_i M_f}(JS, ja) = -g_K\sqrt{2}\begin{pmatrix} J_i & J_f & J \\ M_i & M_f & M \end{pmatrix} \sum_K \langle \mathbf{r}^{L-\eta_K}, K\rangle^* \sum_{\beta_1\beta_2} \begin{Bmatrix} l_1 & \frac{1}{2} & j_1 \\ l_2 & \frac{1}{2} & j_2 \\ L-\eta_K & S & J \end{Bmatrix} \times F_{l_1}^{\beta_1 a_1^*} G^{\beta_2 a_2^*} q^{l_2+1} (-)^S \beta_1^\zeta \delta_{\beta_1 + (2\eta_K - 1)\beta_2}, \tag{40}$$

where $$\langle \mathbf{r}^{L-\eta_K}, K\rangle = \langle f \| (\mathbf{r}^{L-\eta_K}, K)^J \| \mathrm{i}\rangle \tag{41}$$

and $$S = \xi_K.$$

The expression for C_L is

$$C_L = \frac{4\pi^2}{F_0 q^2 p^2} \sum_{M_i M_f} \frac{1}{2J_i+1} \sum_{JM} \sum_{j_1 a_1} \sum_{j_1 a_2} |\, {}^L H_{M_i M_f}(JS, ja)\,|^2. \tag{42}$$

The main steps in the evaluation of C_L are given by Spiers & Blinstoyle. The formula given in appendix I of their paper has been extended and proved quite generally in § 1 A. $\begin{Bmatrix} l_1 & \frac{1}{2} & j_1 \\ l_2 & \frac{1}{2} & j_2 \\ L & S & J \end{Bmatrix}$ has been defined in (17). The Dirac matrices have been factorized in a manner described in § 1 B. For the first-factor matrix, the expression (29) has been used. C_L has been calculated on the assumption that a linear combination of the five forms of interactions, namely,

$$H = G_S S + G_V V + G_T T + G_A A + G_P P,$$

is operative. The shape factor is expressed as a sum of four parts characterized by (i) $\eta_K = \eta_{K'}, \zeta_K = \zeta_{K'}$, (ii) $\eta_K \neq \eta_{K'}, \zeta_K = \zeta_{K'}$, (iii) $\eta_K = \eta_{K'}, \zeta_K \neq \zeta_{K'}$ and (iv) $\eta_K \neq \eta_{K'}$, $\zeta_K \neq \zeta_{K'}$. The values of g, ξ, η and ζ for the different covariants are given in table 4. In table 5 the matrix elements that appear in different interactions are given along with the selection rules on angular momentum. In a general linear combination squares of all the eight terms appear. Product terms occur only between the matrix elements in the same column. The results agree perfectly with those given by Greuling & Pursey.

Thus, if $C_L = \sum_{KK'} C_L(KK')$, we have

(i) $\eta_K = \eta_{K'} = \eta$, $\zeta_K = \zeta_{K'} = \zeta$:

$$C_L(KK') = (-)^{\xi_K+\xi_{K'}} (4\pi)^2 g_K g_{K'} \sum_J \{\langle \mathbf{r}^{L-\eta_K}, K\rangle^{J*} \langle \mathbf{r}^{L-\eta_{K'}}, K'\rangle^J + \text{c.c.}\}$$

$$\times \sum_\nu \Bigg[\sum_{a=\pm1} \begin{Bmatrix} \nu+1 & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta-1 & \frac{1}{2} & L-\nu-\eta-1+\frac{1}{2}a \\ L-\eta & S & J \end{Bmatrix} \begin{Bmatrix} \nu+1 & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta-1 & \frac{1}{2} & L-\nu-\eta-1+\frac{1}{2}a \\ L-\eta & S' & J \end{Bmatrix} \times q^{2(L-\nu-\eta-1)} M_\nu$$

$$+ \sum_{a=\pm1} \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta & \frac{1}{2} & L-\nu-\eta+\frac{1}{2}a \\ L-\eta & S & J \end{Bmatrix} \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta & \frac{1}{2} & L-\nu-\eta+\frac{1}{2}a \\ L-\eta & S' & J \end{Bmatrix} q^{2(L-\nu-\eta)} L_\nu$$

