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On the non-radial oscillations of a star

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A complete theory of the non-radial oscillations of a static spherically symmetric distribution of matter, described in terms of an energy density and an isotropic pressure, is developed, *ab initio*, on the premise that the oscillations are excited by incident gravitational waves. The equations, as formulated, enable the decoupling of the equations governing the perturbations in the metric of the space-time from the equations governing the hydrodynamical variables. This decoupling of the equations reduces the problem of determining the complex characteristic frequencies of the quasi-normal modes of the non-radial oscillations to a problem in the scattering of incident gravitational waves by the curvature of the space-time and the matter content of the source acting as a potential.

The present paper is restricted (for the sake of simplicity) to the case when the underlying equation of state is barotropic. The algorism developed for the determination of the quasi-normal modes is directly confirmed by comparison with an independent evaluation by the extant alternative algorism. Both polar and axial perturbations are considered. Dipole oscillations (which do not emit gravitational waves) are also treated as a particularly simple special case.

Thus, *all* aspects of the theory of the non-radial oscillations of stars find a unified treatment in the present approach.

The reduction achieved in this paper, besides providing a fresh understanding of known physical problems when formulated in the spirit of general relativity, provides also a basis for an understanding, at a deeper level, of Newtonian theory itself.

1. Introduction

During the vintage years of relativistic astrophysics, the two central problems were the radial and the non-radial oscillations of spherical stars: the solutions to the linearized versions of the relevant exact equations of the theory provided unique insights into the physical consequences that may derive from the general theory of relativity without the ambiguity and the uncertainty of *ad hoc* approximative treatments. Thus, the theory of radial oscillations (Chandrasekhar 1964*a*, *b*) revealed a global instability of relativistic origin – an instability that is ultimately the cause for the gravitational collapse of massive stars – while the theory of the non-radial oscillations, initiated by Thorne and his collaborators (Thorne & Compolattaro 1967 and subsequent papers: for a fairly complete bibliography see Lindblom & Detweiler 1983) provided an adequate base for a rigorous resolution of the vexed question of

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the emission of gravitational radiation by non-stationary sources and the damping that ensues.

The theory of radial oscillations is so direct and so simple that its implications were immediately appreciated. But the completion of the theory of non-radial oscillations, on the base provided by Thorne, was slow in coming: only in 1983 did Lindblom & Detweiler succeed in bringing the analytical framework of the theory to a satisfactory enough stage to allow a tabulation of the real and the imaginary parts of the characteristic frequencies of the quasi-normal quadrupole ($l = 2$, by Lindblom & Detweiler 1983) and higher order ($l \geqslant 2$, by Cutler & Lindblom 1987) modes of oscillation for a range of stellar models sufficient for a comparison with the observations on neutron stars.

Pursuant to a suggestion made in an earlier paper (Chandrasekhar & Ferrari 1990, §§8 and 13; this paper will be referred to hereafter as Paper I), we develop in this paper, *ab initio*, a complete unified version of the theory of the non-radial oscillations of a spherical distribution of matter (a star!) that provides not only a different physical base for the origin and the nature of these oscillations, but also simpler algorisms and a simpler set of equations for the numerical evaluation of the complex frequencies of the quasi-normal modes ($l \geqslant 2$) and the real frequencies of the dipole oscillations ($l = 1$).

We consider first the conceptual aspects of the present formulation in the following section.

2. The Newtonian and the relativistic views on the origin of non-radial oscillations

On the Newtonian view, we imagine that an initially static spherically symmetric distribution of matter, representing a star, is disturbed non-radially and set into oscillation by some *unspecified* external agent. We analyse the perturbation in spherical harmonics (Y_{lm}) and consider, individually, the modes of oscillation belonging to the different ls and ms. (In view of the spherical symmetry of the background static distribution, there will be no loss of generality in assuming that the perturbations belonging to the various modes are axisymmetric and are described by appropriate Legendre polynomials – a simplification that is vital in the relativistic context (see §§4 and 11).) In the mathematical treatment of the problem, the variable that is singled out is the Lagrangian displacement, ξ, which an element of mass experiences as we follow it during its motion: for, in its terms, the accompanying changes in the density, the pressure, and the gravitational potential, can all be expressed uniquely. By further supposing that the time dependence of the perturbations is given by $e^{i\sigma t}$, where σ is a constant (generally, but not necessarily, real) the linearized versions of the Poisson and the hydrodynamical equations are reduced to a characteristic value problem for σ (and often, if not invariably, formulated in terms of a variational principle).

In the relativistic theory, as developed by Thorne and others, the Newtonian procedure is followed literally in all details. Since in general relativity we have to consider a minimum of four metric functions (in place of the single Newtonian potential) the resulting theory is naturally complicated.

In the framework of the general theory of relativity, an alternative view of the origin of the non-radial oscillations of a star is possible. Instead of leaving unspecified the agent responsible for setting the star in oscillation, it would be natural to suppose that they are excited by the incidence of gravitational waves of a specified kind – polar or axial, belonging to particular Legendre or Gegenbauer polynomials – as in the treatment of the perturbations of the Schwarzschild space-time. The incident gravitational waves will be scattered (i.e. reflected and absorbed) by the curvature of the space-time; and the problem can be reduced to one in scattering theory. The solution to the problem of the quasi-normal modes (in which we are, of course, primarily interested) can be deduced from the general theory.

By developing the theory of the scattering of gravitational radiation by a spherical distribution of matter in strict analogy with the theory of the scattering by the Schwarzschild black-hole, as set out in *The mathematical theory of black holes* (Chandrasekhar 1983, §§24 and 26, pp. 142–152 and 160–163; this book will be referred to in the sequel as *M.T.*), we find that the perturbation equations allow, very directly, *a decoupling of the equations governing the metric perturbations from the equations governing the hydrodynamical perturbations*. Once the equations governing the metric functions have been solved, the solutions for the hydrodynamical variables follow *algebraically* without any further ado.

The reduction of the problem of the non-radial oscillations of a star to one purely of the scattering of gravitational waves by the static space-time of the star, without any reference to the motions that may or may *not* be induced in the star (as in the case of the axial perturbations considered in §11) is consonant with the physical base of the general theory of relativity. Clearly, this relativistic picture of the origin and the nature of the non-radial oscillations of a star is different from the Newtonian picture. Nowhere is this difference more manifestly brought out than in the total absence of the Lagrangian displacement from the equations determining the quasi-normal modes.

The plan of the paper is the following. In §3, the equations governing the static space-time of the star are quoted in the forms they are written in Paper I†. In §§4–10, the theory of the polar perturbations (the only class of perturbations that are normally considered) is developed; in §4, the linearized versions of the relevant equations of hydrodynamics and of Einstein are separated and a suitable set from them is selected. In §5 it is shown how the decoupling of the equations governing the metric variables from the equations governing the hydrodynamic variables can be effected; and the equations are specialized for the case when the equation of state governing the star is barotropic. (In the rest of the paper we restrict ourselves to this barotropic case.) An attempt to determine, in §6, the behaviour of the solutions near $r = 0$, reveals that the equations derived in §5 are not linearly independent at the origin. The origin of this novel circumstance is explored in §6 and it is shown how by defining a different set of independent variables, which satisfy equations that *are* linearly independent at the origin, we are led to a determinate indicial equation; and the behaviours of the solutions for the chosen variables at $r = 0$ are derived. In §7 it

† The conventions and definitions that are used in this paper are the following:

$$G_{(a)(b)} = +2T_{(a)(b)} \tag{i}$$

and

$$T_{(a)(b)} = (\epsilon+p)\,u_{(a)}\,u_{(b)} - p\eta_{(a)(b)}; \tag{ii}$$

and for the Riemann tensor, the definitions in *M.T.*, ch. 1, are adopted. It should be noted that the conventions used in Paper I are unfortunately 'mixed': while in §9, in writing the equilibrium equations the conventions are the same as in this paper, in §§10–12, equation (i) is used with the *opposite* sign. However, the analysis in §§10–12 has been carried through consistently since the only equilibrium equation that is used is $\nu_{,r} = -p_{,r}/(\epsilon+p)$; and this equation is the same on both conventions.

is shown how a pair of singularity free solutions, allowed by the indicial equation, satisfy the requirements at $r = 0$, and exactly suffice to determine the unique solution (apart from a constant of proportionality) that satisfies the necessary conditions at the boundary of the star. In §8, the manner in which the solution, found for the interior, is to be extended into the vacuum outside the star is considered and how the asymptotic behaviour at infinity of the solution so extended enables the determination of the quasi-normal modes. And finally, in §9, we confirm by considering a specific example – a relativistic polytrope of index 1.5 – that the algorism provided in this paper for determining the *complex* characteristic frequencies of the quasi-normal modes, yields values in almost exact agreement with those given by the different algorism of Lindblom & Detweiler.

In §10, we consider the dipole oscillations of a star. Even though these oscillations are not accompanied by gravitational radiation, the problem is nevertheless amenable to the present treatment as a simple special case. An illustrative numerical example is provided.

And finally we consider the axial perturbations in §11. It is known that the incidence of axial gravitational waves cannot excite any motions in the star. On this account, the reduction of the problem to one of scattering by a potential barrier (see figure 2) is plainly to be expected. Indeed, we find that the wave equation one obtains is a simple generalization of the 'Regge–Wheeler' equation that governs the axial perturbations of the Schwarzschild black-hole.

3. The equations governing the static space-time

The metric of the static space-time of a spherically symmetric distribution of matter, described in terms of an energy density ϵ and an isotropic pressure p, and the equations determining the structure of a star, have been written down in Paper I. (It may be recalled that both in *M.T.* and in Paper I the basic equations are expressed in a tetrad frame.) For convenience of reference, we shall write them out again.

The metric of the space-time, both in the interior of the star and in the vacuum outside, is of the standard form,

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\mu_2}(dr)^2 - e^{2\mu_3}(dx^3)^2 - e^{2\psi}(d\varphi)^2. \tag{1}$$

where, in the present context,

$$e^{2\mu_3} = r^2, \quad e^{2\psi} = r^2 \sin^2\theta, \tag{2}$$

$r = x^2$ is a radial coordinate, and $\theta = x^3$ is a polar angle.

Inside the star, with its centre at $r = 0$, the metric functions ν and μ_2, are determined by

$$\nu_{,r} = -p_{,r}/(\epsilon + p) \tag{3}$$

and

$$e^{-2\mu_2} = 1 - 2M(r)/r, \tag{4}$$

where

$$M(r) = \int_0^r \epsilon r^2 \, dr. \tag{5}$$

The equation of hydrostatic equilibrium is

$$[1 - 2M(r)/r]\, p_{,r} = -(\epsilon + p)\,[pr + M(r)/r^2]. \tag{6}$$

Besides
$$2\epsilon r^2 = +1 - e^{-2\mu_2}(1 - 2r\mu_{2,r}),$$
$$2pr^2 = -1 + e^{-2\mu_2}(1 + 2r\nu_{,r}), \tag{7}$$
and
$$(\epsilon + p)\, r = e^{-2\mu_2}(\nu + \mu_2)_{,r}. \tag{8}$$

It should be noted that in obtaining the solution for ν by the integration of equation (3), we must allow for a constant of integration ν_0: thus,

$$\nu = -\int_0^r \frac{p_{,r}}{(\epsilon + p)}\,dr + \nu_0. \tag{9}$$

The constant ν_0 is to be determined by the condition that, at the boundary $r = r_1$, of the star (where p and, generally, also ϵ vanish),

$$(e^{2\nu})_{r=r_1} = (e^{-2\mu_2})_{r=r_1} = 1 - 2M/r_1, \tag{10}$$

where
$$M = \int_0^{r_1} \epsilon r^2\, dr. \tag{11}$$

denotes the inertial mass of the star. By this choice of ν_0, we ensure that the space-time outside the star is described by the Schwarzschild metric in its standard form.

4. The equations governing the polar perturbations of the star

For the polar perturbations of the star – we postpone to §11 consideration of the axial perturbations – the metric continues to be diagonal and the quantities which describe the perturbation are the amplitudes,

$$\delta\nu,\ \delta\psi,\ \delta\mu_2,\ \delta\mu_3,\ \delta\epsilon \quad \text{and} \quad \delta p, \tag{12}$$

of the oscillations, with a time dependence $e^{i\sigma t}$ where σ is a constant (not necessarily real), in the metric functions ν, ψ, μ_2, and μ_3 and the hydrodynamical variables ϵ and p. Also, we express the tetrad components of the four velocity, $u_{(2)}$ and $u_{(3)}$, of the motions that are induced in the meridian planes, in terms of the Lagrangian displacements, ξ_2 and ξ_3 in the manner (cf. Paper I, equations (97)):

$$\left.\begin{aligned} u_{(2)} &= \delta u_2 = \xi_{2,0} = i\sigma\xi_2, \\ u_{(3)} &= \delta u_3 = \xi_{3,0} = i\sigma\xi_3. \end{aligned}\right\} \tag{13}$$

(*a*) *The linearized hydrodynamical equations*

The linearized hydrodynamical equations for the problem on hand are (cf. Paper I, equations (100) and (101)):

$$-\sigma^2(\epsilon + p)\, e^{-\nu+\mu_2}\xi_2 = \delta p_{,r} + \delta(\epsilon + p)\,\nu_{,r} + (\epsilon + p)\,\delta\nu_{,r}, \tag{14}$$
$$-\sigma^2(\epsilon + p)\, e^{-\nu+\mu_3}\xi_3 = \delta p_{,\theta} + (\epsilon + p)\,\delta\nu_{,\theta}, \tag{15}$$

while the equation governing the conservation of baryon number is (cf. Paper I, equations (107) and (109)):

$$\delta\epsilon = Q\,\delta p + e^{\nu-\mu_2}(\epsilon_{,r} - Qp_{,r})\,\xi_2, \tag{16}$$

where
$$Q = (\epsilon + p)/\gamma p, \tag{17}$$

and γ denotes the adiabatic exponent (defined in Paper I, equation (106)).

