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The stability of the pinch

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The stability of a cylindrical plasma with an axial magnetic field and confined between conducting walls is investigated by solving, for small oscillations about equilibrium, the linearized Boltzmann and Maxwell equations. A criterion for marginal stability is derived; this differs slightly from the one derived by Rosenbluth from an analysis of the particle orbits. However, Rosenbluth's principal results on the possibility of stabilizing the pinch under suitable external conditions are confirmed. In the appendix a dispersion relation appropriate for plane hydromagnetic waves in an infinite medium is obtained; this relation discloses under the simplest conditions certain types of instabilities which may occur in plasma physics.

1. Introduction

The stability of a cylindrical plasma (the 'pinch') with an axial magnetic field has recently been investigated by Kruskal & Tuck (1958), Rosenbluth (1957), Tayler (1957), Shrafranov (1957), and others. Rosenbluth has, in particular, shown that when the plasma is confined between conducting walls, the presence of an axial magnetic field can, under suitable circumstances, stabilize the pinch. Moreover, Rosenbluth has treated the problem not only from the standpoint of conventional hydromagnetics (with the usual assumptions of scalar pressure and adiabatic changes of state), but also from the physically more important standpoint of the orbits described by the ions and electrons in the external magnetic field and under conditions when collisions between particles play no role. The importance of Rosenbluth's treatment from the latter standpoint (along the general lines described by Longmire & Rosenbluth 1957) arises from the fact that under the conditions pinches are usually realized in the laboratory, collisions between ions and electrons do not, indeed, play any significant role.

In this paper we shall re-examine the problem treated by Rosenbluth by going directly to the Boltzmann equation appropriate under the conditions. This method has certain advantages over Rosenbluth's in that certain assumptions justified by him on physical grounds can now be examined for their validity. Also, we are able to treat the general time-dependent problem without being, necessarily, restricted to the marginal case distinguishing stability from instability. Further, the method provides an illustration of a general theory we have recently developed (Chandrasekhar, Kaufman & Watson 1957; this paper will be referred to hereafter as III) for a rigorous treatment for problems of this kind.

2. The method of treatment and the basic equations

Consider a uniform cylindrical plasma of radius r_0 with a constant magnetic field, B_P^0, along the z-axis. The plasma is confined in a cylinder of radius R_0 with conducting walls. In the space between r_0 and R_0 there is a vacuum field with both z- and θ-components; thus,

$$\mathbf{B}_V^0 = B_V^0 \mathbf{1}_z + B_\theta \frac{r_0}{r} \mathbf{1}_\theta \quad (r_0 \leqslant r \leqslant R_0), \tag{1}$$

where $\mathbf{1}_z$ and $\mathbf{1}_\theta$ are unit vectors in the z- and the θ-directions. We are interested in the stability of the plasma under these conditions.

A general perturbation theory for treating small departures from stationary solutions of the Boltzmann equation (in which the collision term is neglected) has been given in III. We shall begin by briefly describing this theory and quoting the basic equations under the simplified conditions of the problem on hand. The essential simplifications are that there are no static drifts (denoted by $\mathbf{V}^0$ in III); also the unperturbed field lines in the plasma are straight.

In treating the perturbation problem, a variable $\boldsymbol{\xi}$ which plays the same role as the Lagrangian displacement in the usual hydromagnetic treatments (cf. Bernstein, Frieman, Kruskal & Kulsrud 1958) is first introduced. In the present theory, $\boldsymbol{\xi}$ is related to the perturbations in the electric ($\mathbf{E}'$) and the magnetic ($\mathbf{B}'$) fields. In the case when the dependence on time of all quantities describing departures from equilibrium is given by

$$e^{\Omega t},$$

these relations are as follows: resolving $\boldsymbol{\xi}$ into two components, $\xi_\parallel$ and $\boldsymbol{\xi}_\perp$, respectively parallel and perpendicular to the direction of the unperturbed magnetic field $\mathbf{B}^0$ (we are suppressing the subscript, P, for the present) we have (cf. III, equations (41), (42) and (46) and note that $\mathbf{U} = \Omega\boldsymbol{\xi}$)

$$E'_\parallel = \frac{m^+\Omega^2}{e}\xi_\parallel, \quad \mathbf{E}'_\perp = -\frac{\Omega}{c}\boldsymbol{\xi}_\perp \times \mathbf{B}^0 \tag{2}$$

and

$$\mathbf{B}' = \operatorname{curl}\left(\boldsymbol{\xi}_\perp \times \mathbf{B}^0 - \frac{m^+c}{e}\Omega\xi_\parallel \mathbf{n}\right), \tag{3}$$

where $\mathbf{n}$ denotes a unit vector in the direction of $\mathbf{B}^0$—the z-axis in the problem on hand; further, m^+ denotes the mass of the ion, e the charge on the ion and c the velocity of light.

Before proceeding further, we may make some general remarks on the notation we shall adopt. The subscripts $\parallel$ and $\perp$ will indicate that the components of the particular vector parallel and perpendicular, respectively, to $\mathbf{n}$ are meant. Similarly, the superscripts $+$ and $-$ will distinguish the quantities referring to the ions and the electrons; when, however, an equation (or a quantity) is to be understood as applying to both ions and electrons, these superscripts will in general be omitted. Finally, superscripts '0' will be used to denote the equilibrium values of the respective quantities while primes will denote the corresponding perturbations.

From equation (2) it follows that

$$\frac{E'_\parallel}{|\mathbf{E}_\perp|} = \frac{m^+c}{eB^0}\Omega \sim \frac{\Omega}{\omega_{\text{Larmor}}}. \tag{2'}$$

Since a basic assumption underlying the present treatment of plasmas (cf. III, § 6) is that the changes which the system undergoes take place in times which are long compared with the Larmor periods of the particle orbits, it is clear that in calculating $\mathbf{B}'$ in accordance with equation (3), we may neglect the term in $\xi_\parallel$ if $\xi_\parallel$ and $|\boldsymbol{\xi}_\perp|$ should be of comparable magnitudes. And even if the term in $\xi_\parallel$ makes no contribution to $\mathbf{B}'$, we shall find that $\xi_\parallel$ does contribute a term to the perturbation in the pressure tensor which cannot be ignored.