$$+ (-)^{\zeta+1} \left(\begin{Bmatrix} \nu+1 & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta-1 & \frac{1}{2} & L-\nu-\eta-\frac{1}{2} \\ L-\eta & S & J \end{Bmatrix} \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta & \frac{1}{2} & L-\nu-\eta-\frac{1}{2} \\ L-\eta & S' & J \end{Bmatrix} \right.$$

$$\left. + \begin{Bmatrix} \nu+1 & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta-1 & \frac{1}{2} & L-\nu-\eta-\frac{1}{2} \\ L-\eta & S' & J \end{Bmatrix} \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta & \frac{1}{2} & L-\nu-\eta-\frac{1}{2} \\ L-\eta & S & J \end{Bmatrix} \right) q^{2(L-\nu-\eta)-1} N_\nu \Bigg].$$

(ii) $\eta_K \neq \eta_{K'}$, $\zeta_K = \zeta_{K'} = \zeta$:

$$C_L(KK') = (-)^{\xi_K+\xi_{K'}}(4\pi)^2 g_K g_{K'} \sum_J \{\mathrm{i}\langle \mathbf{r}^{L-\eta_K}, K\rangle^{J*} \langle \mathbf{r}^{L-\eta_{K'}}, K'\rangle^J + \text{c.c.}\}$$

$$\times \sum_\nu \left[(1-2\eta_K) \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta_K & \frac{1}{2} & L-\nu-\frac{1}{2} \\ L-\eta_K & S & J \end{Bmatrix} \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta_{K'} & \frac{1}{2} & L-\nu-\frac{1}{2} \\ L-\eta_{K'} & S' & J \end{Bmatrix} q^{2(L-\nu)-1} L_\nu \right.$$

$$\left. + (-)^{\zeta_K+1} \sum_{a=\pm 1} \begin{Bmatrix} \nu+1 & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-1 & \frac{1}{2} & L-\nu-1+\frac{1}{2}a \\ L & S & J \end{Bmatrix} \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-1 & \frac{1}{2} & L-\nu-1+\frac{1}{2}a \\ L-1 & S' & J \end{Bmatrix} q^{2(L-\nu-1)} N_\nu \delta_{\eta_K,0} \delta_{\eta_{K'},1} \right].$$

(iii) $\eta_K = \eta_{K'}$, $\zeta_K \neq \zeta_{K'}$:

$$C_L(KK') = (-)^{\xi_K+\xi_{K'}}(4\pi)^2 g_K g_{K'} \sum_J \{\langle \mathbf{r}^{L-\eta}, K\rangle^{J*} \langle \mathbf{r}^{L-\eta}, K'\rangle^J + \text{c.c.}\}$$

$$\times \sum_\nu \left[- \sum_{a=\pm 1} \begin{Bmatrix} \nu+1 & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta-1 & \frac{1}{2} & L-\nu-\eta-1+\frac{1}{2}a \\ L-\eta & S & J \end{Bmatrix} \begin{Bmatrix} \nu+1 & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta-1 & \frac{1}{2} & L-\nu-\eta-1+\frac{1}{2}a \\ L-\eta & S' & J \end{Bmatrix} \times q^{2(L-\nu-\eta-1)} M_\nu^- \right.$$

$$- \sum_{a=\pm 1} \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta & \frac{1}{2} & L-\nu-\eta+\frac{1}{2}a \\ L-\eta & S & J \end{Bmatrix} \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta & \frac{1}{2} & L-\nu-\eta+\frac{1}{2}a \\ L-\eta & S' & J \end{Bmatrix} q^{2(L-\nu-\eta)} L^-$$

$$- (-)^{\zeta_K} \left(\begin{Bmatrix} \nu+1 & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta-1 & \frac{1}{2} & L-\nu-\eta-\frac{1}{2} \\ L-\eta & S & J \end{Bmatrix} \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta & \frac{1}{2} & L-\nu-\eta-\frac{1}{2} \\ L-\eta & S & J \end{Bmatrix} \right.$$

$$\left. \left. - \begin{Bmatrix} \nu+1 & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta-1 & \frac{1}{2} & L-\nu-\eta-\frac{1}{2} \\ L-\eta & S' & J \end{Bmatrix} \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta & \frac{1}{2} & L-\nu-\eta-\frac{1}{2} \\ L-\eta & S & J \end{Bmatrix} \right) q^{2(L-\nu-\eta)-1} N_\nu^- \right].$$