(b) *The linearized Einstein equations*

In Paper I, §11, we have written down the linearized Einstein equations in the forms needed for the purposes of that paper, namely, the derivation of the flux integral that the equations admit (cf. Paper I, equations (132)–(134) with the difference in sign convention noted on p. 249). But for the purposes of this paper, it is more convenient to write down the inhomogeneous equations corresponding to the homogeneous equations (*M.T.*, pp. 145, 146, equations (31)–(34)) that govern the perturbations of the Schwarzschild black-hold. We have:

$$(\delta\psi+\delta\mu_3)_{,r}+\left(\frac{1}{r}-\nu_{,r}\right)(\delta\psi+\delta\mu_3)-\frac{2}{r}\delta\mu_2=-2(\epsilon+p)\,\xi_2\,\mathrm{e}^{\nu+\mu_2}, \tag{18}$$

$$(\delta\psi+\delta\mu_2)_{,\theta}+(\delta\psi-\delta\mu_3)\cot\theta=-2(\epsilon+p)\,\xi_3\,\mathrm{e}^{\nu+\mu_3}, \tag{19}$$

$$(\delta\psi+\delta\nu)_{,r,\theta}+(\delta\psi-\delta\mu_3)_{,r}\cot\theta+\left(\nu_{,r}-\frac{1}{r}\right)\delta\nu_{,\theta}-\left(\nu_{,r}+\frac{1}{r}\right)\delta\mu_{2,\theta}=0, \tag{20}$$

$$\begin{aligned}\mathrm{e}^{-2\mu_2}\left[\frac{2}{r}\delta\nu_{,r}+\left(\frac{1}{r}+\nu_{,r}\right)(\delta\psi+\delta\mu_3)_{,r}-2\left(\frac{1}{r}+2\nu_{,r}\right)\frac{\delta\mu_2}{r}\right]\\+\frac{1}{r^2}[(\delta\psi+\delta\nu)_{,\theta,\theta}+(2\delta\psi+\delta\nu-\delta\mu_3)_{,\theta}\cot\theta+2\delta\mu_3]\\-\mathrm{e}^{-2\nu}(\delta\psi+\delta\mu_3)_{,0,0}=+2\delta p,\end{aligned} \tag{21}$$

and

$$\begin{aligned}\mathrm{e}^{-2\mu_2}\left\{-2\delta\mu_2\left[\nu_{,r,r}+\left(\frac{1}{r}+\nu_{,r}\right)(\nu_{,r}-\mu_{2,r})\right]+(\delta\psi+\delta\nu)_{,r,r}\right.\\\left.+2\delta\psi_{,r}\psi_{,r}+(\nu_{,r}-\mu_{2,r})(\delta\psi+\delta\nu)_{,r}+\left(\frac{1}{r}+\nu_{,r}\right)(\delta\nu-\delta\mu_2)_{,r}\right\}\\+\mathrm{e}^{-2\mu_3}(\delta\nu+\delta\mu_2)_{,3}\cot\theta-\mathrm{e}^{-2\nu}(\delta\psi+\delta\mu_2)_{,0,0}=2\delta p.\end{aligned} \tag{22}$$

(Equation (22) is not included among the equations listed in *M.T.*; it derives from the G_{33}-component of the perturbed field equations.)

(c) *The separation of the variables*

Equations (13)–(16) and (18)–(22) can be separated by the substitutions (cf. *M.T.*, p. 147, equations (36)–(39), originally due to J. Friedman):

$$\left.\begin{aligned}\delta\nu=N(r)\,P_l(\cos\theta),\quad \delta\mu_2=L(r)\,P_l(\cos\theta),\\ \delta\mu_3=T(r)\,P_l+V(r)\,P_{l,\theta,\theta},\quad \delta\psi=T(r)\,P_l+V(r)\,P_{l,\theta}\cot\theta,\\ \xi_2(r,\theta)=\xi_2(r)\,P_l(\cos\theta),\quad \xi_3(r,\theta)=\xi_3(r)\,P_{l,\theta},\\ \delta p=\Pi(r)\,P_l(\cos\theta),\quad\text{and}\quad \delta\epsilon=E(r)\,P_l(\cos\theta).\end{aligned}\right\} \tag{23}$$

According to these substitutions,

$$\left.\begin{aligned}\delta\psi+\delta\mu_3=[2T-l(l+1)\,V]\,P_l,\\ \delta\psi_{,\theta}+(\delta\psi-\delta\mu_3)\cot\theta=(T-V)\,P_{l,\theta},\\ \delta\psi_{,\theta,\theta}+(2\delta\psi-\delta\mu_3)_{,\theta}\cot\theta+2\delta\mu_3=-(l-1)\,(l+2)\,TP_l.\end{aligned}\right\} \tag{24}$$

With the further definitions,

$$\left.\begin{aligned} 2(\epsilon+p)\,e^{\nu+\mu_2}\xi_2(r,\theta) &= U(r)\,P_l, \\ 2(\epsilon+p)\,e^{\nu+\mu_3}\xi_3(r,\theta) &= W(r)\,P_{l,\theta}, \end{aligned}\right\} \tag{25}$$

the hydrodynamical equations (14) and (15) give:

$$U = 2(\epsilon+p)\,e^{\nu+\mu_2}\xi_2 = -\frac{2}{\sigma^2}e^{2\nu}\,[(E+\Pi)\,\nu_{,r}+\Pi_{,r}+(\epsilon+p)\,N_{,r}], \tag{26}$$

$$W = 2(\epsilon+p)\,e^{\nu+\mu_3}\xi_3 = -\frac{2}{\sigma^2}e^{2\nu}\,[\Pi+(\epsilon+p)\,N]. \tag{27}$$

Making use of equations (24), we find from equations (18)–(22):

$$\left[\frac{d}{dr}+\left(\frac{1}{r}-\nu_{,r}\right)\right](2T-\kappa V)-\frac{2}{r}L = -U, \tag{28}$$

$$T-V+L = -W, \tag{29}$$

$$(T-V+N)_{,r}-\left(\frac{1}{r}-\nu_{,r}\right)N-\left(\frac{1}{r}+\nu_{,r}\right)L = 0, \tag{30}$$

$$\frac{2}{r}N_{,r}+\left(\frac{1}{r}+\nu_{,r}\right)(2T-\kappa V)_{,r}-\frac{2}{r}\left(\frac{1}{r}+2\nu_{,r}\right)L$$
$$-\frac{1}{r^2}(2nT+\kappa N)\,e^{2\mu_2}+\sigma^2\,e^{-2\nu+2\mu_2}(2T-\kappa V) = 2e^{2\mu_2}\Pi, \tag{31}$$

$$V_{,r,r}+\left(\frac{2}{r}+\nu_{,r}-\mu_{2,r}\right)V_{,r}+\frac{e^{2\mu_2}}{r^2}(N+L)+\sigma^2\,e^{2\mu_2-2\nu}V = 0, \tag{32}$$

where

$$\kappa = l(l+1), \quad \text{and} \quad 2n = (l-1)\,(l+2) = \kappa-2. \tag{33}$$

Equation (32) is in many ways a remarkable equation. It was first encountered in the form,

$$V_{,r,r}+2\left(\frac{1}{r}+\nu_{,n}\right)V_{,r}+\frac{e^{2\mu_2}}{r^2}(N+L)+\sigma^2\,e^{-4\nu}V = 0 \tag{34}$$

in the context of the perturbations of the Schwarzschild black-hole (*M.T.*, p. 148, equation (51)). It was next encountered in the context of the perturbations of the Reissner–Nordström black-hole (*M.T.*, pp. 232–233). (In these two later cases, $\nu = -\mu_2$, and the reduction of equation (32) to the simpler form (34) is clearly required.) We now encounter equation (32) once again in the present context of the perturbations of the spherically symmetric distribution of matter. The remarkable feature of equation (32) is that it seems to govern quite generally the perturbations of spherically symmetric static space-times independently of the nature of the source. A derivation of equation (34) is provided in Appendix A.

Transcribing equation (16) similarly, we obtain

$$E = Q\Pi+\frac{e^{-2\mu_2}}{2(\epsilon+p)}(\epsilon_{,r}-Qp_{,r})\,U. \tag{35}$$

Next, rewriting equations (27) and (28) in the forms

$$\Pi_{,r}+(E+\Pi)\,\nu_{,r}+(\epsilon+p)\,N_{,r}=-\tfrac{1}{2}\sigma^2\,\mathrm{e}^{-2\nu}U, \tag{36}$$

and

$$-\Pi=\tfrac{1}{2}\sigma^2\,\mathrm{e}^{-2\nu}W+(\epsilon+p)\,N; \tag{37}$$

and eliminating E and Π from equation (35) with the aid of equations (34) and (36), we obtain:

$$\frac{1}{2}\left[\sigma^2\,\mathrm{e}^{-2\nu}+\frac{\mathrm{e}^{-2\mu_2}\nu_{,r}}{\epsilon+p}(\epsilon_{,r}-Qp_{,r})\right]U=(\tfrac{1}{2}\sigma^2\,\mathrm{e}^{-2\nu}W)_{,r}$$
$$+(Q+1)\,\nu_{,r}(\tfrac{1}{2}\sigma^2\,\mathrm{e}^{-2\nu}W)+(\epsilon_{,r}-Qp_{,r})N. \tag{38}$$

Returning to equation (31), we can eliminate Π from the right-hand side of the equation, in favour of the scalars representing the metric functions, by writing equation (36) in the equivalent form (cf. equation (30)),

$$\Pi=\tfrac{1}{2}\sigma^2\,\mathrm{e}^{-2\nu}(T-V+L)-(\epsilon+p)\,N, \tag{39}$$

we find:

$$\frac{2}{r}N_{,r}+\left(\frac{1}{r}+\nu_{,r}\right)(2T-\kappa V)_{,r}-\frac{2}{r^2}(1+2\nu_{,r}\,r)\,L$$
$$-\frac{\mathrm{e}^{2\mu_2}}{r^2}(2nT+\kappa V)+2\mathrm{e}^{2\mu_2}(\epsilon+p)\,N+\sigma^2\,\mathrm{e}^{2\mu_2-2\nu}[T-(\kappa-1)\,V-L]=0. \tag{40}$$

5. The decoupling of the metric perturbations from the hydrodynamical perturbations: the barotropic case

By replacing equation (31) by equation (40) and eliminating from equation (28) with the aid of equation (38), we can clearly decouple the equations governing the metric perturbations from the equations governing the hydrodynamical perturbations. In developing this new approach to the theory of non-radial oscillations, it is convenient, in the first instance, for the sake of simplicity, to restrict our consideration to the *barotropic case*, i.e. when the pressure is a unique function of the energy density and the generally applicable relation,

$$\gamma=\frac{\epsilon+p}{p}\left(\frac{\partial p}{\partial\epsilon}\right)_{\text{entropy=const.}} \tag{41}$$

becomes,

$$\gamma=\frac{\epsilon+p}{p}\frac{\mathrm{d}p}{\mathrm{d}\epsilon}\quad(p\equiv p(\epsilon)). \tag{42}$$

The relation (42) is applicable in the most important contexts that one has in view in developing this theory, namely, white dwarfs and neutron stars.

For an initially static spherical distribution of matter with a barotropic equation of state, equation (42) is equivalent to

$$\gamma=(\epsilon+p)\,p_{,r}/\epsilon_{,r}. \tag{43}$$

For γ given by equation (43), Q as defined in equation (17) becomes,

$$Q=\epsilon_{,r}/p_{,r}, \tag{44}$$

and equation (38) reduces to the very simple form

$$U = W_{,r} + (Q-1)\,\nu_{,r}\,W. \tag{45}$$

The elimination of U from equation (28) now yields:

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r}-\nu_{,r}\right)(2T-\kappa V)-\frac{2}{r}L = W_{,r}+(Q-1)\,\nu_{,r}\,W. \tag{46}$$

Equations (29), (30), (32), (40), and (46) now provide a basic set of equations for the four functions N, L, T, and V that describe the metric perturbations. In considering these equations, it is convenient to eliminate T in favour of W and choose as our independent variables the functions,

$$N, L, X = \tfrac{1}{2}(l-1)\,(l+2)\,V = nV \quad \text{and} \quad W = -(T-L+V). \tag{47}$$

In making this choice, we are following the treatment of the perturbations of the Schwarzschild black-hole as set out in *M.T.* 24(*b*) (pp. 145–152); only in this latter case $W = 0$. In other words, while W is simply related to the Lagrangian displacement ξ_3, it should be considered, here and in the sequel, as an abbreviation for the combination of their scalars $-(T-V+L)$, describing the metric perturbations.

With the choice of variables (47), we obtain, in place of equations (30), (32), (40), and (46), the following set of equations:

$$L_{,r}+\left(\frac{2}{r}-\nu_{,r}\right)L+\left[X_{,r}+\left(\frac{1}{r}-\nu_{,r}\right)X\right]+\tfrac{1}{2}\left\{W_{,r}+\left[\frac{2}{r}-(Q+1)\,\nu_{,r}\right]W\right\}=0, \tag{48}$$

$$N_{,r}-\left(\frac{1}{r}-\nu_{,r}\right)N-\left[L_{,r}+\left(\frac{1}{r}+\nu_{,r}\right)L\right]-W_{,r}=0, \tag{49}$$

$$\begin{aligned} N_{,r}-\mathrm{e}^{2\mu_2}\left[\frac{n+1}{r}-(\epsilon+p)\,r\right]N \\ -\left\{(1+r\nu_{,r})\,L_{,r}-\frac{1}{r}(n\,\mathrm{e}^{2\mu_2}-1)\,L+2\nu_{,r}L+\sigma^2\,\mathrm{e}^{2(\mu_2-\nu)}\,rL\right\} \\ -\left\{(1+r\nu_{,r})\,X_{,r}+\frac{\mathrm{e}^{2\mu_2}}{r}X+\sigma^2\,\mathrm{e}^{2(\mu_2-\nu)}rX\right\} \\ -\left\{(1+r\nu_{,r})\,W_{,r}-\frac{n}{r}\mathrm{e}^{2\mu_2}W+\tfrac{1}{2}\sigma^2\,\mathrm{e}^{2(\mu_2-\nu)}rW\right\}=0, \end{aligned} \tag{50}$$

$$X_{,r,r}+\left(\frac{2}{r}+\nu_{,r}-\mu_{2,r}\right)X_{,r}+n\frac{\mathrm{e}^{2\mu_2}}{r^2}(N+L)+\sigma^2\,\mathrm{e}^{2(\mu_2-\nu)}\,X=0. \tag{51}$$

Equations (48)–(51) provide a system of four coupled linear equations for the four functions N, L, X, and W which describe the perturbations of the space-time. They are of total degree 5 and not 4 as in the reduction of Lindblom & Detweiler in their (conventional) treatment of the problem. It does not however appear that the degree of the system of equations (48)–(51) can be reduced further: the reason derives from the special character of equation (51) to which we have already made reference in §4 following equation (32); and, possibly, also for a more fundamental reason: e.g. the linear dependence of the equations (48)–(51) at $r = 0$ pointed out in §6.