The equations of motion governing $\xi_\parallel$ and $\boldsymbol{\xi}_\perp$ are (III, equations (68) and (73))

$$\rho^0\left(1+\frac{|B^0|^2}{4\pi\rho^0c^2}\right)\Omega^2\boldsymbol{\xi}_\perp = -(\operatorname{div}\mathbf{P}')_\perp+\frac{1}{4\pi}(\operatorname{curl}\mathbf{B}')\times\mathbf{B}^0 \tag{4}$$

and
$$(\Omega^2+\sum_{+,-}\omega_p^2)E'_\parallel = c\Omega\mathbf{n}\,.\operatorname{curl}\mathbf{B}'+4\pi\sum_{+,-}\left\{\frac{e}{m}(\operatorname{div}\mathbf{p}')_\parallel\right\}, \tag{5}$$

where $\mathbf{P}'$ ($\equiv P'_{ij}$) denotes the perturbation in the total pressure tensor† and the summations in equation (5) are over the terms referring to the ions and electrons; also ρ^0 denotes the (unperturbed) density and $\omega_p^\pm$ the plasma frequency,

$$\omega_p^\pm = (4\pi N^{0,\pm}e^2/m^\pm)^{\frac{1}{2}}, \tag{6}$$

and $N^{0,\pm}$ the (unperturbed) concentration of the ions and electrons.

It will be observed that equations (4) and (5) involve the perturbations in the pressure tensor, $\mathbf{p}'^{,\pm}$. We, therefore, need equations for determining $\mathbf{p}'$; and these are provided by the appropriately linearized form of the Boltzmann equation.

Let $f^0(q^2, s^2, \mathbf{r})$ denote the function governing the distribution of the velocities q and $\mathbf{s}$ parallel and perpendicular, respectively, to the direction of $\mathbf{n}$, in the stationary equilibrium state. Then, by expressing the perturbation in the distribution function, $f'(q, \mathbf{s}, \mathbf{r}, t)$ in the manner

$$f'(q,\mathbf{s},\mathbf{r},t) = A_1(q^2,s^2,\mathbf{r},t)+qA_2(q^2,s^2,\mathbf{r},t)+f'_P(q,\mathbf{s},\mathbf{r},t), \tag{7}$$

where f'_P is such that it vanishes when averaged over all directions of $\mathbf{s}$, equations for A_1 and A_2 and an explicit expression for f'_P were derived in III (equations (112), (145), (148) and (150)). Under the simpler conditions of the problem on hand, these equations are

$$f'_P = \frac{2q}{B^0}\left(\frac{\partial f^0}{\partial q^2}-\frac{\partial f^0}{\partial s^2}\right)\mathbf{s}\,.\mathbf{B}', \tag{8}$$

$$\left.\begin{aligned} A_2 = \Omega\Lambda, \quad A_1 &= -q^2\frac{\partial\Lambda}{\partial z}+G_1\\ \text{and}\qquad \Omega^2\Lambda+\frac{\partial A_1}{\partial z} &= -Q_2,\end{aligned}\right\} \tag{9}$$

where (III, equations (127) and (130))

$$G_1 = 2q^2\frac{\partial\xi_\parallel}{\partial z}\frac{\partial f^0}{\partial q^2}+s^2(\boldsymbol{\nabla}_\perp.\boldsymbol{\xi}_\perp)\frac{\partial f^0}{\partial s^2} \tag{10}$$

and
$$Q_2 = s^2\left(\frac{\partial f^0}{\partial q^2}-\frac{\partial f^0}{\partial s^2}\right)\frac{\boldsymbol{\nabla}_\perp.\mathbf{B}'_\perp}{B^0}. \tag{11}$$

† Note that P^0_{ij} and P'_{ij} refer to the total pressure (due to the electrons and the ions) while p^0_{ij} and p'_{ij} with superscripts + and − refer to the pressures due to the ions and electrons, separately; thus $P^0_{ij} = \sum_{+,-} p^0_{ij}$ and $P'_{ij} = \sum_{+,-} p'_{ij}$.

In equations (10) and (11), $\nabla_\perp$ denotes the projection of the usual gradient operator on a plane perpendicular to $\mathbf{n}$:

$$(\nabla_\perp)_i = \frac{\partial}{\partial x_i} - n_i n_j \frac{\partial}{\partial x_j}. \tag{12}$$

In terms of the solution (7) for f', the required perturbation in the pressure tensor, p'_{ij}, is given by

$$p'_{ij} = p'_{\parallel;L} n_i n_j + p'_{\perp;L}(\delta_{ij} - n_i n_j) + p'_{ij;P}, \tag{13}$$

where $$p'_{\parallel;L} = \tfrac{1}{2}m\iint q^2 A_1 \,\mathrm{d}q\,\mathrm{d}s^2; \quad p'_{\perp;L} = \tfrac{1}{4}m\iint s^2 A_1\,\mathrm{d}q\,\mathrm{d}s^2 \tag{14†}$$

and $$p'_{ij;P} = (n_i \mathbf{B}'_{\perp,j} + n_j \mathbf{B}'_{\perp,i})\frac{p^0_\parallel - p^0_\perp}{B^0}. \tag{15}$$

3. The solution of the perturbation equations for the plasma

We shall now show how the various equations governing the perturbed plasma can be solved. In solving these equations we shall suppose that the disturbance has been analyzed into normal modes and that the dependence on z and θ of all the perturbed quantities is given by

$$e^{i(kz+m\theta)}, \tag{16}$$

where k is the wave number of the disturbance and m is an integer (positive, zero or negative). Apart from the factor (16), the various quantities are functions only of r, the distance from the z-axis. For solutions having the dependence on z and θ given by (16),

$$\frac{\partial}{\partial z} = ik \quad \text{and} \quad \frac{\partial}{\partial\theta} = im; \tag{17}$$

also, $$\nabla_\perp = \mathbf{1}_r \frac{\partial}{\partial r} + \mathbf{1}_\theta \frac{im}{r}. \tag{18}$$

(a) *The perturbations in the electric and the magnetic fields*

Let $$F' = \frac{c}{\Omega}E'_\parallel \quad \text{so that} \quad \xi_\parallel = \frac{e}{m^+ c\Omega}F'. \tag{19}$$

In terms of F' the expression for $\mathbf{B}'$ is (cf. equation (3))

$$\mathbf{B}' = \operatorname{curl}(\boldsymbol{\xi}_\perp \times \mathbf{B}^0) + \mathbf{n} \times \operatorname{grad} F'. \tag{20}$$