(iv) $\eta_K \neq \eta_{K'}$, $\zeta_K \neq \zeta_{K'}$:

$$C_L(KK') = (-)^{\xi_K+\xi_{K'}}(4\pi)^2 g_K g_{K'} \sum_J \{\mathrm{i}\,\langle \mathbf{r}^{L-\eta_K}, K\rangle^{J*} \langle \mathbf{r}^{L-\eta_{K'}}, K'\rangle^J + \text{c.c.}\}$$

$$\times \sum_\nu \left[(2\eta_K-1) \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta_K & \frac{1}{2} & L-\nu-\frac{1}{2} \\ L-\eta_K & S & J \end{Bmatrix} \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-\eta_{K'} & \frac{1}{2} & L-\nu-\frac{1}{2} \\ L-\eta_{K'} & S' & J \end{Bmatrix} q^{2(L-\nu)-1} L_\nu^- \right.$$

$$\left. + (-)^{\zeta_K} \sum_{a=\pm 1} \begin{Bmatrix} \nu+1 & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-1 & \frac{1}{2} & L-\nu-1+\frac{1}{2}a \\ L & S & J \end{Bmatrix} \begin{Bmatrix} \nu & \frac{1}{2} & \nu+\frac{1}{2} \\ L-\nu-1 & \frac{1}{2} & L-\nu-1+\frac{1}{2}a \\ L-1 & S' & J \end{Bmatrix} q^{2(L-\nu-1)} N_\nu^- \delta_{\eta_K,0} \delta_{\eta_K,1} \right].$$

The relations between the reduced matrix elements used here and the Cartesian tensors of Greuling are as follows:

$$\Sigma\,|\,Q_{L+1}(\mathbf{A},\mathbf{B})/(L+1)!\,|^2 = [N(L)]^2 \frac{4\pi}{3}\,|\,\langle \mathbf{B}^L, \mathbf{A}\rangle^{L+1}\,|^2,$$

$$\Sigma\,|\,Q_L(\mathbf{A})/L!\,|^2 = [N(L)]^2\,|\,\langle \mathbf{A}^L\rangle^L\,|^2,$$

$$\Sigma|\,Q_L(\mathbf{A}\times\mathbf{B},\mathbf{B})/L!\,|^2 = \frac{4\pi}{3}[N(L)]^2 \frac{L+1}{L}\,|\langle \mathbf{B}^L, \mathbf{A}\rangle^L\,|^2,$$

$$\Sigma \frac{\mathrm{i}Q_L^*(\mathbf{A},\mathbf{B})\,Q_L(\mathbf{C}\times\mathbf{B},\mathbf{B}) + \text{c.c.}}{(L!)^2}$$

$$= \frac{4\pi}{3} N(L)\,N(L-1) \sqrt{\left(\frac{L+1}{L}\right)}\,[\langle \mathbf{B}^{L-1}, \mathbf{A}^L\rangle^{L*} \langle \mathbf{B}^L, \mathbf{C}\rangle + \text{c.c.}],$$

etc.

The quantities L, M and N are defined as follows:

$$L_\nu = \frac{1}{2p^2F_0}(|F_\nu^{-11}|^2+|F_\nu^{1-1}|^2) = \frac{g_\nu^2+f_{-\nu-2}^2}{2p^2F_0R^{2\nu}} = \frac{F_\nu}{F_0}\left(\frac{2^\nu\nu!}{(2\nu+1)!}p^\nu\right)^2\frac{\nu+1+s_\nu}{2\nu+2},$$

$$L_\nu^- = \frac{1}{2p^2F_0}(|F_\nu^{-11}|^2-|F_\nu^{1-1}|^2) = \frac{g_\nu^2-f_{-\nu-2}^2}{2p^2F_0R^{2\nu}} = \frac{F_\nu}{F_0}\frac{2^\nu\nu!}{(2\nu+1)!}p^2\Big)^2\frac{S_\nu(\nu+1+s_\nu)}{2(\nu+1)^2}\frac{1}{W},$$