6. The transformation of equations (48)–(51) to a system linearly independent at the origin: the indicial equation and the behaviour of the solutions at $r = 0$

A novel feature of equations (48)–(51) – their linear dependence at the origin – becomes manifest when we examine the behaviour of the solutions at $r = 0$.

For establishing the behaviour of the solutions for $r \to 0$, it is, of course, necessary to know the behaviours of the various quantities that describe the static space-time of the star. We list below the expansions that we shall need.

$$\left.\begin{aligned}
&\epsilon = \epsilon_0 - \epsilon_2 r^2 + \dots, \quad p = p_0 - p_2 r^2 + \dots, \quad Q = \epsilon_{,r}/p_{,r} = Q_0 + O(r^2);\\
&Q_0 = \epsilon_2/p_2, \quad M(r) = \tfrac{1}{3}\epsilon_0 r^3 - \tfrac{1}{5}\epsilon_2 r^5 + \dots, \quad \sigma_0 = \sigma e^{-\nu_0},\\
&e^{-2\mu_2} = 1 - 2M(r)/r = 1 - \tfrac{2}{3}\epsilon_0 r^2 + \tfrac{2}{5}\epsilon_2 r^4,\\
&e^{2\mu_2} = 1 + br^2 + b_2 r^4, \quad e^{2(\mu_2 - \nu)} = e^{-2\nu_0}[1 + (b-a) r^2],\\
&\mu_{2,r} = r(b + b_2^{(2)} r^2), \quad \nu_{,r} = r(a + a_2 r^2),\\
&a = p_0 + \tfrac{1}{3}\epsilon_0, \quad a_2 = \tfrac{2}{3}\epsilon_0(p_0 + \tfrac{1}{3}\epsilon_0) - (p_2 + \tfrac{1}{5}\epsilon_2),\\
&b = \tfrac{2}{3}\epsilon_0, \quad b_2 = \tfrac{4}{9}\epsilon_0^2 - \tfrac{2}{5}\epsilon_2, \quad b_2^{(2)} = \tfrac{4}{9}\epsilon_0^2 - \tfrac{4}{5}\epsilon_2.
\end{aligned}\right\} \tag{52}$$

We may, parenthetically, note the relations,

$$a + b = \epsilon_0 + p_0 \quad \text{and} \quad a - b = p_0 - \tfrac{1}{3}\epsilon_0. \tag{53}$$

We now seek the behaviour of the solutions, N, L, X, and W of equations (48)–(51) via an indicial equation for the exponent x in the substitution,

$$(N, L, X, W) = (N_0, L_0, X_0, W_0)\, r^x + O(r^{x+2}). \tag{54}$$

By this substitution, we obtain a system of linear homogeneous equations for N_0, L_0, X_0, and W_0; and the required vanishing of the determinant of the system leads to the indicial equation:

$$\begin{vmatrix}
0 & x+2 & x+1 & \tfrac{1}{2}x+1\\
x-1 & -(x+1) & 0 & -x\\
x-(n+1) & -x+(n-1) & -(x+1) & -x+n\\
n & n & x(x+1) & 0
\end{vmatrix} \equiv 0. \tag{55}$$

The determinant vanishes identically and any value of the exponent x is allowed! The reason for this paradoxical result is that equations (48)–(51) are *not* linearly independent at the origin (though they are at finite distances from the origin). This possible feature of a system of coupled linear equations does not seem to have been considered in the extant standard treatments of the subject. However, in the present context, the impasse can be overcome by a judicious choice of independent variables when we do obtain equations which *are* linearly independent at the origin. We proceed as follows.

First, separating the terms in equation (50) that are of the lowest order in r (as

$r \to 0$) (and which contribute to the indicial equation (55)) from the rest, we can rewrite it in the form,

$$N_{,r} - \frac{n+1}{r}N - \left[\frac{n+1}{r}(e^{2\mu_2}-1) - e^{2\mu_2}(\epsilon+p)\,r\right]N$$
$$-\left(L_{,r} - \frac{n-1}{r}L\right) - \left[r\nu_{,r}L_{,r} - \frac{n}{r}(e^{2\mu_2}-1)\,L + 2\nu_{,r}L + \sigma^2\,e^{2(\mu_2-\nu)}rL\right]$$
$$-\left(X_{,r} + \frac{1}{r}X\right) - \left[r\nu_{,r}X_{,r} + \frac{1}{r}(e^{2\mu_2}-1)\,X + \sigma^2\,e^{2(\mu_2-\nu)}rX\right]$$
$$-\left(W_{,r} - \frac{n}{r}W\right) - \left[r\nu_{,r}W_{,r} - \frac{n}{r}(e^{2\mu_2}-1)\,W + \tfrac{1}{2}\sigma^2\,e^{2(\mu_2-\nu)}rW\right] = 0, \tag{56}$$

or, in the abridged form,

$$N_{,r} - \frac{n+1}{r}N - \left(L_{,r} - \frac{n-1}{r}L\right) - \left(X_{,r} + \frac{1}{r}X\right) - \left(W_{,r} - \frac{n}{r}W\right) - \text{III (say)} = 0 \tag{57}$$

Similarly, rewriting equation (49) in the form,

$$N_{,r} - \frac{1}{r}N - \left(L_{,r} + \frac{1}{r}L\right) - W_{,r} + \nu_{,r}(N-L) = 0, \tag{58}$$

and subtracting it from equation (57), we obtain

$$-n(N-L-W) - (rX_{,r} + X) = r[\text{III} + \nu_{,r}(N-L)] = r^2\mathscr{G}\ \text{(say)}. \tag{59}$$

From this last equation, it follows that

$$(r^2\mathscr{G})_{,r} = -n(N-L-W)_{,r} - (rX_{,r,r} + 2X_{,r}). \tag{60}$$

On the other hand, according to equations (49) and (51),

$$(N-L-W)_{,r} = \frac{1}{r}(N+L) - \nu_{,r}(N-L). \tag{61}$$

and

$$X_{,r,r} + \frac{2}{r}X_{,r} + \frac{n}{r^2}(N+L) + \frac{n}{r^2}(e^{2\mu_2}-1)\,(N+L) + (\nu_{,r} - \mu_{2,r})\,X_{,r} + \sigma^2\,e^{2(\mu_2-\nu)}X = 0. \tag{62}$$

Substituting from these last two equations for the terms on the right-hand side of equation (60), we obtain

$$(r^2\mathscr{G})_{,r} = n\nu_{,r}(N-L) + \frac{n}{r}(e^{2\mu_2}-1)\,(N+L) + r(\nu_{,r} - \mu_{2,r})\,X_{,r} + \sigma^2\,e^{2(\mu_2-\nu)}rX. \tag{63}$$

Returning to $\mathscr{G}$ as defined in equations (57) and (59), namely,

$$\mathscr{G} = [\text{III} + \nu_{,r}(N-L)]\,r^{-1}, \tag{64}$$

and evaluating it accordingly, we find

$$\mathscr{G} = \nu_{,r}(L+X+W)_{,r} + \frac{1}{r^2}(e^{2\mu_2}-1)\,[(n+1)N - n(L+W) + X]$$
$$+ \frac{\nu_{,r}}{r}(N+L) - e^{2\mu_2}(\epsilon+p)\,N + \sigma^2\,e^{2(\mu_2-\nu)}(L+X+\tfrac{1}{2}W). \tag{65}$$

We shall find it suitable for our purposes to simplify the foregoing expression for $\mathscr{G}$ with the aid of the equation (cf. equation (61))

$$\nu_{,r}(N-L-W)_{,r}-\frac{\nu_{,r}}{r}(N+L)+(\nu_{,r})^2(N-L)=0. \qquad (66)$$

We find:

$$\mathscr{G}=\nu_{,r}[(N+X)_{,r}+\nu_{,r}(N-L)]+\frac{1}{r^2}(e^{2\mu_2}-1)[(n+1)N-n(L+W)+X]$$
$$-e^{2\mu_2}(\epsilon+p)N+\sigma^2 e^{2(\mu_2-\nu)}(L+X+\tfrac{1}{2}W). \qquad (67)$$

We shall now show that by considering

$$N, L, X, \quad \text{and} \quad \mathscr{G} \qquad (68)$$

as the independent variables, we can obtain a set of equations for them which are linearly independent at the origin. For the required elimination of W we make use of the equation (cf. equation (59))

$$W=\frac{r^2}{n}\mathscr{G}+(N-L)+\frac{1}{n}(rX_{,r}+X). \qquad (69)$$

Substituting for W from this equation in equation (67), we obtain after some additional simplifications,

$$\mathscr{G}=\nu_{,r}[(N+X)_{,r}+\nu_{,r}(N-L)]+\frac{1}{r^2}(e^{2\mu_2}-1)(N-rX_{,r}-r^2\mathscr{G})$$
$$-e^{2\mu_2}(\epsilon+p)N+\tfrac{1}{2}\sigma^2 e^{2(\mu_2-\nu)}\left\{N+L+\frac{r^2}{n}\mathscr{G}+\frac{1}{n}[rX_{,r}+(2n+1)X]\right\}. \qquad (70)$$

Equations (51), (63), and (70) provide three of the required equations. A fourth equation is obtained by eliminating $W_{,r}$ from equations (48) and (49) and substituting for W from equation (69). We find in this fashion,

$$(L+N+2X)_{,r}+\left(\frac{1}{r}-\nu_{,r}\right)(-N+3L+2X)$$
$$+\left[\frac{2}{r}-(Q+1)\nu_{,r}\right]\left\{N-L+\frac{r^2}{n}\mathscr{G}+\frac{1}{n}(rX_{,r}+X)\right\}=0. \qquad (71)$$

The basic set of equations, when considering X, $\mathscr{G}$, N, and L as the independent variables, is

$$X_{,r,r}+\left(\frac{2}{r}+\nu_{,r}-\mu_{2,r}\right)X_{,r}+\frac{n}{r^2}e^{2\mu_2}(N+L)+\sigma^2 e^{2(\mu_2-\nu)}X=0, \qquad (72)$$

$$(r^2\mathscr{G})_{,r}=n\nu_{,r}(N-L)+\frac{n}{r}(e^{2\mu_2}-1)(N+L)+r(\nu_{,r}-\mu_{2,r})X_{,r}+\sigma^2 e^{2(\mu_2-\nu)}rX, \qquad (73)$$

$$-\nu_{,r}N_{,r}=-\mathscr{G}+\nu_{,r}[X_{,r}+\nu_{,r}(N-L)]+\frac{1}{r^2}(e^{2\mu_2}-1)(N-rX_{,r}-r^2\mathscr{G})$$
$$-e^{2\mu_2}(\epsilon+p)N+\tfrac{1}{2}\sigma^2 e^{2(\mu_2-\nu)}\left\{N+L+\frac{r^2}{n}\mathscr{G}+\frac{1}{n}[rX_{,r}+(2n+1)X]\right\}, \qquad (74)$$

$$-L_{,r} = (N+2X)_{,r} + \left(\frac{1}{r} - \nu_{,r}\right)(-N+3L+2X) + \left[\frac{2}{r} - (Q+1)\,\nu_{,r}\right]\left[N - L + \frac{r^2}{n}\mathcal{G} + \frac{1}{n}(rX_{,r}+X)\right]. \quad (75)$$

The indicial equation

By the manner of their derivation, we may expect that equations (72)–(75) are linearly independent at $r = 0$. Indeed, as we shall now show, they lead to a determinate indicial equation which provides the required number of linearly independent solutions at the origin. Thus, by assuming a behaviour of the form,

$$(X, \mathcal{G}, N, L) = (X_0, \mathcal{G}_0, N_0, L_0)\, r^x + O(r^{x+2}), \quad (76)$$

and making use of the expansions listed in equations (52), we find for the exponent x the indicial equation,

$$(x+1)\begin{vmatrix} x(x+1) & 0 & n & n \\ (a-b)\,x+\sigma_0^2 & -(x+2) & n(a+b) & -n(a-b) \\ (a-b)\,x+\dfrac{\sigma_0^2}{2n}(x+2n+1) & -1 & a(x-1)+\tfrac{1}{2}\sigma_0^2 & \tfrac{1}{2}\sigma_0^2 \\ l(l+1) & 0 & n & n \end{vmatrix} = 0, \quad (77)$$

where in reducing the determinant to the foregoing form, we have made use of the first of the two relations (53). By expanding the determinant, we find

$$na(x+1)\,(x-l)^2(x+l+1)^2 = 0. \quad (78)$$

We conclude that *equations* (72–(75) *allow two linearly independent singularity-free solutions with the behaviour r^l at the origin*. Indeed, for $x = l$, the homogeneous system of equations for X_0, $\mathcal{G}_0$, N_0, and L_0, that follows from equation (77), are:

$$\left.\begin{aligned} l(l+1)\,X_0 + n(L_0+N_0) &= 0, \\ [(a-b)\,l+\sigma_0^2]\,X_0 - (l+2)\,\mathcal{G}_0 + n(a+b)\,N_0 - n(a-b)\,L_0 &= 0, \\ \left[(a-b)\,l + \frac{\sigma_0^2}{2n}(l^2+2l-1)\right]X_0 - \mathcal{G}_0 + a(l-1)\,N_0 + \tfrac{1}{2}\sigma_0^2(N_0+L_0) &= 0; \end{aligned}\right\} \quad (79)$$

and a fourth equation which is a repetition of the first.