When $\mathbf{B}^0$ is in the direction of the z-axis, the foregoing becomes

$$\mathbf{B}' = B^0\{ik\boldsymbol{\xi}_\perp - (\nabla_\perp . \boldsymbol{\xi}_\perp)\mathbf{n}\} + (\mathbf{n} \times \nabla_\perp) F', \tag{21}$$

where we have made use of the fact that when $\mathbf{n}$ is a constant vector,

$$\nabla . \mathbf{A}_\perp \equiv \nabla_\perp . \mathbf{A} \equiv \nabla_\perp . \mathbf{A}_\perp,$$

where $\mathbf{A}$ is an arbitrary vector. Accordingly,

$$B'_\parallel = -B^0 \nabla_\perp . \boldsymbol{\xi}_\perp \quad \text{and} \quad \mathbf{B}'_\perp = ikB^0 \boldsymbol{\xi}_\perp + (\mathbf{n} \times \nabla_\perp) F'. \tag{22}$$

† In writing (14) we have assumed that f^0 is so normalized that, for example,

$$N^0 = \frac{1}{2}\int_0^\infty \int_0^\infty f^0 \,\mathrm{d}q\,\mathrm{d}s^2.$$

We may note here (for later use) that for $\mathbf{B}'$ given by equation (21)

$$\operatorname{curl}\mathbf{B}' = B^0\left\{ik\operatorname{curl}\boldsymbol{\xi}_\perp + (\mathbf{n}\times\nabla_\perp)(\nabla_\perp.\boldsymbol{\xi}_\perp) + \frac{1}{B^0}(\mathbf{n}\nabla^2 - ik\nabla)F'\right\}, \tag{23}$$

so that
$$\mathbf{n}\times\operatorname{curl}\mathbf{B}' = B^0\left\{ik\mathbf{n}\times\operatorname{curl}\boldsymbol{\xi}_\perp - \nabla_\perp(\nabla_\perp.\boldsymbol{\xi}_\perp) - \frac{ik}{B^0}(\mathbf{n}\times\nabla_\perp)F'\right\} \tag{24}$$

and
$$\mathbf{n}.\operatorname{curl}\mathbf{B}' = B^0\left\{ik\mathbf{n}.\operatorname{curl}\boldsymbol{\xi}_\perp + \frac{1}{B^0}\nabla_\perp^2 F'\right\}. \tag{25†}$$

(b) *The solution for* A_1

According to equations (9) and (17)

$$A_1 = -ikq^2\Lambda + G_1 \tag{26}$$

and
$$\Omega^2\Lambda + ikA_1 = -Q_2. \tag{27}$$

From these equations, we find

$$A_1 = \frac{\Omega^2}{\Omega^2+k^2q^2}G_1 + \frac{ikq^2}{\Omega^2+k^2q^2}Q_2; \tag{28}$$

and substituting for G_1 and Q_2 from equations (10) and (11), we have

$$A_1 = \frac{\Omega^2}{\Omega^2+k^2q^2}\left\{2q^2\frac{\partial f^0}{\partial q^2}(ik\xi_\parallel) + s^2\frac{\partial f^0}{\partial s^2}(\nabla_\perp.\boldsymbol{\xi}_\perp)\right\} + \frac{ikq^2s^2}{\Omega^2+k^2q^2}\left(\frac{\partial f^0}{\partial q^2} - \frac{\partial f^0}{\partial s^2}\right)\frac{\nabla_\perp.\mathbf{B}'_\perp}{B^0}. \tag{29}$$

(c) *The pressure tensor* p'_{ij} *and its divergence*

On inserting for A_1 from (29) in equations (14), we observe that $p'_{\parallel;L}$ and $p'_{\perp;L}$ can be expressed in the forms

$$p'_{\parallel;L} = ik\xi_\parallel I_1 + (\nabla_\perp.\boldsymbol{\xi}_\perp)I_2 + ik\frac{\nabla_\perp.\mathbf{B}'_\perp}{B^0}I_3 \tag{30}$$

and
$$p'_{\perp;L} = ik\xi_\parallel J_1 + (\nabla_\perp.\boldsymbol{\xi}_\perp)J_2 + ik\frac{\nabla_\perp.\mathbf{B}'_\perp}{B^0}J_3, \tag{31}$$

where
$$\begin{pmatrix}I_1\\J_1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\\frac{1}{4}\end{pmatrix}m\iint\frac{2\Omega^2q^2}{\Omega^2+k^2q^2}\begin{pmatrix}q^2\\s^2\end{pmatrix}\frac{\partial f^0}{\partial q^2}\,dq\,ds^2, \tag{32}$$

$$\begin{pmatrix}I_2\\J_2\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\\frac{1}{4}\end{pmatrix}m\iint\frac{\Omega^2s^2}{\Omega^2+k^2q^2}\begin{pmatrix}q^2\\s^2\end{pmatrix}\frac{\partial f^0}{\partial s^2}\,dq\,ds^2 \tag{33}$$

and
$$\begin{pmatrix}I_3\\J_3\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\\frac{1}{4}\end{pmatrix}m\iint\frac{q^2s^2}{\Omega^2+k^2q^2}\begin{pmatrix}q^2\\s^2\end{pmatrix}\left(\frac{\partial f^0}{\partial q^2} - \frac{\partial f^0}{\partial s^2}\right)dq\,ds^2. \tag{34}$$

Now substituting for $\xi_\parallel$ and $\mathbf{B}'_\perp$ from equations (19) and (22) in the expressions for $p'_{\parallel;L}$ and $p'_{\perp;L}$, we obtain

$$p'_{\parallel;L} = (I_2 - k^2I_3)\nabla_\perp.\boldsymbol{\xi}_\perp + ik\frac{eI_1}{mc\Omega}F' \tag{35}$$

and
$$p'_{\perp;L} = (J_2 - k^2J_3)\nabla_\perp.\boldsymbol{\xi}_\perp + ik\frac{eJ_1}{mc\Omega}F'. \tag{36}$$

† In deriving this, use has been made of the relation $\nabla^2 - ik\mathbf{n}.\nabla = \nabla^2 + k^2 = \nabla_\perp^2$.