$$M_\nu = \frac{1}{2p^2F_0}(|F_{\nu+1}^{11}|^2+|F_{\nu+1}^{-1-1}|^2) = \frac{f_\nu^2+g_{-\nu-2}^2}{2p^2F_0R^{2\nu+2}} = \frac{F_\nu}{F_0}\left(\frac{2^{\nu+1}\nu!}{(2\nu+2)!}p^\nu\right)^2\left[\frac{2\nu+2}{\nu+1+s_\nu}\left(\frac{\alpha Z}{2R}\right)^2\right.$$
$$+\left(\frac{s_\nu}{2s_\nu+1}\frac{p^2}{W}\frac{(2\nu+1)(\alpha z)^2W}{(2s_\nu+1)(\nu+1+s_\nu)}\right)\left(\frac{\alpha Z}{R}\right)+\frac{(\nu+1)(s_\nu-\nu)s_\nu}{(2s_\nu+1)^2}\left(1-\frac{4s_\nu+3}{s_\nu(s_\nu+1)}\alpha^2z^2\right)p^2$$
$$\left.+\left(1+\frac{\nu(4s_\nu+3)}{(s_\nu+1)(\nu+1+s_\nu)}\alpha^2Z^2\right)\left(\frac{\alpha Z}{2s_\nu+1}\right)^2\right],$$

$$M_\nu^- = \frac{1}{2p^2F_0}(|F_{\nu+1}^{-1-1}|^2-|F_{\nu+1}^{11}|^2) = \frac{g_{-\nu-2}^2-f_\nu^2}{2p^2F_0R^{2\nu+2}}$$
$$= \frac{F_\nu}{F_0}\left(\frac{2^{\nu+1}\nu!}{(2\nu+2)!}p^\nu\right)^2\left[-\frac{2s_\nu}{\nu+1+s_\nu}\left(\frac{\alpha Z}{2R}\right)^2\frac{1}{W}+\frac{\alpha^2Z^2}{\nu+1+s_\nu}\frac{Z\alpha}{R}+\frac{s_\nu(\nu+1)}{(2s_\nu+1)^2}\right.$$
$$\left.\times\left[1+\frac{s_\nu\alpha^2Z^2}{(\nu+1)(\nu+1+s_\nu)}\right]\frac{p^2}{W}+\left[1-\frac{(4s_\nu+3)\alpha^2Z^2}{\nu+1+s_\nu}\right]\left(\frac{\alpha Z}{2s_\nu+1}\right)^2W\right],$$

$$N_\nu = \frac{\mathrm{i}}{2p^2F_0}(F_{\nu+1}^{11*}F_\nu^{-11}+F_{\nu+1}^{-1-1*}F_\nu^{1-1}) = \frac{f_\nu g_\nu-f_{-\nu-2}g_{-\nu-2}}{2p^2F_0R^{2\nu+1}}$$
$$= \frac{F_\nu}{F_0}\left(\frac{2^\nu\nu!}{(2\nu+1)!}p^\nu\right)^2\frac{1}{\nu+1}\left[\frac{\alpha Z}{2R}+\frac{s_\nu}{2s_\nu+1}\frac{p^2}{W}-\frac{2\alpha^2Z^2}{2s_\nu+1}W\right],$$

$$N_\nu^- = \frac{\mathrm{i}}{2p^2F_0}(F_{\nu+1}^{11*}F_\nu^{-11}-F_{\nu+1}^{-1-1*}F_\nu^{1-1}) = \frac{f_\nu g_\nu+f_{-\nu-2}g_{-\nu-2}}{2p^2F_0R^{2\nu+1}}$$
$$= \frac{F_\nu}{F_0}\left(\frac{2^\nu\nu!}{(2\nu+1)!}p^\nu\right)^2\frac{1}{(\nu+1)^2}\left[-\frac{s_\nu}{W}\left(\frac{Z\alpha}{2R}\right)+Z^2\alpha^2\right],$$

where
$$F_\nu(W,Z) = \left[\frac{(2\nu+2)!}{\nu!}\right]^2(2pR)^{2(s_\nu-\nu-1)}\,\mathrm{e}^{\pi y}\frac{|\Gamma(s_\nu+\mathrm{i}y)|^2}{\Gamma^2(1+2s_\nu)},$$

$$S_\nu = \sqrt{\{(\nu+1)^2-\alpha^2Z^2\}} \quad\text{and}\quad y = \frac{\alpha ZW}{p}.$$