From these equations, it readily follows that we may take for the two linearly independent solutions with the behaviour r^l at the origin, the ones derived from

$$L_0 = 0, \quad X_0 = -\frac{n}{l(l+1)}N_0,$$

$$\mathcal{G}_0 = +\tfrac{1}{2}(l-1)\left\{a+b-\frac{1}{l(l+1)}[(a-b)\,l+\sigma_0^2]\right\}N_0; \quad (80)$$

and

$$N_0 = 0, \quad X_0 = -\frac{n}{l(l+1)}L_0,$$

$$\mathcal{G}_0 = -\tfrac{1}{2}(l-1)\left\{a-b+\frac{1}{l(l+1)}[(a-b)\,l+\sigma_0^2]\right\}L_0. \quad (81)$$

Expansions including terms of $O(r^{l+2})$ can readily be found by standard procedures. In particular, by assuming an expansion of the form

$$(X, \mathscr{G}, N, L) = (X_0, \mathscr{G}_0, N_0, L_0)\, r^l + (X_2, \mathscr{G}_2, N_2, L_2)\, r^{l+2} + O(r^{l+4}) \tag{82}$$

we find

$$\left.\begin{aligned}
&(l+2)(l+3)X_2 + n(N_2+L_2) + l(a-b)X_0 + nb(N_0+L_0) + \sigma_0^2 X_0 = 0,\\
&[(l+2)(a-b)+\sigma_0^2]X_2 - (l+4)\mathscr{G}_2 + n(a+b)N_2 - n(a-b)L_2\\
&\qquad + n(a_2+b_2)N_0 - n(a_2-b_2)L_0 + l(a_2-b_2^{(2)})X_0 + \sigma_0^2(b-a)X_0 = 0,\\
&-\mathscr{G}_2 + [(l+1)a + \tfrac{1}{2}\sigma_0^2]N_2 + \tfrac{1}{2}\sigma_0^2 L_2 + \left[(l+2)(a-b) + \frac{(l+1)^2}{2n}\sigma_0^2\right]X_2\\
&\qquad + [a_2 l + a^2 + b_2 - b(\epsilon_0+p_0) + (\epsilon_2+p_2) + \tfrac{1}{2}\sigma_0^2(b-a)]N_0\\
&\qquad + [\tfrac{1}{2}\sigma_0^2(b-a) - a^2]L_0 + \left[l(a_2-b_2) + \frac{\sigma_0^2}{2n}(b-a)(l^2+2l-1)\right]X_0 + \left(\frac{\sigma_0^2}{2n} - b\right)\mathscr{G}_0 = 0,\\
&(l+3)\left[N_2 + L_2 + \frac{l(l+1)}{n}X_2\right] - a\left\{Q_0 N_0 - (Q_0-2)L_0 + \left[2 + (l+1)\frac{Q_0+1}{n}\right]X_0\right\} + \frac{2}{n}\mathscr{G}_0 = 0.
\end{aligned}\right\} \tag{83}$$

where we may insert for $(X_0, \mathscr{G}_0, N_0, L_0)$ either of the two solutions (80) or (81).

7. The interior solution

A solution that describes the interior of a star must satisfy certain requirements at the centre, ($r = 0$), and at the boundary, ($r = r_1$), where the pressure p of the static star vanishes.

At the centre the conditions are that $\delta\epsilon$ and δp (i.e. E and Π) both vanish. (A somewhat milder requirement may suffice: but the stronger conditions that $E = \Pi = 0$ and singularity free, are certainly sufficient.) By equations (35) and (37), the conditions are met by the requirement that W and N vanish at $r = 0$. Of the solutions allowed by the indicial equation (78), only the two linearly independent solutions with the r^l-behaviour at the origin are compatible with the physical requirements. These solutions are explicitly determined by the expansions specified in equations (80)–(83). (It may be noted here that the r^l-behaviour of the functions N, L, X, and W assures the required flatness of the space-time at the origin.)

At the boundary, $r = r_1$, of the star two conditions must be met: Π must vanish and we must also ensure that the space-time is continuous with the vacuum that prevails outside the star. (Strictly we need only require that Π vanishes on the displaced moving boundary of the oscillating star. This distinction between the static boundary and the moving boundary is relevant only if $\epsilon \neq 0$ on $r = r_1$. Allowance for this contingency is readily made: but we shall not digress now to consider this eventuality.) The vanishing of Π clearly requires (under the circumstances envisaged, namely that ϵ and p vanish on $r = r_1$)

$$W = -(T - V + L) = 0 \quad \text{at} \quad r = r_1 \tag{84}$$

and this is also one of the conditions that the metric perturbations are required to satisfy (cf. *M.T.*, p. 147, eqn (43)) in order to match continuously with the exterior metric perturbations – we consider the remaining conditions in §8 below.

The condition (84) on W follows from either of the equations (27) and (37). More stringent conditions on W follow from equations (26) and (45). The former equation, relating U with the Lagrangian displacement ξ_2, requires U to vanish on the boundary of the star. At the same time, it follows from entirely general considerations, explained in Appendix C, that

$$\left.\begin{aligned} &\nu_{,r} \text{ tends to a finite limit } \nu_1', \\ \text{and} \quad &Q = \epsilon_{,r}/p_{,r} \rightarrow Q_1/(r_1-r) \quad (r \rightarrow r_1-0), \end{aligned}\right\} \tag{85}$$

where Q_1 is some constant. By virtue of these behaviours, equation (45) gives,

$$U \rightarrow W_{,r} + \frac{Q_1 \nu_1'}{r_1-r} W \quad (r \rightarrow r_1-0). \tag{86}$$

Therefore, the governing equations *already* require that

$$W \rightarrow \text{const.}\,(r_1-r) \quad \text{for} \quad r \rightarrow r_1-0; \tag{87}$$

a result that is directly derived in Appendix C. We must *in addition* require (as a boundary condition that we must impose)

$$W_{,r} = 0 \quad \text{at} \quad r = r_1. \tag{88}$$

To satisfy this further condition at $r = r_1$, *a superposition of the two linearly independent solutions, belonging to the double root $x = l$ of the indicial equation, is needed*. For, along the solutions belonging to this double root, $W_{,r}$ tends to different constants as $r \rightarrow r_1 - 0$ – a fact that is established in Appendix C. We can therefore find for each, assigned σ^2 (> 0), a superposition of the two solutions, unique apart from a single constant of proportionality, that satisfies the condition (84) on the boundary.

In obtaining the required superposition of the two linearly independent solutions, belonging to the double root, $x = l$, we shall adopt the following prescription. Denoting by

$$[N_0 = 1; L_0 = 0] \quad \text{and} \quad [L_0 = 1; N = 0], \tag{89}$$

the two linearly independent solutions derived from the expansions (80)–(83) at the origin, we determine the constant, q in the superposition,

$$q[N_0 = 1; L_0 = 0] + [L_0 = 1; N_0 = 0], \tag{90}$$

by the requirement that $W_{,r} = 0$ at $r = r_1$. Solutions for the interior, $0 \leqslant r \leqslant r_1$, obtained in this fashion, satisfying the necessary boundary conditions at $r = 0$ and $r = r_1$ for different assigned σ^2, are 'normalized' so that

$$L_0 = 1 \tag{91}$$

for all of them.

8. The exterior solution and the procedure for determining the quasi-normal modes

In the vacuum that prevails outside the star ($r > r_1$), the equations that govern the metric perturbations are (*M.T.*, pp. 147–150, equations (43), (58), (62) and (63)):

$$T - V + L = 0 \tag{92}$$

– an equation which we have already considered in §7 as a boundary condition to be imposed on the interior solution for $r \rightarrow r_1 - 0$ – and the *Zerilli equation*,

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right) Z = V^{(+)} Z, \quad \text{where} \quad Z = \frac{r}{nr+3M}\left(\frac{3M}{n} X - rL\right). \tag{93}$$

$$\left.\begin{aligned} V^{(+)} &= \frac{2\Delta}{r^5(nr+3M)^2}[n^2(n+1)\,r^3+3Mn^2r^2+9M^2nr+9M^3], \\ \Delta &= r^2-2Mr, \frac{\mathrm{d}}{\mathrm{d}r_*} = \frac{\Delta}{r^2}\frac{\mathrm{d}}{\mathrm{d}r} = \left(1-\frac{2M}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r} \end{aligned}\right\} \tag{94}$$

and M is the mass of the star enclosed inside $r = r_1$. In integrating equation (93) into the vacuum, we must, to ensure the continuity of the space–time at $r = r_1$, use as starting values for Z and $Z_{,r_*}$, the values they have as $r \to r_1 - 0$ from the interior, i.e.

$$Z(r = r_1) = \lim_{r \to r_1 - 0}\left\{\frac{r}{nr+3M}\left(\frac{3M}{n}X - rL\right)\right\}, \tag{95}$$

and
$$Z_{,r_*}(r = r_1) = \left(1-\frac{2M}{r_1}\right)\lim_{r \to r_1 - 0}\left\{\left[\frac{r}{nr+3M}\left(\frac{3M}{n}X - rL\right)\right]\right\}_{,r}, \tag{96}$$

where

$$Z_{,r}(r = r_1) = \left\{\frac{1}{(nr+3M)^2}\left[\frac{3M}{n}(nr+3M)\,rX_{,r}+\frac{9M^2}{n}X - (nr+6M)\,rL - r^2(nr+3M)\,L_{,r}\right]\right\}_{r=r_1-0}. \tag{97}$$

With the starting values for Z and $Z_{,r_*}$, at $r = r_1$, determined, as described, from the interior solution, we must integrate the Zerilli equation forward to determine its asymptotic form.

One readily finds from the Zerilli equation that for $r \to \infty$, Z must have the asymptotic form (cf. Chandrasekhar & Detweiler 1975, equations (51) and (52); Lindblom & Detweiler 1983, equations (A 37)–(A 40)):

$$\begin{aligned} Z \to &+\left\{\alpha_0 - \frac{n+1}{\sigma}\frac{\beta_0}{r} - \frac{1}{2\sigma^2}\left[n(n+1)\,\alpha_0 - \tfrac{3}{2}M\sigma\left(1+\frac{2}{n}\right)\beta_0\right]\frac{1}{r^2}+\ldots\right\}\cos\sigma r_* \\ &-\left\{\beta_0 + \frac{n+1}{\sigma}\frac{\alpha_0}{r} - \frac{1}{2\sigma^2}\left[n(n+1)\,\beta_0 + \tfrac{3}{2}M\sigma\left(1+\frac{2}{n}\right)\alpha_0\right]\frac{1}{r^2}+\ldots\right\}\sin\sigma r_*. \end{aligned} \tag{98}$$

appropriate for *standing waves* and real σ. By integrating the Zerilli equation forward from $r = r_1$ to a sufficiently large r, we can determine α_0 and β_0 by matching the numerically integrated solution with that prescribed by equation (98).

A method for determining the characteristic frequencies of the quasi-normal modes

We have shown that, for any assigned real positive value of σ, the behaviour of the solution at infinity is that of standing gravitational waves with a determinate amplitude, $(\alpha_0^2+\beta_0^2)^{\frac{1}{2}}$. From the manner of its determination, it is clear that $\alpha_0^2+\beta_0^2$ is a unique function of σ, if the interior solutions, belonging to the double root $x = l$, are determined in the fashion described in the last paragraph of §7 (so that $L_0 = 1$ for all of them). An examination of this function will provide a method for determining the *complex characteristic frequency*, $\sigma_0 + \mathrm{i}\sigma_{\mathrm{i}}$ ($\sigma_{\mathrm{i}} > 0$) of the quasi-normal mode.

It follows from some general considerations, set out by Thorne (1969), in accordance with the well-known Breit–Wigner formula (cf. Landau & Lifshitz 1977, pp. 603–611), that $\alpha_0^2+\beta_0^2$, as a function of σ (>0, say) must exhibit a deep minimum at some determinate σ_0 with the behaviour,

$$\alpha_0^2+\beta_0^2 = \text{const.}\,[(\sigma-\sigma_0)^2+1/\tau^2] \quad (\tau = 1/\sigma_{\text{i}}), \tag{99}$$

in the neighbourhood of σ_0. The origin of this behaviour is simply that, if the star were set in oscillation with some general frequency σ, very little of the energy of excitation will be converted into the mechanical energy of oscillation and almost all of it will escape to infinity as gravitational waves. If on the other hand, the star were set in oscillation with its 'resonant' natural frequency σ_0, most of the energy of excitation will be converted into the mechanical energy of oscillation and very little will escape to infinity. (There are some further restrictions for the validity of the formula (99): e.g. the non-existence of other resonant frequencies in the neighbourhood of σ_0 and $\sigma_{\text{i}} \ll \sigma_0$ – restrictions which obtain in the contexts considered.)

A method for determining the complex frequency of the quasi-normal modes that is suggested, then, is to determine $\alpha_0^2+\beta_0^2$ as a function of σ (>0) and match it with the predicted behaviour (99). Examples are provided in §9 below.

9. A direct numerical confirmation of the algorism developed in this paper

The algorism developed in this paper for determining the quasi-normal modes of oscillation of a star, is founded on a physical base sufficiently different from that of extant methods, that a numerical confirmation may be useful. We have, therefore, carried through the algorism, numerically, to its completion, for a specific model. The model chosen was a polytrope of index 1.5 with $\alpha_0(=\epsilon_0/p_0) = 9$. For this polytrope,

$$r_1 = 1.576412485, \quad M = 0.20541, \quad \text{and} \quad \bar{\epsilon} = 3M/r_1^3 = 0.1573, \tag{100}$$

where r_1 is the radius of the polytrope in the unit $\sqrt{\epsilon_0}$. (The need for r_1 to be known to so many decimals is explained in Appendix D.) For this model, $\alpha_0^2+\beta_0^2$ was numerically evaluated for various initially assigned values of σ by the procedures described in §§7 and 8 for the quadrupole mode $l = 2$: they are listed in table 1 and $\alpha_0^2+\beta_0^2$ as a function σ (>0) is further exhibited in figure 1. Table 1 and figure 1 also include a comparison with the matching parabola,

$$\alpha_0^2+\beta_0^2 = \text{const.}\,[(\sigma-0.3248)^2+(1.026\times10^{-4})^2] \tag{101}$$

in the neighbourhood of its minimum. The agreement of the computed curve with the matching parabola is within the accuracy of our calculations. We conclude that for the model considered

$$\sigma_0 = 0.3248 \quad \text{and} \quad \sigma_{\text{i}} = 1.026\times10^{-4}. \tag{102}$$

Dr Lee Lindblom very kindly undertook to determine σ_0 and σ_{i} for this same model and for $l = 2$ by the method described by him and Detweiler in their paper on *The quadrupole oscillations of neutron stars* (1982); and he finds

$$\sigma_0 = 0.3248 \quad \text{and} \quad \sigma_{\text{i}} = 1.00\times10^{-4}, \tag{103}$$

in astonishingly good agreement with the values (100).