Using the definitions of I_2 and I_3, we readily find that

$$I_2 - k^2 I_3 = \tfrac{1}{2}m \iint q^2 s^2 \left\{\frac{\partial f^0}{\partial s^2} - \frac{\partial f^0}{\partial q^2} + \frac{\Omega^2}{\Omega^2 + k^2 q^2}\frac{\partial f^0}{\partial q^2}\right\} \mathrm{d}q\,\mathrm{d}s^2$$
$$= (p^0_\perp - p^0_\parallel) + J_1. \tag{37}$$

Similarly,

$$J_2 - k^2 J_3 = -\tfrac{1}{4}m \iint s^4 \left(\frac{\partial f^0}{\partial q^2} - \frac{\partial f^0}{\partial s^2}\right) \mathrm{d}q\,\mathrm{d}s^2 + \tfrac{1}{4}m\Omega^2 \iint \frac{s^4}{\Omega^2 + k^2 q^2}\frac{\partial f^0}{\partial q^2}\mathrm{d}q\,\mathrm{d}s^2; \tag{38}$$

or, letting

$$S\iint s^2 f^0\,\mathrm{d}q\,\mathrm{d}s^2 = \iint s^4 \left(\frac{\partial f^0}{\partial q^2} - \frac{\partial f^0}{\partial s^2}\right) \mathrm{d}q\,\mathrm{d}s^2 - \Omega^2 \iint \frac{s^4}{\Omega^2 + k^2 q^2}\frac{\partial f^0}{\partial q^2}\mathrm{d}q\,\mathrm{d}s^2, \tag{39}$$

we can write

$$J_2 - k^2 J_3 = -S p^0_\perp. \tag{40}$$

Thus,

$$p'_{\parallel;L} = (J_1 + p^0_\perp - p^0_\parallel)\,\nabla_\perp . \boldsymbol{\xi}_\perp + \mathrm{i}k \frac{eI_1}{mc\Omega} F' \tag{41}$$

and

$$p'_{\perp;L} = -S p^0_\perp (\nabla_\perp . \boldsymbol{\xi}_\perp) + \mathrm{i}k \frac{eJ_1}{mc\Omega} F'. \tag{42}$$

For later use we shall obtain here the divergence of the tensor p'_{ij}. For p'_{ij} given by equation (13) we may write (cf. III, equations (85) to (88))

$$\operatorname{div}\mathbf{p}' = \nabla_\perp p'_{\perp;L} + \mathrm{i}k p'_{\parallel;L}\mathbf{n} + \operatorname{div} p'_P. \tag{43}$$

By making use of equation (15), we can reduce the last term in equation (43) as follows:

$$\frac{\partial}{\partial x_j} p'_{ij;P} = \left[n_i \frac{\partial}{\partial x_j} B'_{\perp,j} + n_j \frac{\partial}{\partial x_j} B'_{\perp,i}\right] \frac{p^0_\parallel - p^0_\perp}{B^0}$$
$$= \left[\mathrm{i}k B'_{\perp,i} + n_i \frac{\partial}{\partial x_j}(B'_j - n_j n_l B'_l)\right] \frac{p^0_\parallel - p^0_\perp}{B^0}$$
$$= \mathrm{i}k (B'_{\perp,i} - n_i B'_\parallel) \frac{p^0_\parallel - p^0_\perp}{B^0}; \tag{44}$$

and substituting for $\mathbf{B}'_\perp$ and $B'_\parallel$ from (22), we obtain

$$\operatorname{div}\mathbf{p}'_P = \mathrm{i}k\left\{\mathrm{i}k\boldsymbol{\xi}_\perp + \frac{1}{B^0}(\mathbf{n}\times\nabla_\perp)F' + (\nabla_\perp . \boldsymbol{\xi}_\perp)\mathbf{n}\right\}(p^0_\parallel - p^0_\perp). \tag{45}$$

Finally, combining equations (41), (42), (43) and (45), we have

$$\operatorname{div}\mathbf{p}' = \nabla_\perp\left\{-S p^0_\perp(\nabla_\perp . \boldsymbol{\xi}_\perp) + \mathrm{i}k\frac{eJ_1}{mc\Omega}F'\right\} + \mathrm{i}k\left\{[J_1 + (p^0_\perp - p^0_\parallel)](\nabla_\perp . \boldsymbol{\xi}_\perp) + \mathrm{i}k\frac{eI_1}{mc\Omega}F'\right\}\mathbf{n}$$
$$+ \mathrm{i}k\left\{\mathrm{i}k\boldsymbol{\xi}_\perp + \frac{1}{B^0}(\mathbf{n}\times\nabla_\perp)F' + \mathbf{n}(\nabla_\perp . \boldsymbol{\xi}_\perp)\right\}(p^0_\parallel - p^0_\perp). \tag{46}$$

Hence

$$(\operatorname{div}\mathbf{p}')_\parallel = \mathrm{i}k\left\{J_1(\nabla_\perp . \boldsymbol{\xi}_\perp) + \mathrm{i}k\frac{eI_1}{mc\Omega}F'\right\} \tag{47}$$

and

$$(\operatorname{div}\mathbf{p}')_\perp = \nabla_\perp\left\{-S p^0_\perp(\nabla_\perp . \boldsymbol{\xi}_\perp) + \mathrm{i}k\frac{eJ_1}{mc\Omega}F'\right\} + \left\{k^2\boldsymbol{\xi}_\perp - \frac{\mathrm{i}k}{B^0}(\mathbf{n}\times\nabla_\perp)F'\right\}(p^0_\perp - p^0_\parallel). \tag{48}$$

(*d*) *The reduction of the equations governing F' and $\boldsymbol{\xi}_\perp$*

With the pressure tensor $\mathbf{p}'$ determined, we can now proceed to the solution of the equations governing $\xi_\parallel$ and $\boldsymbol{\xi}_\perp$.

Considering first equation (4) for $\boldsymbol{\xi}_\perp$, and making use of equations (24) and (48), we have

$$\rho^*\Omega^2\boldsymbol{\xi}_\perp = (\sum_{+,-} Sp_\perp^0)\nabla_\perp(\nabla_\perp.\boldsymbol{\xi}_\perp) - \mathrm{i}k\left(\sum_{+,-}\frac{eJ_1}{mc\Omega}\right)\nabla_\perp F'$$
$$-\left\{k^2\boldsymbol{\xi}_\perp - \frac{\mathrm{i}k}{B^0}(\mathbf{n}\times\nabla_\perp)F'\right\}(P_\perp^0 - P_\parallel^0)$$
$$-\frac{|B^0|^2}{4\pi}\left\{\mathrm{i}k\mathbf{n}\times\operatorname{curl}\boldsymbol{\xi}_\perp - \nabla_\perp(\nabla_\perp.\boldsymbol{\xi}_\perp) - \frac{\mathrm{i}k}{B^0}(\mathbf{n}\times\nabla_\perp)F'\right\}, \quad (49)$$

where
$$\rho^* = \rho^0\left(1+\frac{|B^0|^2}{4\pi\rho^0c^2}\right). \quad (50)$$