TABLE 4

interaction	K	g_K	ξ_K	η_K	ζ_K
S	β	G_S	0	0	0
V	1	G_V	0	0	1
	$\boldsymbol{\alpha}$	$-G_V\sqrt{\frac{4\pi}{3}}$	1	1	1
T	$\beta\boldsymbol{\sigma}$	$G_T\sqrt{\frac{4\pi}{3}}$	1	0	0
	$\beta\boldsymbol{\alpha}$	$G_T\sqrt{\frac{4\pi}{3}}$	1	1	0
A	$\boldsymbol{\sigma}$	$G_A\sqrt{\frac{4\pi}{3}}$	1	0	1
	γ_5	$-G_A$	0	1	1
P	$\beta\gamma_5$	G_P	0	1	0

As an example of the use of the equations (43) and the tables 4 and 5, the expression for C_L is calculated for the STP combination. The various products of the tensors that appear are given in table 6. The diagonal terms arise from the squares of the pure interactions and the off-diagonal terms from interference of the different interactions. The coefficient associated with each term is given in table 7. It is important to note that besides $|\langle\beta\boldsymbol{\sigma},\mathbf{r}^L\rangle^{L+1}|^2$, $\mathrm{i}\langle\beta\boldsymbol{\sigma},\mathbf{r}^L\rangle^{L+1*}\langle\beta\gamma_s\mathbf{r}^{L+1}\rangle^{L+1}+\text{c.c.}$ will also contribute to the unique forbidden transitions. While the terms omitted on the basis of the criterion (*c*) are smaller by a factor 10^{-4}, $\mathrm{i}\langle\beta\boldsymbol{\sigma},\mathbf{r}^L\rangle^{L+1*}\langle\beta\gamma_5\mathbf{r}^{L+1}\rangle^{L+1}+\text{c.c.}$ is only 10^{-2} times smaller. So that it may not altogether be neglected.

$\langle\beta\sigma,\gamma\rangle^0$ and $\langle\beta\gamma_s\rangle^0$ give rise to the first forbidden $(0\leftrightarrow 0,\text{ yes})$ transition (cf. the case of Ra*E*).

TABLE 5

	K	$\Delta J=\pm(L-1)$	$\Delta J=\pm L,\ \pm(L-1)$	$\Delta J=\pm(L+1),\ \pm L$
S	β	—	$\langle\beta,\mathbf{r}^L\rangle^L$	—
V	1	—	$\langle\mathbf{r}\rangle^L$	—
	$\boldsymbol{\alpha}$	$\langle\boldsymbol{\alpha},\mathbf{r}^{L-1}\rangle^{L-1}$	$\langle\boldsymbol{\alpha},\mathbf{r}^{L-1}\rangle^L$	—
T	$\beta\boldsymbol{\sigma}$	$\langle\beta\boldsymbol{\sigma},\mathbf{r}^L\rangle^{L-1}\delta_{L,1}$	$\langle\beta\boldsymbol{\sigma},\mathbf{r}^L\rangle^L$	$\langle\beta\boldsymbol{\sigma},\mathbf{r}^L\rangle^{L+1}$
	$\beta\boldsymbol{\alpha}$	$\langle\beta\boldsymbol{\alpha},\mathbf{r}^{L-1}\rangle^{L-1}$	$\langle\beta\boldsymbol{\alpha},\mathbf{r}^{L-1}\rangle^L$	—
A	$\boldsymbol{\sigma}$	$\langle\boldsymbol{\sigma},\mathbf{r}^L\rangle^{L-1}\delta_{L,1}$	$\langle\boldsymbol{\sigma},\mathbf{r}^L\rangle^L$	$\langle\boldsymbol{\sigma},\mathbf{r}^L\rangle^{L+1}$
	γ_5	$\langle\gamma_5,\mathbf{r}^{L-1}\rangle^{L-1}$	—	—
P	$\beta\gamma_5$	$\langle\beta\gamma_5,\mathbf{r}^{L-1}\rangle^{L-1}$	—	—