10. The dipole oscillations

The oscillations of a star with a dipole character presents a singular case in the theory of non-radial oscillations: they are not accompanied by the emission of gravitational radiation for the simple reason that gravitational waves with dipole symmetry do not exist.

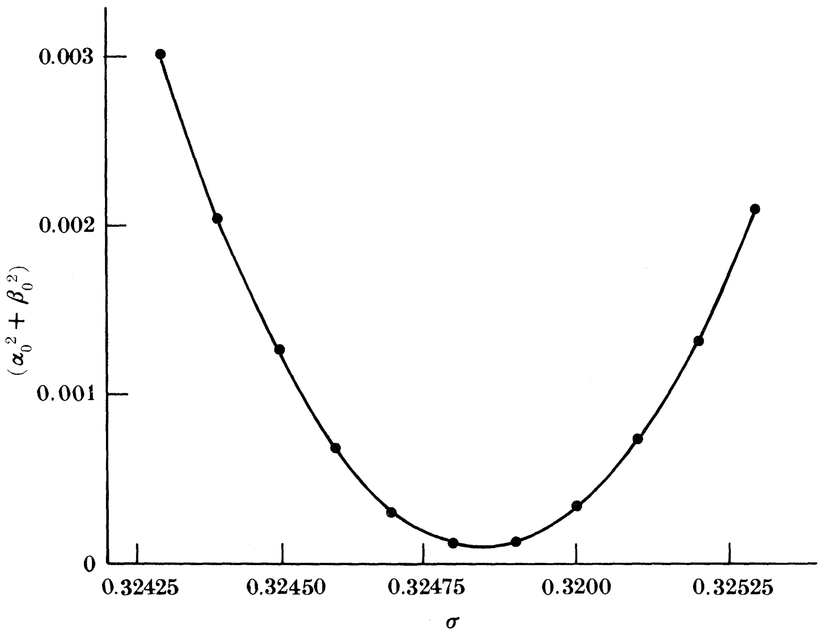

Figure 1. A comparison of the variation with σ of the flux of radiation, $(\alpha_0^2+\beta_0^2)$, of standing gravitational waves at infinity with a matching parabola. The deep minimum occurs at the real part of the characteristic frequency of the quasi-normal mode of oscillation and the width of the resonance is a measure of the reciprocal of the imaginary part. (The calculations are for the quadrupole oscillations of a relativistic polytrope of index 1.5 and $\epsilon_0/p_0 = 9$.)

Table 1. *The parabolic variation of* $\sigma^2+\beta^2$ *as a function* σ, *for a relativistic polytrope of index* 1.5 *and* $\epsilon_0/p_0 = 9$, *in the neighbourhood of its resonant frequency* σ_0 (≈ 0.3248)

σ	$\alpha_0^2+\beta_0^2$ *	$\alpha_0^2+\beta_0^2$ †	σ	$\alpha_0^2+\beta_0^2$ *	$\alpha_0^2+\beta_0^2$ †
0.3243	0.3006×10^{-2}	0.2988×10^{-2}	0.3248	0.1219×10^{-3}	0.1219×10^{-3}
0.3244	0.2037×10^{-2}	0.2026×10^{-2}	0.3249	0.1317×10^{-3}	0.1317×10^{-3}
0.3245	0.1263×10^{-2}	0.1259×10^{-2}	0.3250	0.3349×10^{-3}	0.3357×10^{-3}
0.3246	0.6876×10^{-3}	0.6853×10^{-3}	0.3251	0.7323×10^{-3}	0.7341×10^{-3}
0.3247	0.3065×10^{-3}	0.3065×10^{-3}	0.3252	0.1316×10^{-2}	0.1327×10^{-2}

* As derived from the asymptotic behaviour of the solutions at infinity.
† As derived from the matching parabola (101).

Dipole oscillations in relativistic astrophysics were first considered by Campolattaro & Thorne (1970) and by others since. But the theory was completed satisfactorily only recently by Lindblom & Splinter (1989). In the context of the present development, the theory of these oscillations follows as a particularly simple special case.

Because of the singular character of these oscillations, it is necessary that we start, *ab initio*, with the equations (48)–(51) and specialize them to the case

$$l = 1, \quad n = 0 \quad \text{and} \quad \kappa = 2. \tag{104}$$

Clearly

$$X \equiv 0, \tag{105}$$

and we are left with the three equations:

$$L_{,r}+\left(\frac{2}{r}-\nu_{,r}\right)L+\tfrac{1}{2}\left\{W_{,r}+\left[\frac{2}{r}-(Q+1)\,\nu_{,r}\right]W\right\}=0, \tag{106}$$

$$N_{,r}-\left(\frac{1}{r}-\nu_{,r}\right)N-\left[L_{,r}+\left(\frac{1}{r}+\nu_{,r}\right)L\right]-W_{,r}=0, \tag{107}$$

$$N_{,r}-\mathrm{e}^{2\mu_2}\left[\frac{1}{r}-(\epsilon+p)\,r\right]N-\left\{(1+r\nu_{,r})\,L_{,r}+\left(\frac{1}{r}+2\nu_{,r}\right)L+\sigma^2\,\mathrm{e}^{2(\mu_2-\nu)}rL\right\}$$
$$-\{(1+r\nu_{,r})\,W_{,r}+\tfrac{1}{2}\sigma^2\,\mathrm{e}^{2(\mu_2-\nu)}rW\}=0. \tag{108}$$

(It is important to observe that for the present case, $l=1$, equation (32) for V, as an independent equation, does *not* exist, as follows from its derivation in Appendix A.)

Equations (106)–(108) are linearly dependent at $r=0$. However, the equation,

$$N\left\{\frac{1}{r}(\mathrm{e}^{2\mu_2}-1)+\nu_{,r}-r(\epsilon+p)\,\mathrm{e}^{2\mu_2}\right\}+r\left\{\nu_{,r}\left(L_{,r}+\frac{L}{r}\right)+\sigma^2\,\mathrm{e}^{2(\mu_2-\nu)}L\right\}$$
$$+r(\nu_{,r}\,W_{,r}+\tfrac{1}{2}\sigma^2\,\mathrm{e}^{2(\mu_2-\nu)}W)=0, \tag{109}$$

obtained by subtracting equation (107) from (108), together with equations (106) and (107), do provide a system linearly independent at $r=0$. Thus, by considering at the origin, a behaviour of the form

$$(N,L,W)=(N_0,L_0,W_0)\,r^x+O(r^{x+2})\quad(r\to 0), \tag{110}$$

we obtain the indicial equation,

$$\begin{vmatrix} 0 & x+2 & \tfrac{1}{2}(x+2) \\ x-1 & -(x+1) & -x \\ 0 & a(x+1)+\sigma_0^2 & ax+\tfrac{1}{2}\sigma_0^2 \end{vmatrix}=-\tfrac{1}{2}a(x-1)^2(x+2). \tag{111}$$

We therefore obtain two singularity-free solutions corresponding to the double root $x=1$. And we find that from the corresponding homogeneous equations for (N_0,L_0,W_0) that

$$N_0\quad\text{is undetermined and}\quad W_0=-2L_0. \tag{112}$$

The two linearly independent solutions at $r=0$ become determinate when we seek expansions including terms of $O(r^3)$. Thus, by assuming the expansions,

$$(N,L,W)=(N_0,L_0,W_0)\,r+(N_2,L_2,W_2)\,r^3+O(r^5)\quad(r\to 0), \tag{113}$$

we find that N_2, L_2, and W_2 are determined in terms of N_0, L_0, and W_0 by the equations,

$$\left.\begin{aligned} 2L_2+W_2&=-\tfrac{2}{5}aQ_0L_0,\\ 2N_2-4L_2-3W_2&=a(L_0-N_0),\\ (4a+\sigma_0^2)\,L_2+(3a+\tfrac{1}{2}\sigma_0^2)\,W_2&=-[b_2+a_2-(\epsilon_0+p_0)\,b+(\epsilon_2+p_2)]\,N_0 \end{aligned}\right\} \tag{114}$$

Explicitly, we may take for the two linearly independent solutions at $r = 0$ the following:

$$\left.\begin{aligned} &1.\ L_0 = 0, \quad N_0 = 1: \\ &N = r - \frac{1}{2a}[a^2 + (b_2 + a_2) - (\epsilon_0 + p_0)\, b + (\epsilon_2 + p_2)]\, r^3 \\ &L = \frac{1}{2a}[(b_2 + a_2) - (\epsilon_0 + p_0)\, b + (\epsilon_2 + p_2)]\, r^3, \\ &W = -\frac{1}{a}[(b_2 + a_2) - (\epsilon_0 + p_0)\, b + (\epsilon_2 + p_2)]\, r^3, \end{aligned}\right\} \tag{115}$$

and

$$\left.\begin{aligned} &2.\ L_0 = 1, \quad N_0 = 0: \\ &N = \tfrac{1}{2}(a + \tfrac{1}{5}\sigma_0^2 Q_0)\, r^3, \\ &L = +r - \tfrac{1}{5}(3a + \tfrac{1}{2}\sigma_0^2)\, Q_0\, r^3, \\ &W = -2r + \tfrac{1}{5}(4a + \sigma_0^2)\, Q_0\, r^3. \end{aligned}\right\} \tag{116}$$

Turning next to the behaviour of the solutions of equations (106), (107), and (109) at the boundary, $r = r_1$, of the star, we conclude by the same arguments as in Appendix C, that it is of the form,

$$(L, N, W) \rightarrow (L_1, N_1, yW_1)\, e^{\alpha y} \quad (y = r_1 - r \rightarrow 0), \tag{117}$$

where α is the root of the characteristic equation,

$$\begin{vmatrix} -\alpha - \nu_1' + 2/r_1 & 0 & -\tfrac{1}{2}(1 + Q_1 \nu_1') \\ +\alpha - \nu_1' - 1/r_1 & -\alpha + \nu_1' - 1/r_1 & 1 \\ r_1[\nu_1'(-\alpha + 1/r_1) + \sigma^2 e^{-4\nu_1}] & \nu_1' + (e^{-2\nu_1} - 1)/r_1 & r_1 \nu_1' \end{vmatrix} = 0. \tag{118}$$

Therefore, quite generally,

$$(L, N, Wy^{-1}) \rightarrow \text{finite limits}\, (L_1, N_1, W_1) \quad \text{as} \quad r \rightarrow r_1 - 0. \tag{119}$$

Besides the conditions,

$$W = W_{,r} = 0 \quad \text{at} \quad r = r. \tag{120}$$

that we must impose, as in §7 (equations (84) and (88)), we must also require in the present instance ($l = 1$) that

$$N = L = 0 \quad \text{at} \quad r = r, \tag{121}$$

to ensure that, consistently with the dipole character of the oscillations, no perturbations in the space-time extend into the vacuum outside the star. The vanishing of L, N, and W at $r = r_1$ is sufficient to ensure the *identical* vanishing of the perturbations in the space-time for $r \geqslant r_1$, since from the linear homogeneous character of the governing equations,

$$(L, N, W) = 0 \quad \text{at} \quad r = r_1 \rightarrow L_{,r} = N_{,r} = W_{,r} = 0 \quad \text{at} \quad r = r_1. \tag{122}$$

The remaining requirements,

$$L = N = W_{,r} = 0 \quad \text{at} \quad r = r_1, \tag{123}$$

would appear to be one too many. But this is not the case: for, as we shall now show,

Table 2. *Illustrating the algorism for determining the fundamental characteristic frequency of dipole oscillations for a relativistic polytrope of index* 1.5 *and* $\epsilon_0/p_0 = 9$

(For each σ, the first row gives the values of $W_{,r}$, L, and N, at the boundary, $r = r_1$, for the solution $[N_0 = 1;\ L_0 = 0]$ and the second row gives the corresponding values for the solution $[L_0 = 1;\ N_0 = 0]$. (Note the simultaneous vanishing of $W_{,r}$ for the two solutions for $\sigma = 0.4786$.))

σ	$W_{,r}$	L	N	σ	$W_{,r}$	L	N
0.2	0.1972×10^{1}	-0.5735	0.2989	*0.4786*	-0.1362×10^{-4}	-0.3475	*0.5890*
	0.6262	-0.4726	0.1150		-0.5422×10^{-5}	-0.2654	*0.4497*
0.3	0.1102×10^{1}	-0.4981	0.3934	0.479	-0.1331×10^{-2}	-0.3472	0.5894
	0.2363	-0.3973	0.2294		0.1015×10^{-3}	-0.2651	0.4502
0.45	0.1083	-0.3708	0.5580	0.5	-0.6300×10^{-1}	-0.3307	0.6117
	-0.1883×10^{-2}	-0.2840	0.4162		-0.7270×10^{-2}	-0.2523	0.4740
0.478	0.1972×10^{-2}	-0.3480	0.5883	0.6	-0.1795	-0.2600	0.7098
	-0.1356×10^{-3}	-0.2657	0.4490		0.6140×10^{-1}	-0.2015	0.5764

any solution of equations (106), (107), *and* (109) *for which* $W = W_{,r} = 0$ *at* $r = r_1$, N *and* L *become determinate multiples of one another.*

The result stated follows from the fact that for any solution for which $W = W_{,r} = 0$ at $r = r_1$, equations (106), (107), and (109) tend to the vacuum equations given in *M.T.*, pp. 151 and 152. therefore, the solutions for L and N, in these cases, approach those for the vacuum. In particular, from *M.T.*, equations (65), (71) (or (73)) and (75) on pp. 151 and 152, it follows that

$$L = \text{const.}\frac{e^{\nu}}{r^2} = \frac{3M}{r^2}\Phi, \tag{124}$$

and

$$N = \left(M - \frac{M^2+\sigma^2 r^4}{r}e^{-2\nu}\right)\frac{\Phi}{r^2}. \tag{125}$$

Therefore,

$$N = \tfrac{1}{3}\left(1 - \frac{M^2+\sigma^2 r_1^4}{r_1 M}e^{-2\nu_1}\right)L \quad (r\to r_1-0). \tag{126}$$

i.e. one becomes a determinate multiple of the other, as stated. (We have numerically verified the validity of the relation (126).)