Letting
$$\sum_{+,-} Sp_\perp^0 = S^+p_\perp^{0,+} + S^-p_\perp^{0,-} = \bar{S}p_\perp^0 \quad \text{(say)}, \quad (51)$$

and noting that $\mathbf{n}\times\operatorname{curl}\boldsymbol{\xi}_\perp = -\mathrm{i}k\boldsymbol{\xi}_\perp$, we find after some rearranging that

$$\left(\bar{S}P_\perp^0 + \frac{|B^0|^2}{4\pi}\right)\nabla_\perp(\nabla_\perp.\boldsymbol{\xi}_\perp) - k^2\left(P_\perp^0 - P_\parallel^0 + \frac{|B^0|^2}{4\pi} + \rho^*\frac{\Omega^2}{k^2}\right)\boldsymbol{\xi}_\perp$$
$$= -\left(P_\perp^0 - P_\parallel^0 + \frac{|B^0|^2}{4\pi}\right)\frac{\mathrm{i}k}{B^0}(\mathbf{n}\times\nabla_\perp)F' + \mathrm{i}k\left(\sum_{+,-}\frac{eJ_1}{mc\Omega}\right)\nabla_\perp F'. \quad (52)$$

Similarly, by combining equations (5), (19), (25) and (47), we obtain

$$(\Omega^2 + \sum_{+,-}\omega_p^2)\frac{\Omega}{c}F' = c\Omega\nabla_\perp^2 F' + \mathrm{i}kcB^0\Omega\mathbf{n}.\operatorname{curl}\boldsymbol{\xi}_\perp$$
$$+4\pi\mathrm{i}k\left(\sum_{+,-}\frac{e}{m}J_1\right)\nabla_\perp.\boldsymbol{\xi}_\perp - k^2\left(\sum_{+,-}\frac{4\pi e^2}{m^2c\Omega}I_1\right)F', \quad (53)$$

or, equivalently (cf. equation (6))

$$\left\{\Omega^2 + \sum_{+,}\omega_p^2\left(1+\frac{k^2}{mN^0}\frac{I_1}{\Omega^2}\right)\right\}\frac{\Omega}{c}F'$$
$$= c\Omega\nabla_\perp^2 F' + \mathrm{i}kcB^0\Omega\mathbf{n}.\operatorname{curl}\boldsymbol{\xi}_\perp + 4\pi\mathrm{i}k\left(\sum_{+,-}\frac{e}{m}J_1\right)\nabla_\perp.\boldsymbol{\xi}_\perp. \quad (54)$$

Making use of the definition of I_1 (equation (32)) we find,

$$1+\frac{k^2I_1}{mN^0\Omega^2} = 1+\frac{1}{N^0}\iint\frac{k^2q^4}{\Omega^2+k^2q^2}\frac{\partial f^0}{\partial q^2}\,\mathrm{d}q\,\mathrm{d}s^2$$
$$= 1+\frac{1}{N^0}\iint q^2\frac{\partial f^0}{\partial q^2}\,\mathrm{d}q\,\mathrm{d}s^2 - \frac{1}{N^0}\iint\frac{\Omega^2q^2}{\Omega^2+k^2q^2}\frac{\partial f^0}{\partial q^2}\,\mathrm{d}q\,\mathrm{d}s^2$$
$$= -\frac{\Omega^2}{N^0}\iint\frac{q^2}{\Omega^2+k^2q^2}\frac{\partial f^0}{\partial q^2}\,\mathrm{d}q\,\mathrm{d}s^2 = -\Omega^2K \quad \text{(say)}. \quad (55)$$

Equation (54) may, therefore, be rewritten as

$$c\Omega\left\{\frac{\Omega^2}{c^2}(1-\sum_{+,-}\omega_p^2K) - \nabla_\perp^2\right\}F' = \mathrm{i}k\left\{c\Omega B^0\mathbf{n}.\operatorname{curl}\boldsymbol{\xi}_\perp + 4\pi\left(\sum_{+,-}\frac{e}{m}J_1\right)\nabla_\perp.\boldsymbol{\xi}_\perp\right\}. \quad (56)$$

(*e*) *The solution of the equations for F' and $\boldsymbol{\xi}_\perp$ in the limit $kc/\Omega \to \infty$*

Under most conditions of physical interest, the fluid velocities are very small compared with the velocity of light and $kc/\Omega \gg 1$. In the limit $kc/\Omega \to \infty$, equations (52) and (56) simplify somewhat and we are left with

$$a\boldsymbol{\nabla}_\perp(\boldsymbol{\nabla}_\perp.\boldsymbol{\xi}_\perp) = k^2\left(b+\rho\frac{\Omega^2}{k^2}\right)\boldsymbol{\xi}_\perp - \mathrm{i}k\frac{b}{B^0}(\mathbf{n}\times\boldsymbol{\nabla}_\perp)F' + \mathrm{i}k\left(\sum_{+,-}\frac{eJ_1}{mc\Omega}\right)\boldsymbol{\nabla}_\perp F' \tag{57}$$

and
$$\nabla_\perp^2 F' = -\mathrm{i}k\left\{B^0\mathbf{n}.\operatorname{curl}\boldsymbol{\xi}_\perp + 4\pi\left(\sum_{+,-}\frac{eJ_1}{mc\Omega}\right)\boldsymbol{\nabla}_\perp.\boldsymbol{\xi}_\perp\right\} - \frac{\Omega^2}{c^2}\left(\sum_{+,-}\omega_p^2 K\right)F', \tag{58}$$

where we have introduced the abbreviations

$$a = \bar{S}P_\perp^0 + \frac{|B^0|^2}{4\pi} \quad\text{and}\quad b = P_\perp^0 - P_\parallel^0 + \frac{|B^0|^2}{4\pi}. \tag{59}$$