TABLE 6

	S	T	P
S	$\|\langle\beta,\mathbf{r}^L\rangle^L\|^2$	—	—
T	$\mathrm{i}\langle\beta,\boldsymbol{\gamma}^L\rangle^{L*}\langle\beta\boldsymbol{\alpha},\mathbf{r}^{L-1}\rangle^L+\text{c.c.}$	$\|\langle\beta\boldsymbol{\sigma},\mathbf{r}^L\rangle^{L+1}\|^2$	$\mathrm{i}\langle\beta\boldsymbol{\sigma},\mathbf{r}^L\rangle^{L+1*}\langle\beta\gamma_5,\mathbf{r}^{L+1}\rangle^{L+1}+\text{c.c.}$
	$\langle\beta,\boldsymbol{\gamma}^L\rangle^{L*}\langle\beta\boldsymbol{\sigma},\mathbf{r}^L\rangle^L+\text{c.c.}$	$\|\langle\beta\boldsymbol{\sigma},\mathbf{r}^L\rangle^L\|^2$	—
	—	$\|\langle\beta\boldsymbol{\sigma},\mathbf{r}^L\rangle^{L-1}\|^2\delta_{L,1}$	$\mathrm{i}\langle\beta\boldsymbol{\sigma},\mathbf{r}^L\rangle^{L-1*}\langle\beta\gamma_5,\mathbf{r}^{L-1}\rangle^{L-1}\delta_{L,1}+\text{c.c.}$
	—	$\|\langle\beta\boldsymbol{\alpha},\mathbf{r}^{L-1}\rangle^L\|^2$	—
	—	$\|\langle\beta\boldsymbol{\alpha},\mathbf{r}^{L-1}\rangle^{L-2}\|^2\delta_{L,2}$	—
	—	$\mathrm{i}\langle\beta\boldsymbol{\sigma},\mathbf{r}^L\rangle^{L*}\langle\beta\boldsymbol{\alpha},\boldsymbol{\gamma}^{L-1}\rangle+\text{c.c.}$	—
P	—	—	$\|\langle\beta\gamma_5,\mathbf{r}^{L-1}\rangle^{L-1}\|^2\delta_{L,1}$

The most interesting feature about the expression for C_L is the appearance of the $1/W$ term when there is a mixing of operators having different values of ζ. The advantage of formulating in this manner is that on the basis of experimental evidence regarding the absence of the $1/W$ term, at least, in the allowed and the first forbidden $(\Delta J=\pm 1, 0, \text{yes})$ spectra (Mahmoud & Konopinski 1952; Konopinski & Langer 1953; Davidson & Peaslee 1953; Peaslee 1953), we can at once conclude that the interaction in β-decay must be either a STP combination $(\zeta=0)$ or a VA combination $(\zeta=1)$. So it appears that the proper way to set up the β-decay Hamiltonian

$$H=\sum_K g_K(\Psi_f^\dagger K\Psi_i)(\psi^\dagger K\phi)$$

is to impose on K's the condition that they should either all commute with γ_5 (STP) or all anti-commute with it (VA), so that with reference to a space-time reflexion they should either all retain the same sign or all change sign.

Comparing with the condition of Fierz for selecting the β-decay Hamiltonian, we find that our condition is less stringent than that of Fierz in the sense that g_S, g_T, etc., may be arbitrary, whereas Fierz's condition imposes definite relationship between them. On the other hand, our condition is more stringent when we remember that the Fierz condition allows other combinations also besides STP and VA, e.g. SPAV may be allowed.