By virtue of the relation (126), we are left only with the boundary conditions,

$$L \quad \text{or} \quad N = 0 \quad \text{and} \quad W_{,r} = 0 \quad \text{at} \quad r = r_1, \tag{127}$$

to satisfy, since $W = 0$ at $r = r_1$ for *all* solutions. A solution satisfying the boundary conditions (126) can be found as follows.

We have shown that for any assigned σ^2, along all solutions of equations (106), (107) and (109) and, therefore, in particular along the two singularity-free linearly independent solutions derived from the expansions (115) and (116), N, L, and Wy^{-1} will tend to determinate finite limits, N_1, L_1, and W_1 as y $(= r_1 - r)$ tends to zero (cf. equation (117)). By varying σ^2, we can find solutions, as in §7, for which $W_{,r}$ vanishes. Since, in specifying the solutions for which $W_{,r}$ vanishes, no member of the two-parameter family of the singularity-free solutions is distinguished. $W_{,r}$ *will vanish simultaneously, for the two linearly independent solutions derived from the expansions*

(114) *and* (115). (This conclusion, while it is intuitively manifest, can be established directly with the aid of equation (118).) And, finally, by a linear superposition of the two linearly independent solutions for which $W_{,r}$ vanishes, simultaneously, at $r = r_1$, we can find one for which N and, therefore, also L vanishes at $r = r_1$; and the solution will be completed.

An illustrative example

The algorism we have described for determining the characteristic frequencies of dipole oscillations of a star, is illustrated in table 2 for the same polytropic model ($n = 1.5$ and $\epsilon_0/p_0 = 9$) considered in §8. The values attained by $W_{,r}$, L, and N at the boundary, $r = r_1$, are tabulated for the two solutions.

$$[N_0 = 1; L_0 = 0] \quad \text{and} \quad [L_0 = 1; N_0 = 0], \tag{128}$$

derived, respectively, from the expansions (114), (115) and (116). It will be noticed that, as predicted, $W_{,r}$ vanishes simultaneously for a particular σ (≈ 0.4786 in the example considered). We readily find from the tabulated values that for the solution,

$$[N_0 = 1; L_0 = 0] - 1.309\,[L_0 = 1; N_0 = 0] \quad \text{for} \quad \sigma = 0.4786, \tag{129}$$

the boundary values attained by $W_{,r}$, L, and N are:

$$W_{,r} = -5.2\times10^{-4}, \quad L \approx 4\times10^{-4}, \quad \text{and} \quad N \approx 5\times10^{-4}. \tag{130}$$

We conclude that the solution (129) satisfies all the required boundary conditions as accurately as one would wish.

Dr Lindblom and Dr Splinter have (at our request) determined, for the same polytrophic model, by their alternative algorism, the characteristic frequency $\sigma = 0.4786$ in 'exact' agreement with our determination.

11. The axial perturbations

We now turn to the axial perturbations of a star. For such perturbations, the metric is of the form,

$$\mathrm{d}s^2 = \mathrm{e}^{2\nu}(\mathrm{d}t)^2 - \mathrm{e}^{2\psi}(\mathrm{d}\varphi - \omega\,\mathrm{d}t - q_2\,\mathrm{d}x^2 - q_3\,\mathrm{d}x^3)^2 - \mathrm{e}^{2\mu_2}(\mathrm{d}x^2)^2 - \mathrm{e}^{2\mu_3}(\mathrm{d}x^3)^2, \tag{131}$$

where ν, ψ, μ_2, and μ_3 retain their unperturbed values (given in equations (2)–(9)) while ω, q_2, and q_3, defining the perturbations, are functions of t ($= x^0$), x^2 ($= r$), and x^3 ($= \theta$). Besides, these perturbations, by definition, are not accompanied by any motions in the r- and the θ-directions. Accordingly, the components of the four-velocity, $u_{(2)}$ and $u_{(3)}$ (in the tetrad-frame) vanish identically. In the first instance, only the φ-component, $u_{(1)}$, is allowed to be non-zero (though as we shall presently show, it too vanishes).

Since for the perturbations considered,

$$T_{(1)(2)} = (\epsilon + p)\,u_{(1)}\,u_{(2)} = T_{(1)(3)} = (\epsilon + p)\,u_{(1)}\,u_{(3)} = 0, \tag{132}$$

it follows from the field equations (*M.T.*, p. 143, equations (11) and (12)) that

$$(\mathrm{e}^{3\psi+\nu-\mu_2-\mu_3}Q_{23})_{,\theta} + \mathrm{e}^{3\psi-\nu-\mu_2+\mu_3}Q_{02,0} = 0, \tag{133}$$

$$(\mathrm{e}^{3\psi+\nu-\mu_2-\mu_3}Q_{23})_{,r} - \mathrm{e}^{3\psi-\nu+\mu_2-\mu_3}Q_{03,0} = 0, \tag{134}$$

where $$Q_{AB} = q_{A,B} - q_{B,A} \quad \text{and} \quad Q_{A0} = q_{A,0} - \omega_{,A} \quad (A = 2, 3). \tag{135}$$

The integrability condition of equations (133) and (134) is

$$(e^{3\psi-\nu-\mu_2+\mu_3}Q_{02})_{,r,0}+(e^{3\psi-\nu+\mu_2-\mu_3}Q_{03})_{,\theta,0}=0. \tag{136}$$

For a time-dependence $e^{i\sigma t}$ of the perturbations, equation (136) is equivalent to

$$(e^{3\psi-\nu-\mu_2+\mu_3}Q_{02})_{,r}+(e^{3\psi-\nu+\mu_2-\mu_3}Q_{03})_{,\theta}=0. \tag{137}$$

By the (01)-component of the field-equations (*M.T.*, p. 141, equation (d)), equation (137) implies that

$$R_{(0)(1)}=+2T_{(0)(1)}=+2(\epsilon+p)\,u_{(0)}\,u_{(1)}=0. \tag{138}$$

In other words,

$$u_{(1)}=0. \tag{139}$$

We conclude that *incident gravitational waves do not excite any fluid motions in the star*. The star simply scatters the incident radiation by the potential barrier (derived from the curvature of the space-time of the star) that it presents.

Making use of the relation (cf. equation (110)),

$$Q_{20,\theta}-Q_{30,r}=(q_{23}-q_{32})_{,0}=Q_{23,0}, \tag{140}$$

we derive from equations (133) and (134) the wave-equation,

$$[e^{-3\psi+\nu-\mu_2+\mu_3}(e^{3\psi+\nu-\mu_2-\mu_3}Q_{23})_{,r}]_{,r}+[e^{-3\psi+\nu+\mu_2-\mu_3}(e^{3\psi+\nu-\mu_2-\mu_3}Q_{23})_{,\theta}]_{,\theta}=Q_{23,0,0}. \tag{141}$$

The further reduction of this equation proceeds exactly as in *M.T.*, §24(*a*), pp. 143–144. Thus, equation (141) can be separated by the substitution,

$$e^{3\psi+\nu-\mu_2-\mu_3}Q_{23}=X(r)\,C_{l+2}^{-\frac{3}{2}}(\theta), \tag{142}$$

where C_n^{ν} denotes the Gegenbauer function as defined in *M.T.*, p. 144, equation (20). We find:

$$r^2\,e^{\nu-\mu_2}\left(\frac{e^{\nu-\mu_2}}{r^2}X_{,r}\right)_{,r}-2n\frac{e^{2\nu}}{r^2}X+\sigma^2X=0,\quad [\mu^2=2n=(l-1)(l+2)], \tag{143}$$

where we have made use of the relations,

$$e^{-3\psi+\nu-\mu_3+\mu_2}=e^{\nu+\mu_2}r^{-4}\operatorname{cosec}^3\theta \quad\text{and}\quad e^{-3\psi+\nu-\mu_2+\mu_3}=e^{\nu-\mu_2}r^{-2}\operatorname{cosec}^3\theta. \tag{144}$$

Equation (143) can be reduced to the form,

$$X_{,r,r}-\frac{e^{2\mu_2}}{r}\{2+r^2[\epsilon-p-6M(r)/r^3]\}X_{,r}-e^{2\mu_2}\frac{\mu^2}{r^2}X+\sigma^2\,e^{2(\mu_2-\nu)}X=0. \tag{145}$$

It is of interest to notice that by defining the variable,

$$r_*=\int_0^r e^{-\nu+\mu_2}\,dr \tag{146}$$

analogous to the variable r_* in the treatment of the perturbations of the Schwarzschild black-hole (*M.T.*, p. 144, equation (25)) and letting

$$X=rZ, \tag{147}$$

equation (145) can be brought to the form,

$$\left(\frac{d^2}{dr_*^2}+\sigma^2\right)Z=VZ, \tag{148}$$

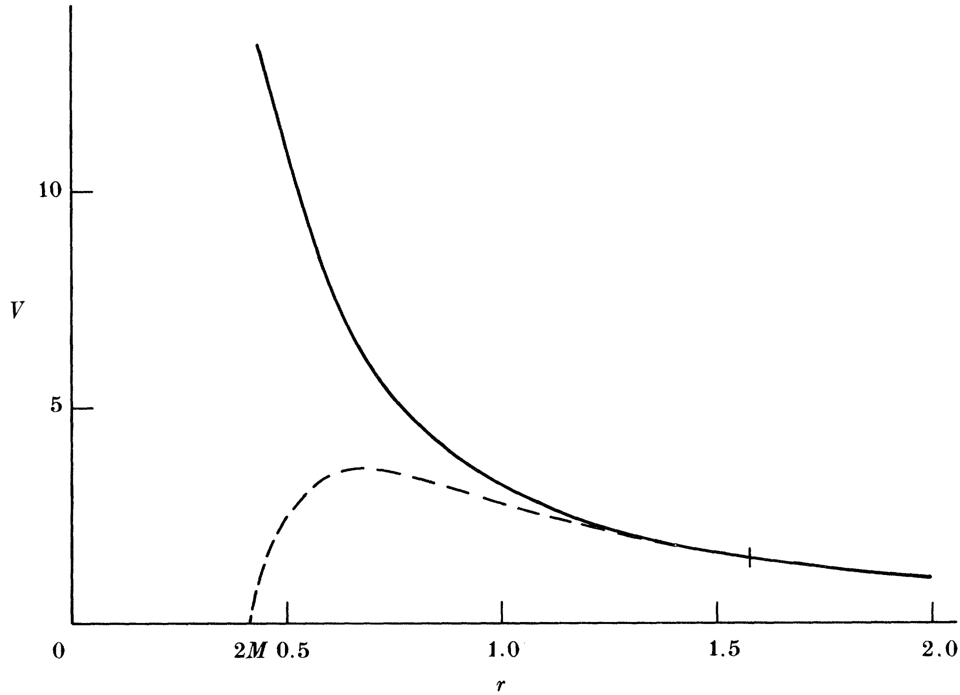

Figure 2. A comparison of the potential barriers, presented to incident axial gravitational waves, by a relativistic polytrope of index 1.5 and $\epsilon_0/p_0 = 9$ and by a black hole of the same mass. In the vacuum exterior to the boundary of the star (at $r \geqslant r_1$) the potentials are the same. They are different for $r < r_1$; the potential for the black hole vanishes at its horizon $r = 2M$ while that of the polytrope tends to infinity for $r \rightarrow 0$.

where

$$V = \frac{e^{2\nu}}{r^3}[(\mu^2+2)r + r^3(\epsilon - p) - 6M(r)]. \tag{149}$$

Outside the star, where $p = \epsilon = 0$ and $M(r) = M$, the potential V reduces to the 'Regge–Wheeler' potential of the Schwarzschild black-hole for axial perturbations (see equation (151) below). It should, however, be noted that, unlike in the case of the black hole when r_* ranges from $-\infty$ to $+\infty$, the variable r_*, as defined in equation (146) ranges only from 0 to $+\infty$. The potential V defined in equation (149) is strictly a central field; and equation (145) describes a scattering problem for a 'soft-core' Coulomb-potential. (See figure 2 in which a comparison is made of the potential for the relativistic polytrope, $n = 1.5$ and $\epsilon_0/p_0 = 9$, with that for the Schwarzschild black-hole of the same mass.)

A study of the scattering of axial gravitational waves by a star, with the aid of equation (145), for the interior of the star ($r \leqslant r_1$), and the 'Regge–Wheeler' equation,

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right)Z = V^{(-)}Z, \tag{150}$$

where

$$V^{(-)} = \frac{\Delta}{r^5}[(\mu^2+2)r - 6M], \tag{151}$$

for the vacuum exterior to the star ($r > r_1$), presents no difficulty.

The solution of equation (145), free of singularity at the origin, has the expansion,

$$X = r^{l+2} + \frac{1}{2(2l+3)}\{(l+2)\left[\tfrac{1}{3}(2l-1)\,\epsilon_0 - p_0\right] - \sigma_0^2\}\,r^{l+4} + \dots . \tag{152}$$

With the aid of this expansion, equation (145) can be integrated forward to the boundary, $r = r_1$, of the star. The integration can then be continued into the vacuum outside the star with the Regge–Wheeler equation (150) with the starting values,

$$Z(r = r_1) = \lim_{r \to r_1 - 0} (X/r) \tag{153}$$

and

$$Z_{,r_*}(r = r_1) = \left(1 - \frac{2M}{r_1}\right) \lim_{r \to r_1 - 0} \left[\frac{1}{r^2}(rX_{,r} - X)\right]. \tag{154}$$

The integration of equation (150) must be continued to a sufficiently large r that, matching with the asymptotic expansion (cf. equation (98)),

$$\begin{aligned} Z \to +&\left\{\alpha_0 - \frac{n+1}{\sigma}\frac{\beta_0}{r} - \frac{1}{2\sigma^2}[n(n+1)\,\alpha_0 - 3M\sigma\beta_0]\frac{1}{r^2} + \dots\right\}\cos\sigma r_* \\ -&\left\{\beta_0 + \frac{n+1}{\sigma}\frac{\alpha_0}{r} - \frac{1}{2\sigma^2}[n(n+1)\,\beta_0 + 3M\sigma\alpha_0]\frac{1}{r^2} + \dots\right\}\sin\sigma r_*, \end{aligned} \tag{155}$$

will enable us to determine α_0 and β_0. Determining α_0 and β_0, in the present context, is equivalent to determining the 'phase-shift' in the standard terminology.