Letting
$$\frac{\mathrm{i}k}{B^0}F' = \phi, \quad \boldsymbol{\nabla}_\perp.\boldsymbol{\xi}_\perp = \chi, \tag{60}$$

$$\Sigma_1 = B^0\sum_{+,-}\frac{eJ_1}{mc\Omega}, \quad \Sigma_2 = \frac{\Omega^2}{c^2}\left(\sum_{+,-}\omega_p^2 K\right) \tag{61}$$

and
$$\gamma^2 = \frac{1}{a}\left(b+\rho\frac{\Omega^2}{k^2}\right), \tag{62}$$

we can rewrite equations (57) and (58) more conveniently in the forms

$$\boldsymbol{\nabla}_\perp\chi = k^2\gamma^2\boldsymbol{\xi}_\perp - \frac{b}{a}(\mathbf{n}\times\boldsymbol{\nabla}_\perp)\phi + \frac{\Sigma_1}{a}\boldsymbol{\nabla}_\perp\phi \tag{63}$$

and
$$\nabla_\perp^2\phi = k^2\left\{\mathbf{n}.\operatorname{curl}\boldsymbol{\xi}_\perp + \frac{4\pi}{|B^0|^2}\Sigma_1\chi\right\} - \Sigma_2\phi. \tag{64}$$

Taking the vector product of equation (63) with $\boldsymbol{\nabla}_\perp$, we get

$$k^2\gamma^2\boldsymbol{\nabla}_\perp\times\boldsymbol{\xi}_\perp = \frac{b}{a}\mathbf{n}\nabla_\perp^2\phi. \tag{65}$$

Multiplying this equation scalarly by $\mathbf{n}$ and making use of equation (64), we obtain

$$k^2\gamma^2\mathbf{n}.\operatorname{curl}\boldsymbol{\xi}_\perp = \frac{b}{a}k^2\left\{\mathbf{n}.\operatorname{curl}\boldsymbol{\xi}_\perp + \frac{4\pi}{|B^0|^2}\Sigma_1\chi\right\} - \frac{b}{a}\Sigma_2\phi. \tag{66}$$

On further simplification (in which use is made of equation (62)) equation (66) reduces to

$$\rho\Omega^2\mathbf{n}.\operatorname{curl}\boldsymbol{\xi}_\perp = b\left\{\frac{4\pi k^2\Sigma_1}{|B^0|^2}\chi - \Sigma_2\phi\right\}. \tag{67}$$

The terms in $\boldsymbol{\xi}_\perp$ in this equation are of relative order (cf. equations (32), (59) and (61))

$$\frac{\rho\Omega^2}{k^2\Sigma_1}\frac{b}{|B^0|^2} = O\left(\frac{\rho\Omega^2}{k^2J_1}\frac{mc\Omega}{eB^0}\right) = O\left(\frac{mc\Omega}{eB^0}\right) = O\left(\frac{\Omega}{\omega_{\text{Larmor}}}\right); \tag{68}$$

and quantities of this order are neglected in our present treatment (cf. the remarks following equation (2′)). Accordingly, we may equate the quantity on the right-hand side of equation (67) to zero and obtain

$$\phi = \frac{4\pi k^2}{|B^0|^2}\frac{\Sigma_1}{\Sigma_2}\chi. \tag{69}$$

Returning to equation (63), we observe that of the two terms in ϕ on the right-hand side of this equation, the first can be neglected since it is of order $(\Omega/\omega)\,(\omega_p^2/k^2c^2)$ relative to the second. Thus, we can write

$$\nabla_\perp \chi = k^2\gamma^2 \boldsymbol{\xi}_\perp + \frac{\Sigma_1}{a}\nabla_\perp \phi. \tag{70}$$

Eliminating ϕ from this equation by making use of equation (69), we obtain

$$\left(1 - \frac{4\pi k^2}{|B^0|^2}\frac{\Sigma_1^2}{a\Sigma_2}\right)\nabla_\perp \chi = k^2\gamma^2 \boldsymbol{\xi}_\perp. \tag{71}$$

On substituting for Σ_1, Σ_2 and ω_p in accordance with equations (6) and (61), we find that the quantity in parentheses on the left-hand of equation (71) simplifies to

$$1 - \frac{k^2}{a\Omega^4}\frac{\{\sum_{+,-}(eJ_1/m)\}^2}{\sum_{+,-} e^2(N^0K/m)}. \tag{72}$$

Letting

$$\gamma^2 = \Gamma^2\left\{1 - \frac{k^2}{a\Omega^4}\frac{\{\sum_{+,-}(eJ_1/m)\}^2}{\sum_{+,-} e^2(N^0K/m)}\right\}, \tag{73}$$

we can rewrite equation (71) in the form

$$\nabla_\perp \chi = k^2\Gamma^2 \boldsymbol{\xi}_\perp. \tag{74}$$

Taking the scalar product of this equation with $\nabla_\perp$ and remembering that $\nabla_\perp . \boldsymbol{\xi}_\perp = \chi$, we obtain

$$\nabla_\perp^2 \chi = k^2\Gamma^2\chi. \tag{75}$$

The general solution of equation (75) which has no singularity at $r = 0$ is a multiple of

$$I_m(\Gamma kr)\,\mathrm{e}^{\mathrm{i}m\theta} = X(r)\,\mathrm{e}^{\mathrm{i}m\theta} \quad \text{(say)}, \tag{76}$$

where $I_m(x)$ is the Bessel function of order m for a purely imaginary argument. (In this paper we shall adopt Watson's (1952) notation regarding Bessel functions.) With the solution for X known (apart from an arbitrary constant factor which we shall ignore), we can, in accordance with equation (74), write

$$\boldsymbol{\xi}_\perp = \nabla_\perp(X\,\mathrm{e}^{\mathrm{i}m\theta}). \tag{77}$$

Thus

$$\xi_r = X' = k\Gamma I_m'(\Gamma kr) \tag{78}$$

and

$$\xi_\theta = \frac{\mathrm{i}m}{r}X = \frac{\mathrm{i}m}{r}I_m(\Gamma kr), \tag{79}$$

where (as in all other cases) we have suppressed the (common) factor $\mathrm{e}^{\mathrm{i}(kz+m\theta)}$ in the expressions for ξ_r and ξ_θ.

(*f*) *The perturbation in the magnetic field*

In terms of the solution for $\boldsymbol{\xi}_\perp$ obtained in the preceding subsection, the perturbation in the magnetic field $\mathbf{B}'_P$ (given by equation (21)) becomes

$$\mathbf{B}'_P = \mathrm{i}kB^0_P\left(X'\mathbf{1}_r + \frac{\mathrm{i}m}{r}X\mathbf{1}_\theta\right) - k^2\Gamma^2B^0_PX\mathbf{1}_z, \tag{80}$$

where we have restored the subscript P to indicate that this solution refers to the interior of the plasma. (Note that to the order of accuracy of the present treatment, the term in F' in equation (21) does not make any contribution to $\mathbf{B}'_P$ (cf. the remarks following equation (2')).)