Table 7

$\lvert\langle\beta, \mathbf{r}^L\rangle^L\rvert^2$	$G_S^2N^2(L)\sum_{\nu=0}^{L}[A_{L\nu}q^{2L-2\nu-2}M_\nu+2C_{L\nu}q^{2(L-\nu)-1}N_\nu+D_{L\nu}q^{2L-2\nu}L_\nu]$
$\lvert\langle\beta\boldsymbol{\sigma}, \mathbf{r}^L\rangle^{L+1}\rvert^2$	$G_T^2N^2(L)\frac{4\pi}{3}\sum_{\nu=0}^{L}B_{L\nu}q^{2L-2\nu}L_\nu$
$\lvert\langle\beta\boldsymbol{\sigma}, \mathbf{r}^L\rangle^{L}\rvert^2$	$G_T^2N^2(L)\frac{4\pi}{3}\frac{L+1}{L}\sum_{\nu=0}^{L}\left[A_{L\nu}q^{2(L-\nu-1)}M_\nu-2C_{L\nu}q^{2L-2\nu-1}N_\nu+\left(D_{L\nu}-\frac{B_{L\nu}}{L+1}\right)q^{2L-2\nu}L_\nu\right]$
$\lvert\langle\beta\boldsymbol{\sigma}, \mathbf{r}^L\rangle^{L-1}\rvert^2\delta_{L,1}$	$G_T^2N^2(1)\,4\pi(\frac{1}{9}q^2L_0+\frac{2}{3}qN_0+M_0)$
$\lvert\langle\beta\boldsymbol{\alpha}, \mathbf{r}^{L-1}\rangle^{L}\rvert^2$	$G_T^2N^2(L-1)\frac{4\pi}{3}\sum_{\nu=0}^{L}A_{L\nu}q^{2L-2\nu-2}L_\nu$
$\lvert\langle\beta\boldsymbol{\alpha}, \mathbf{r}^{L-1}\rangle^{L-2}\rvert^2\,\delta_{L,2}$	$G_T^2N(1)\,4\pi(\frac{1}{9}q^2L_0+\frac{2}{3}qN_0+M_0)$
$\mathrm{i}\langle\beta\boldsymbol{\sigma}, \mathbf{r}^L\rangle^{L*}\langle\beta\boldsymbol{\alpha}, \mathbf{r}^{L-1}\rangle^{L}+\text{c.c.}$	$G_T^2N(L)\,N(L-1)\sqrt{\left(\frac{L+1}{L}\right)}\frac{4\pi}{3}\sum_{\nu=0}^{L}[-A_{L\nu}q^{2L-2\nu-2}N_\nu+C_{L\nu}q^{2L-2\nu-1}L_\nu]$
$\lvert\langle\beta\gamma_5, \mathbf{r}^{L-1}\rangle^{L-1}\rvert^2\delta_{L,1}$	$G_P^2N^2(0)\,L_0$
$\mathrm{i}\langle\beta\mathbf{r}^L\rangle^{L*}\langle\beta\boldsymbol{\alpha}, \mathbf{r}^{L-1}\rangle^{L}+\text{c.c.}$	$G_SG_TN(L)\,N(L-1)\sqrt{\left(\frac{4\pi}{3}\right)}\sum_{\nu=0}^{L}[A_{L\nu}q^{2L-2\nu-2}N_\nu+C_{L\nu}q^{2L-2\nu-1}L_\nu]$
$\langle\beta\mathbf{r}^L\rangle^{L*}\langle\beta\boldsymbol{\sigma}, \mathbf{r}^{L}\rangle^{L}+\text{c.c.}$	$G_SG_TN^2(L)\sqrt{\left(\frac{L+1}{L}\frac{4\pi}{3}\right)}\sum_{\nu=0}^{L}[A_{L\nu}q^{2L-2\nu-2}M_\nu+(D_{L\nu}-B_{L\nu})\,q^{2L-2\nu}L_\nu]$
$\mathrm{i}\langle\beta\boldsymbol{\sigma}, \mathbf{r}^L\rangle^{L+1*}\langle\beta\gamma_5, \mathbf{r}^{L+1}\rangle^{L+1}+\text{c.c.}$	$G_PG_TN(L)\,N(L+1)\sqrt{\left(\frac{4\pi}{3}\right)}\sum_{\nu=0}^{L+1}[A_{L+1,\nu}q^{2L-2\nu}N_\nu+C_{L+1,\nu}q^{2L-2\nu+1}L_\nu]$
$\mathrm{i}\langle\beta\boldsymbol{\sigma}, \mathbf{r}^L\rangle^{L-1*}\langle\beta\gamma_5, \mathbf{r}^{L-1}\rangle^{L-1}\delta_{L,2}+\text{c.c.}$	$G_PG_TN(1)\,N(0)\sqrt{(4\pi)}\,(\frac{1}{3}qL_0+N_0)$

At the present stage more emphasis is put on the STP combination, as Marshak & Petschek (1952) pointed out that the presence of P is required in order to explain the spectrum of RaE.

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