To exhibit the differing scattering properties of a star for the axial and the polar perturbations, we present in figure 3, for comparison with figure 1, the variation of $(\alpha_0^2 + \beta_0^2)$ with σ, for $l = 2$ and the same polytropic model.

It is not an uncommon view that the axial perturbations of a star present but an uninviting subject for study since the incidence of axial gravitational waves does not excite any fluid motions in the star. But this view overlooks certain facts germane to the 'conformity of the parts' of general relativity 'to one another and to the whole.' It is known, for example, that the scattering of axial and polar gravitational waves by the Schwarzschild black-hole (and, indeed, also by the Reissner–Nordström black-hole) is *isospectral* in that the reflection and the transmission coefficients for both types of waves are the same. While we do not, of course, expect that the scattering of axial and polar gravitational waves by spherical distributions of matter will be as closely related as they are in the case of black holes, it would, nevertheless, strike a discordant note if the scattering of axial gravitational waves depended only on the static space-time and it was not so for polar gravitational waves. It *is* consonant with the character of general relativity that the two cases share the same property in this important respect. Also, the particular simplicity of the analysis for the barotropic case must be attributed to the circumstance that in this case the treatment of the perturbation problem requires no additional information (such as, for example, the conservation of baryon number) beyond that needed for the specification of the static space-time.

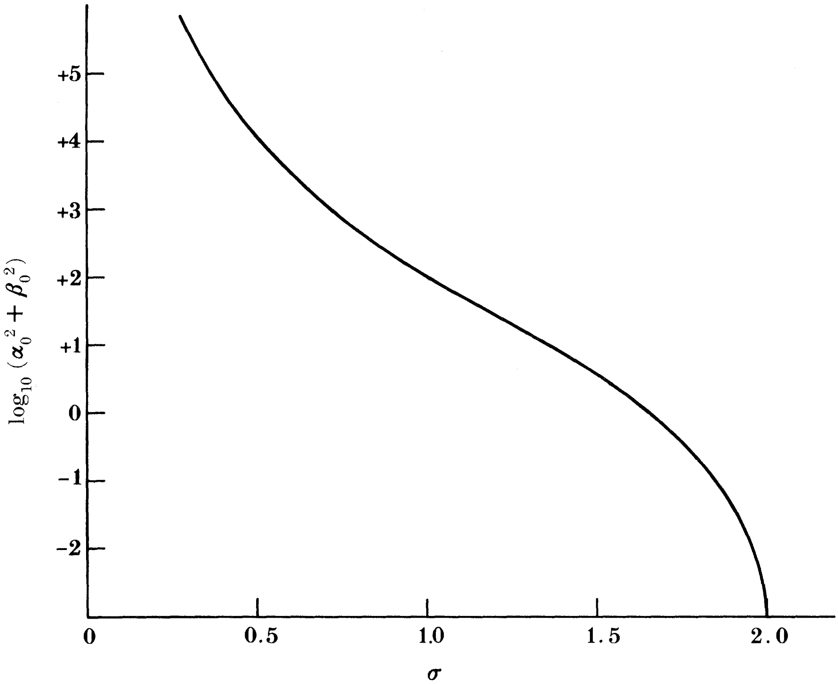

Figure 3. The variation with σ of the flux of radiation, $(\alpha_0^2+\beta_0^2)$, at infinity of standing axial gravitational waves for the same relativistic polytrope considered for polar perturbations in figure 1.

12. Concluding remarks

We shall dispense with what one normally expects in the concluding section of a paper: a restatement of the underlying ideas, an outline of the development, and a final summing. The two long introductory sections, the detailed summary, and the substantive comments in §§ 10 and 11, amply take their place. And we shall also not enumerate the many directions in which the present investigation can be extended: they are numerous and they are obvious. We shall confine our remarks, instead, to an aspect of the theory to which we have not made any reference so far: a comparison with the Newtonian theory.

The simplicity of the algorism developed in this paper for determining the complex characteristic frequencies of the non-radial modes of oscillation of a star, relative to the traditional ones, requires no emphasis in the relativistic context. What *is* astonishing is its simplicity relative to the corresponding Newtonian algorism (cf. Chandrasekhar 1964; Hurley *et al.* 1966). Indeed, by contrasting with the Newtonian treatment of non-radial oscillation, one becomes aware of a source of even greater puzzlement.

On the relativistic theory, the frequencies of oscillation of the non-radial modes (as we have shown) depend only on the distribution of the energy-density and the pressure in the static configuration and the equation of state only to the extent of its adiabatic exponent. If this is a true representation of the physical situation, then it *must* be valid in the Newtonian theory as well: the true nature of an object cannot change with the mode and manner of one's perception. In the relativistic picture, the independence of the frequencies of the non-radial modes of oscillation of a star, on anything except its characterization in terms of its equilibrium structure, is to be understood in terms of the scattering of incident gravitational waves by the curvature of the static space-time and its matter content acting as a potential. But what are the counterparts of these same concepts in the Newtonian framework? Perhaps they lie concealed in the meanings that are to be attached, in the *Newtonian theory*, to the four metric functions (*and* their perturbations) that describe a spherically symmetric static space-time (*and* their polar perturbations). It is known that the Newtonian gravitational potential, in some sense, replaces the metric function g_{tt}. Are there similar meanings to be attached to g_{rr}, $g_{\theta\theta}$, and $g_{\varphi\varphi}$? That is the predominant question to which the present investigation seems to lead.

We are grateful to several colleagues for their patience in discussing with us different aspects of the problem considered in this paper: to John Friedman and Robert Wald on the physical aspects and to Norman Lebovitz and Sotirios Persides on the linear dependence, at the origin, of the basic system of differential equations; to Kip Thorne for extremely valuable decisive critical remarks; and to Jesus Ibañez for a careful scrutiny of the entire analysis. We are also very grateful to Bernard F. Whiting who independently pointed out to us a serious error which vitiated an earlier version of this paper.

But our greatest indebtedness is to Lee Lindblom for generously sharing with us his experience with his alternative treatment of the quadrupole oscillations of neutron stars; and particularly for determining for us afresh by his methods the quasi-normal quadrupole mode of oscillation of the polytrophic model considered in §9. We are similarly grateful to Lee Lindblom and Randall Splinter for determining the characteristic frequency for dipole oscillations for the same polytrophic model. The research reported in this paper has, in part, been supported by grants from the National Science Foundation under Grant PHY-89-18388 with the University of Chicago. We are also grateful for a grant from the Division of Physical Sciences of the University of Chicago which has enabled our continued collaboration by making possible periodic visits by Valeria Ferrari to the University of Chicago.

Appendix A. The derivation of equation (32)

We have already remarked in §4*c* the absence of any explicit derivation (or an outline of a derivation) of equation (32) in spite of its remarkable character: its apparent independence of the nature of the source of the static space-time.

We start with equation (22). Making use of the relation,

$$\nu_{,r,r}+\left(\frac{1}{r}+\nu_{,r}\right)(\nu_{,r}-\mu_{2,r}) = 2p\,\mathrm{e}^{2\mu_2} \qquad (\mathrm{A}\ 1)$$

(which follows from the unperturbed equation, $G_{33} = 2p$) and substituting from equations (23) for the various quantities describing the perturbation, we find after some rearrangement of the terms:

$$\begin{aligned}&\left\{-4pL\,\mathrm{e}^{2\mu_2}+(T+N)_{,r,r}+(\nu_{,r}-\mu_{2,r})\,(T+N)_{,r}+\frac{2}{r}T_{,r}\right.\\&\left.\quad+\left(\frac{1}{r}+\nu_{,r}\right)(N-L)_{,r}-2\,\mathrm{e}^{2\mu_2}\Pi+\sigma^2\,\mathrm{e}^{2(\mu_2-\nu)}(T+L)\right\}P_l\\&\quad+\left\{V_{,r,r}+\left(\frac{2}{r}+\nu_{,r}-\mu_{2,r}\right)V_{,r}+\frac{\mathrm{e}^{2\mu_2}}{r^2}(N+L)+\sigma^2\,\mathrm{e}^{2(\mu_2-\nu)}V\right\}P_{l,\theta}\cot\theta = 0, \qquad (\mathrm{A}\ 2)\end{aligned}$$

i.e. a linear combination of terms with the angular factors P_l and $P_{l,\theta}\cot\theta$ vanishes.

We can accordingly equate separately the terms with the two factors. (Exactly the same remarks are made in Chandrasekhar & Xanthopoulos (1979) following equation (71) in the context of the perturbations of the Reissner–Nordström black-hole.) Perhaps an explanation for this statement is needed.

Consider quite generally an equation in the form

$$A(r)\,P_l(\mu)-B(r)\,\mu P_{l,\mu}=0 \quad (\mu=\cos\theta). \tag{A 3}$$

By making use of the known recurrence relations among the Legendre polynomials, we can rewrite equation (A 3) in the form

$$(A-lB)\,P_l-B(r)\,P_{l-1,\mu}=0. \tag{A 4}$$

Since $P_{l-1,\mu}$ is a polynomial of degree $l-2$ in μ, it is orthogonal to P_l: and, therefore,

$$\left.\begin{aligned} A-lB=0 \quad \text{and} \quad B=0 \quad (l>1),\\ \text{and} \qquad A-B=0 \quad (l=1). \end{aligned}\right\} \tag{A 5}$$

We conclude that
$$A=B=0 \quad \text{if} \quad l>1. \tag{A 6}$$

Applying this last result to equation (A 2) we conclude that

$$V_{,r,r}+\left(\frac{2}{r}+\nu_{,r}-\mu_{2,r}\right)V_{,r}+\frac{e^{2\mu_2}}{r^2}(N+L)+\sigma^2\,e^{2(\mu_2-\nu)}V=0, \tag{A 7}$$

so long as $l>1$. It is to be particularly noted that *equation* (A 7), *as an independent equation, does not exist for* $l=1$.

It can be readily verified that the vanishing of the terms in equation (A 2) with the factor P_l $(l>1)$ contains no more information than equation (30).

Appendix B. Relativistic polytropes

Relativistic polytropes have been considered extensively, in various contexts, since the early sixties (cf. Chandrasekhar 1964; Tooper 1964). Nevertheless, some of the essential relations that we needed for the numerical illustrations in §§9 and 10 were not readily available in the extant literature (at least in the forms we needed them). The present Appendix is provided on that account.

The fundamental assumption of the theory is that the energy density ϵ and the pressure p are expressible in the manner,

$$\epsilon=\epsilon_0\,\Theta^n \quad \text{and} \quad p=p_0\,\Theta^{n+1}, \tag{B 1}$$

where ϵ_0 and p_0 denote the values of ϵ and p at the centre and n is the polytropic index (not to be confused with the 'n' defined in equation (25)). By measuring r (and t) in the unit $\surd\epsilon_0$, the relativistic equation of hydrostatic equilibrium (equation (6)) takes the form:

$$\left[1-2\frac{M(r)}{r}\right](n+1)\,\Theta_{,r}=-(\alpha_0+\Theta)\left[\frac{M(r)}{r^2}+\frac{1}{\alpha_0}\Theta^{n+1}r\right], \tag{B 2}$$

where, in accordance with our present convention regarding units,

$$\epsilon=\Theta^n, \quad p=\Theta^{n+1}\alpha_0^{-1}, \quad M(r)=\int_0^n\Theta^n r^2\,\mathrm{d}r, \quad \text{and} \quad \alpha_0=\epsilon_0/p_0. \tag{B 3}$$

Equation (3) for $\nu_{,r}$ now gives

$$\nu_{,r} = -\frac{p_{,r}}{\epsilon+p} = -(n+1)\frac{\Theta_{,r}}{\alpha_0+\Theta}, \tag{B 4}$$

or, after integration,

$$e^{-\nu} = e^{-\nu_0}\left(\frac{\alpha_0+\Theta}{\alpha_0+1}\right)^{n+1}. \tag{B 5}$$

The constant of integration, ν_0, is to be determined by the condition (cf. equation (10))

$$e^{2\nu_0}\left(\frac{\alpha_0+1}{\alpha_0}\right)^{2(n+1)} = 1-2\frac{M}{r_1}, \tag{B 6}$$

where r_1 denotes the radius of the polytrope where Θ vanishes. Also, it may be noted that Q as defined in equation (44) is now given by

$$Q = \frac{\alpha_0 n}{(n+1)\,\Theta}. \tag{B 7}$$

At $r = r_1$, Θ vanishes linearly as $(r_1 - r)$ with its derivative tending to a finite negative limit Θ_1':

$$\Theta \rightarrow |\Theta_1'|\,(r_1-r) \quad (r \rightarrow r_1 - 0). \tag{B 8}$$

The value of $|\Theta_1'|$ is an important constant of the theory. It is best determined by the equation,

$$M = \frac{r_1^2(n+1)\,|\Theta_1'|}{\alpha_0+2(n+1)\,r_1|\Theta_1'|}, \tag{B 9}$$

for M (which follows directly from equation (B 2)), since $M(r)$ attains its limiting value long before r_1 (to a sufficient accuracy). Using equation (B 9), we can rewrite equation (B 6) more conveniently in the form,

$$e^{2\nu_0} = \left(\frac{\alpha_0}{\alpha_0+1}\right)^{2(n+1)}\frac{\alpha_0}{\alpha_0+2(n+1)\,r_1|\Theta_1'|}. \tag{B 10}$$

From equations (B 4) and (B 7) it follows that

$$\nu_{,r} \rightarrow (n+1)\frac{|\Theta_1'|}{\alpha_0} = \nu_1' \quad \text{(say)}. \tag{B 11}$$

Also, by the choice of ν_0 (by equations (10) and (B 6)),

$$\mu_2 \rightarrow -\nu, \quad \mu_{2,r} \rightarrow -\nu_{,r}, \tag{B 12}$$

and

$$e^{2\nu} \rightarrow e^{2\nu_1} = 1-2M/r_1 \quad (r \rightarrow r_1 - 0). \tag{B 13}$$

Besides, in accordance with equations (B 7) and (B 8)

$$Q \rightarrow Q_1/(r_1-r) \quad (r \rightarrow r_1 - 0), \tag{B 14}$$

where

$$Q_1 = n\alpha_0/(n+1)|\Theta_1'|. \tag{B 15}$$

A useful relation which follows from equations (B 11) and (B 15) is

$$Q_1\,\nu_1' = n. \tag{B 16}$$

Finally, we may note the following series expansions for Θ, ϵ, and p at $r=0$:

$$\left.\begin{aligned}\Theta &= 1+\theta_2 r^2+\theta_4 r^4+\dots,\\ \epsilon &= \epsilon_0\{1+n\theta_2 r^2+[n\theta_4+\tfrac{1}{2}n(n-1)\,\theta_2^2]\,r^4+\dots\},\\ p &= p_0\{1+(n+1)\,\theta_2 r^2+[(n+1)\,\theta_4+\tfrac{1}{2}(n+1)\,n\theta_2^2]\,r^4+\dots\},\end{aligned}\right\} \qquad \text{(B 17)}$$

where

$$\left.\begin{aligned}\theta_2 &= -\epsilon_0\frac{1+\alpha_0}{2(n+1)}\left(\frac{1}{3}+\frac{1}{\alpha_0}\right),\\ \theta_4 &= +\epsilon_0\frac{\theta_2}{4(n+1)}\left\{\tfrac{4}{3}(n+1)-\left[\frac{1}{3}+\frac{1}{\alpha_0}+(\alpha_0+1)\left(\tfrac{1}{5}n+\frac{n+1}{\alpha_0}\right)\right]\right\}.\end{aligned}\right\} \qquad \text{(B 18)}$$

(In writing the foregoing expansions, we have, for convenience, abandoned the convention $\sqrt{\epsilon_0}=1$.)