(*g*) *The perturbation in the pressure tensor*

The perturbation in the transverse component of the pressure tensor can now be found. According to equations (42), (60) and (69), we have

$$p'_{\perp;L} = \left\{-Sp^0_\perp + \left(\frac{eB^0}{mc\Omega}J_1\right)\frac{4\pi k^2}{|B^0|^2}\frac{\Sigma_1}{\Sigma_2}\right\}\boldsymbol{\nabla}_\perp.\boldsymbol{\xi}_\perp. \tag{81}$$

On substituting for Σ_1 and Σ_2 from (61), we find that the second term in braces on the right-hand side of equation (81) can be simplified to the form

$$\frac{k^2}{\Omega^4}\frac{(eJ_1/m)\sum_{+,-}(eJ_1/m)}{\sum_{+,-}(e^2N^0K/m)} = -Rp^0_\perp \quad \text{(say)}. \tag{82}$$

Thus,
$$p'_{\perp;L} = -(S+R)\,p^0_\perp\boldsymbol{\nabla}_\perp.\boldsymbol{\xi}_\perp = -k^2\Gamma^2(S+R)\,p^0_\perp X. \tag{83}$$

From equations (82) and (83) it follows that

$$P'_{\perp;L} = -k^2\Gamma^2(\bar{S}+\bar{R})\,P^0_\perp X, \tag{84}$$

where (cf. equations (62) and (73))

$$\bar{R}P^0_\perp = -\frac{k^2}{\Omega^4}\frac{\{\sum_{+,-}(eJ_1/m)\}^2}{\sum_{+,-}(e^2N^0K/m)} = \left(\frac{\gamma^2}{\Gamma^2}-1\right)a, \tag{85}$$

and $\bar{S}$ has the same meaning as in equation (51).

From equation (85) we can derive an alternative formulae for Γ^2. We have

$$(\bar{S}+\bar{R})\,P^0_\perp\Gamma^2 = (\bar{S}P^0_\perp - a)\,\Gamma^2 + \gamma^2a; \tag{86}$$

or making use of equations (59), we have

$$(\bar{S}+\bar{R})\,P^0_\perp\Gamma^2 = -\frac{|B^0|^2}{4\pi}\Gamma^2 + P^0_\perp - P^0_\parallel + \rho\frac{\Omega^2}{k^2} + \frac{|B^0|^2}{4\pi}. \tag{87}$$

Thus
$$\Gamma^2 = \frac{P^0_\perp - P^0_\parallel + |B^0|^2/4\pi + \rho\Omega^2/k^2}{(\bar{S}+\bar{R})\,P^0_\perp + |B^0|^2/4\pi}. \tag{88}$$

4. The solution for the perturbed field in the vacuum

The unperturbed field outside the plasma is given by equation (1). Let $\mathbf{B}'_V$ denote the perturbation in the field. Since no currents can flow in a vacuum, we can derive $\mathbf{B}'_V$ from a scalar potential Ψ; thus

$$\mathbf{B}'_V = \nabla\Psi, \quad \text{where} \quad \nabla^2\Psi = 0. \tag{89}$$

For solutions which have the $e^{i(kz+m\theta)}$ dependence on z and θ, the appropriate form for Ψ is

$$\Psi = B_\theta r_0[C_1 I_m(kr) + C_2 K_m(kr)]\, e^{i(kz+m\theta)}, \tag{90}$$

where C_1 and C_2 are constants to be determined, and I_m and K_m are the Bessel functions of order m and of the two kinds for a purely imaginary argument (cf. Watson 1952). The constant factor $B_\theta r_0$ has been introduced in (90) for later convenience. The required solution for $\mathbf{B}'_V$ can, therefore, be expressed in the form

$$\mathbf{B}'_V = B_\theta r_0\left(\psi'\mathbf{1}_r + \frac{im}{r}\psi\mathbf{1}_\theta + ik\psi\mathbf{1}_z\right), \tag{91}$$

where

$$\psi(r) = C_1 I_m(kr) + C_2 K_m(kr), \tag{92}$$

and the factor $e^{i(kz+m\theta)}$ has again been suppressed.

At the outer boundary, $r = R_0$, the radial component of $\mathbf{B}'_V$ must vanish. This leads to the relation

$$C_1 I'_m(kR_0) + C_2 K'_m(kR_0) = 0. \tag{93}$$

A further relation arises from applying the boundary conditions at the surface of the plasma. These latter conditions are considered in the following section.

5. The characteristic equation for Ω^2

The solutions of the equations governing the departures from equilibrium of the plasma have been obtained in §§ 3 and 4. It remains to satisfy the boundary conditions which must be met at the surface of the plasma. As we shall presently see, these conditions will lead to an equation for Ω^2 and thus to the criterion for stability.

The boundary conditions which must in general be satisfied at a surface of discontinuity have been formulated in III, § 14. The principal requirements are that the normal components of the magnetic field and the stress tensor, T_{ij}, are continuous across the boundary. Thus, if $\mathbf{N}$ denotes the unit outward normal on a surface of discontinuity, then

$$N_j\Delta[B_j] = 0 \tag{94}$$

and

$$N_j\Delta[T_{ij}] = 0, \tag{95}$$

where $\Delta[X]$ is the jump experienced by a quantity X at the surface. The conditions (94) and (95) must, of course, be satisfied for both the perturbed and the unperturbed problems.