Appendix C. The behaviour of the solutions at the boundary of the star

In obtaining the behaviours of the solutions of equations (48)–(51) for r tending to the boundary r_1 of the star, it is important to note that quite generally,

$$\nu_{,r} \text{ tends to a finite limit as } r\rightarrow r_1-0. \qquad \text{(C 1)}$$

Proof. By combining equations (3) and (6), we have the relation,

$$\nu_{,r} = \frac{pr+M(r)/r^2}{1-2M(r)/r}, \qquad \text{(C 2)}$$

and therefore,

$$\nu_{,r}\rightarrow\frac{M/r_1^2}{1-2M/r_1} = \nu_1' \quad \text{(say) for} \quad r\rightarrow r_1-0. \qquad \text{(C 3)}$$

(It can be verified that ν_1' given by this equation agrees with equation (B 11) for polytropes.) Also, by the choice of ν_0 in equation (9),

$$\mu_2 = -\nu \quad \text{and} \quad \mu_{2,r} = -\nu_{,r} \quad \text{at} \quad r=r_1 \qquad \text{(C 4)}$$

On the other hand, we may expect on entirely general grounds that Q $(=\epsilon_{,r}/p_{,r})$ which occurs in equation (48) has the behaviour,

$$Q\rightarrow Q_1/(r_1-r) \quad (r\rightarrow r_1-0), \qquad \text{(C 5)}$$

where Q_1 is some constant. For polytropes, Q_1 has the value

$$Q_1 = n\alpha_0/(n+1)|\Theta_1'| \quad \text{while} \quad Q_1\nu_1' = n. \qquad \text{(C 6)}$$

From the behaviours (C 3)–(C 5) of the various coefficients that occur in equations (48)–(51), we conclude that near the boundary, $r=r_1$, the solutions take the forms,

$$(L,N,X,W)\rightarrow(L_1,N_1,X_1,yW_1)\,\mathrm{e}^{\alpha y} \quad (y=r_1-r), \qquad \text{(C 7)}$$

where α, L_1, N_1, X_1, and W_1 are constants. (We have already remarked in §7 on the necessity of W vanishing at the boundary.) We may note that according to the substitution (C 7),

$$W_{,r}\rightarrow -W_1\,\mathrm{e}^{\alpha y} \quad (y\rightarrow 0). \qquad \text{(C 8)}$$

Making the substitution (C 7) in equations (48)–(51) we obtain, by virtue of equations (C 3)–(C 5) and (C 9), the following characteristic equation for α:

$$\begin{array}{cccc} L_1 & N_1 & X_1 & W_1 \end{array}$$

$$\begin{vmatrix} -\alpha-\nu_1'+2/r_1 & 0 & -\alpha-\nu_1'+1/r_1 & -\tfrac{1}{2}(1+Q_1\,\nu_1') \\ +\alpha-\nu_1'-1/r_1 & -\alpha+\nu_1'-1/r_1 & 0 & 1 \\ \begin{array}{c}-\alpha(1+r\nu_1')+2\nu_1' \\ +\sigma^2\,\mathrm{e}^{-4\nu_1}r_1-(n\,\epsilon^{-2\nu}-1)/r_1\end{array} & +\alpha+(n+1)\,\mathrm{e}^{-2\nu_1}/r_1 & \begin{array}{c}-\alpha(1+r\nu_1')+\epsilon^{-2\nu_1}/r_1 \\ +\sigma^2\,\epsilon^{-4\nu_1}r_1\end{array} & +(1+r\nu_1') \\ n\,\mathrm{e}^{-2\nu_1}/r_1 & n\,\mathrm{e}^{-2\nu_1}/r_1 & \begin{array}{c}r_1[\alpha^2-2\alpha(\nu_1'+1/r_1)] \\ +\sigma^2\,\mathrm{e}^{-4\nu_1}r_1\end{array} & 0 \end{vmatrix} = 0. \qquad \text{(C 10)}$$

This is a quintic for α. In terms of its five roots, α_j ($j = 1, \ldots, 5$), the behaviour of the solutions at the boundary will be given by a superposition of the basic solutions,

$$(L_j, N_j, X_j, y W_j)\,\mathrm{e}^{\alpha_j y}, \qquad \text{(C 11)}$$

where (L_j, N_j, X_j, W_j) represents a characteristic vector belonging to α_j. It follows that in general L, N, X, and $W_{,r}$ *tend to finite determinate constants as* $r \rightarrow r_1 - 0$.

Appendix D. Some details on the procedures adopted in the numerical integrations

In this Appendix we shall give some details on the procedures adopted in the numerical integrations to achieve the necessary accuracy in the derived characteristic frequencies of the quadrupole ($l = 2$) and the dipole ($l = 1$) modes of oscillation: they may be useful for others who may wish to use the algorisms developed in this paper.

The integration of the equations for the polytrope $n = 1.5$ and $\epsilon_0/p_0 = 9$ was performed by a simple version of the standard Runge–Kutta routine with a constant integration step; and all the calculations were carried out in double precision. The integration was started at $r_0 = 0.01$ by using the series expansions given in §§6 and 8. Since all the metric functions have the behaviour,

$$r^l(\alpha + \beta r^2), \qquad \text{(D 1)}$$

the percentage error that is made by neglecting the next term, γr^4 (say), in the series expansion is

$$\gamma r^4/\alpha \approx 10^{-8}. \qquad \text{(D 2)}$$

(*a*) *The quadrupole case*

Starting from r_0 ($= 0.01$) we integrate the two linearly independent solutions derived with the expansions (80)–(83). Our object, of course, is to find by superposition of these two solutions (by the prescription (90)) a solution for which $W_{,r}$ vanishes at the boundary $r = r_1$ where the polytropic function Θ vanishes. And we need also the values of the functions X and L and their derivatives at $r = r_1$.

Table A 1. (For explanation see text.)

r		X	$X_{,r}$	L	$L_{,r}$	W	$W_{,r}$
1.55	(a)	-0.5501568	-0.3401758	-0.1419385	-0.7029284	-0.3022427×10^{-1}	1.2871274
	(b)	-1.0166848	-1.1600380	$+0.2857576$	-1.8964062	-0.7022729×10^{-1}	2.5341350
1.5764	(a)	-0.5589986	-0.3846175	-0.1563445	-0.1685542	-0.9675857×10^{-5}	0.7800915
	(b)	-1.0472447	-1.1549196	$+0.2341168$	-1.8805155	-0.3306309×10^{-4}	2.6514366
1.5764123	(a)	-0.5590026	-0.3297308	-0.1563465	-0.1550421	-0.1419701×10^{-6}	0.7665871
	(b)	-1.0472589	-1.1549169	$+0.2340938$	-1.8716880	-0.4897413×10^{-6}	2.6426765

Table A 2. (For explanation see text.)

$\mathrm{d}r = 10^{-9}$, $r = 1.576412485$:	(a)	$L_{,r} = -0.1533122$,	$W_{,r} = 0.7648572$,
	(b)	$L_{,r} = -1.8705368$,	$W_{,r} = 2.6415263$;
$\mathrm{d}r = 10^{-10}$, $r = 1.5764124853$:	(a)	$L_{,r} = -0.1531958$,	$W_{,r} = 0.7647808$,
	(b)	$L_{,r} = -1.8704600$,	$W_{,r} = 2.6414495$;
$\mathrm{d}r = 10^{-11}$, $r = 1.57641248534$:	(a)	$L_{,r} = -0.1531701$,	$W_{,r} = 0.7647151$,
	(b)	$L_{,r} = -1.8704437$,	$W_{,r} = 2.6414332$.

It was found that the determination of the values which the various functions attain at $r = r_1$ is a very delicate matter. And to check whether the desired accuracy was reached, the range of integration, $r_0 < r < r_1$, was divided into four intervals:

$$\left.\begin{aligned} &(1)\ r_0 \leqslant r \leqslant r_a; \quad r_a = 1.55, && \mathrm{d}r = 10^{-4};\\ &(2)\ r_a \leqslant r \leqslant r_b; \quad r_b = 1.5764, && \mathrm{d}r = 10^{-6};\\ &(3)\ r_b \leqslant r \leqslant r_c; \quad r_c = 1.5764123, && \mathrm{d}r = 10^{-8};\\ &(4)\ r_c \leqslant r \leqslant r_1; \quad r_1 = 1.5764124853, && \mathrm{d}r = 10^{-11}.\end{aligned}\right\} \tag{D 3}$$

The criterion that was adopted in the choice of the step-size dr, for each interval, was that the same integration carried out with a step ten times smaller affected the values of the functions and their derivatives, at the end points of the intervals, only in the eighth significant figure. An illustrative example is provided in table A 1. The entries in the rows (labelled (a) and (b)) are the values of X, L, and W and their derivatives at the end points of the first three intervals (with the respective step-sizes 10^{-4}, 10^{-6} and 10^{-8}) for the two solutions derived, respectively, with the expansions (80)–(83). It will be observed that by the end point of the third interval $W \approx 0$ and X and L have attained their limiting values to the desired accuracy. Thus,

$$\left.\begin{aligned} &(a)\ X = -0.5590027, \quad L = -0.1563465, \quad W \sim 10^{-7};\\ &(b)\ X = -1.0472591, \quad L = 0.2340934, \quad W \sim 10^{-7}.\end{aligned}\right\} \tag{D 4}$$

But it will be noticed (cf. table 1 A) that $L_{,r}$ and $W_{,r}$ continue to change. We must therefore continue to integrate beyond the end point of the third interval. The results of further integrations with decreasing step-sizes are shown in table A 2. From the results in table A 2 we conclude that a further reduction in the step-size will affect only the sixth significant figure and that a step-size 10^{-11} is needed in the last interval.

With the necessary accuracy achieved for the functions and their derivatives at $r = r_1$, the integration of the Zerilli equation for $r > r_1$ was carried out with a step-size $\mathrm{d}r_* = 0.03$. For $r_* > 25/\sigma$ the sinusoidal behaviour, in accordance with equation

(98), is well established and the determination of $\alpha_0^2+\beta_0^2$ does not present any difficulty. All the values given in table 1 are therefore reliable to the number of decimals retained.

(*b*) *The dipole case*

We integrate the two independent solutions with the initial conditions (115) and (116) using the same procedure described for the quadrupole case. In this case the characteristic frequency is identified by the simultaneous vanishing of $W_{,r}$ for both solutions at the boundary. And it turns out that in order to obtain a result accurate up to the sixth significant figure we again have to use the same values for the integration steps used in the quadrupole case.

References

Campolattaro, A. & Thorne, K. S. 1970 *Astrophys. J.* **159**, 847–858.

Chandrasekhar, S. 1964*a* *Astrophys. J.* **140**, 417–433.

Chandrasekhar, S. 1964*b* *Phys. Rev. Lett.* **12**, 114–116; 437–438.

Chandrasekhar, S. 1964*c* *Astrophys. J.* **139**, 664–674.

Chandrasekhar, S. 1983 *The mathematical theory of black holes.* Oxford: Clarendon Press.

Chandrasekhar, S. & Detweiler, S. 1975 *Proc. R. Soc. Lond.* A **344**, 441–452.

Chandrasekhar, S. & Ferrari, V. 1990 *Proc. R. Soc. Lond.* A **428**, 325–349.

Chandrasekhar, S. & Xanthopoulos, B. C. 1979 *Proc. R. Soc. Lond.* A **367**, 1–14.

Cutler, C. & Lindblom, L. 1987 *Astrophys. J.* **314**, 234–241.

Hurley, M., Roberts, P. H. & Wright, K. 1966 *Astrophys. J.* **143**, 535–551.

Landau, L. D. & Lifshitz, E. M. 1977 *Quantum mechanics: non-relativistic theory.* London: Pergamon Press.

Lindblom, L. & Detweiler, S. 1983 *Astrophys. J. Suppl.* **53**, 73–92.

Lindblom, L. & Splinter, R. J. 1989 *Astrophys. J.* **345**, 925–930.

Thorne, K. S. & Campolattaro, A. 1967 *Astrophys. J.* **149**, 591–611.

Thorne, K. S. 1969 *Astrophys. J.* **158**, 1–16.

Tooper, R. F. 1964 *Astrophys. J.* **140**, 434–459.

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