(a) *The continuity of the normal component of* $\mathbf{B}$

Consider first the condition (94). Since $\mathbf{N}^0 = \mathbf{1}_r$, the unperturbed fields

$$\mathbf{B}^0_P = B^0_P\mathbf{1}_z = \alpha_P B_\theta \mathbf{1}_z \quad (r \leqslant r_0) \tag{96}$$

and

$$\mathbf{B}^0_V = B_\theta\left(\alpha_V\mathbf{1}_z + \frac{r_0}{r}\mathbf{1}_\theta\right) \quad (r_0 \leqslant r \leqslant R_0), \tag{97}$$

(where $\alpha_P = B_P^0/B_\theta$ and $\alpha_V = B_V^0/B_\theta$) clearly satisfy the required condition. That the condition be satisfied for the perturbed problem, as well, requires that

$$\mathbf{1}_r.\Delta[\mathbf{B}'] + \delta\mathbf{N}.\Delta[\mathbf{B}^0] = 0, \tag{98}$$

where $\delta\mathbf{N}$ is the change in the direction of the outward normal caused by the perturbation. This latter change can be inferred from the equation of motion relating $\mathbf{N}$ to the velocity of displacement, $\boldsymbol{\alpha}$, of the surface, namely (III, equation (178))

$$\frac{\partial\mathbf{N}}{\partial t} + (\boldsymbol{\alpha}.\boldsymbol{\nabla})\,\mathbf{N} = \mathbf{N}\times[\mathbf{N}\times\{(\boldsymbol{\nabla}\boldsymbol{\alpha}).\mathbf{N}\}]. \tag{99}$$

Since there are no static drifts in the problem under consideration, the appropriate, linearized form of equation (99) for the perturbed problem is

$$\Omega\delta\mathbf{N} = \mathbf{1}_r\times[\mathbf{1}_r\times\{(\boldsymbol{\nabla}\boldsymbol{\alpha}).\mathbf{1}_r\}]. \tag{100}$$

The velocity of displacement of the surface of the plasma is clearly given by

$$\boldsymbol{\alpha} = \Omega\boldsymbol{\xi}_\perp. \tag{101}$$

From equations (100) and (101) we deduce that

$$\delta\mathbf{N} = -\frac{\mathrm{i}m}{r_0}\xi_r(r_0)\,\mathbf{1}_\theta - \mathrm{i}k\xi_r(r_0)\,\mathbf{1}_z. \tag{102}$$

Now substituting for $\mathbf{B}'$, $\mathbf{B}^0$ and $\delta\mathbf{N}$ in accordance with equations (80), (91), (96), (97) and (102) in equation (98), we find, after some reductions, that

$$\mathrm{i}\xi_r(r_0)\,(\alpha_V y + m) = r_0^2\psi'(r_0), \tag{103}$$

where

$$y = kr_0 \tag{104}$$

is the wave number of the disturbance in the unit $1/r_0$. The explicit form of equation (103) is (cf. equations (78) and (92))

$$\mathrm{i}k\Gamma I_m'(\Gamma y)\,(\alpha_V y + m) = r_0^2 k[C_1 I_m'(y) + C_2 K_m'(y)]. \tag{105}$$

Equations (92) and (105) now determine the constants C_1 and C_2. We find:

$$\left.\begin{aligned} C_1 &= -\frac{\mathrm{i}}{r_0^2}\frac{\Gamma I_m'(\Gamma y)}{I_m'(y)}\frac{G_{m,\zeta}(y)}{1-G_{m,\zeta}(y)}(\alpha_V y + m),\\ C_2 &= +\frac{\mathrm{i}}{r_0^2}\frac{\Gamma I_m'(\Gamma y)}{K_m'(y)}\frac{1}{1-G_{m,\zeta}(y)}(\alpha_V y + m),\end{aligned}\right\} \tag{106}$$

where

$$G_{m,\zeta}(y) = \frac{I_m'(y)\,K_m'(R_0 y/r_0)}{K_m'(y)\,I_m'(R_0 y/r_0)} = \frac{I_m'(y)\,K_m'(\zeta y)}{K_m'(y)\,I_m'(\zeta y)}, \tag{107}$$

where $\zeta = R_0/r_0$.

(b) *The continuity of the normal component of T_{ij}*

Consider next the requirement of the continuity of the normal component of T_{ij}. Since

$$T_{ij} = -P_{ij} + \frac{1}{4\pi}(B_iB_j + E_iE_j) - \frac{\delta_{ij}}{8\pi}(|\,\mathbf{E}\,|^2 + |\,\mathbf{B}\,|^2), \tag{108}$$

the condition for the unperturbed problem is

$$\Delta\left[P^0_\perp + \frac{|\mathbf{B}|^2}{8\pi}\right] = 0. \tag{109}$$

According to equations (96) and (97) this condition requires that

$$P^0_\perp + \alpha_P^2 \frac{B_\theta^2}{8\pi} = \frac{1}{8\pi}(\alpha_V^2 B_\theta^2 + B_\theta^2) \tag{110}$$

or

$$\frac{4\pi}{B_\theta^2} P^0_\perp = \tfrac{1}{2}(1 + \alpha_V^2 - \alpha_P^2); \tag{111}$$

this is a condition for the equilibrium of the stationary plasma.

The continuity of $N_j T_{ij}$ for the perturbed problem on the displaced surface of the plasma leads to the single condition (cf. III, equation (207))

$$\Delta\left[\xi_r \frac{\partial}{\partial r}\left(P^0_\perp + \frac{|\mathbf{B}^0|^2}{8\pi}\right) + P'_\perp + \frac{1}{4\pi}\mathbf{B}^0 . \mathbf{B}'\right] = 0. \tag{112}$$

According to equations (96) and (97)

$$\Delta\left[\xi_r \frac{\partial}{\partial r}\left(P^0_\perp + \frac{|\mathbf{B}^0|^2}{8\pi}\right)\right] = -\frac{B_\theta^2}{4\pi}\left[\xi_r \frac{\partial}{\partial r}\left(\frac{r_0}{r}\right)\right]_{r=r_0} = \frac{B_\theta^2}{4\pi}\left(\frac{\xi_r}{r}\right)_{r=r_0}. \tag{113}$$

Similarly, from equations (80), (91), (96) and (97) we find

$$\Delta[\mathbf{B}^0 . \mathbf{B}'] = \mathbf{B}^0_P . \mathbf{B}'_P - \mathbf{B}^0_V . \mathbf{B}'_V = -B_\theta^2[\alpha_P^2 \nabla_\perp . \boldsymbol{\xi}_\perp + \mathrm{i}\psi(\alpha_V y + m)]_{r=r_0}; \tag{114}$$

also (cf. equation (84)),

$$\Delta[P'_\perp] = -(\bar{S} + \bar{R}) P^0_\perp \nabla_\perp . \boldsymbol{\xi}_\perp. \tag{115}$$

Substituting for the various terms in equation (112) in accordance with the foregoing equations, we obtain

$$-(\bar{S} + \bar{R}) P^0_\perp \nabla_\perp . \boldsymbol{\xi}_\perp = \frac{B_\theta^2}{4\pi}\left[\alpha_P^2 \nabla_\perp . \boldsymbol{\xi}_\perp + \mathrm{i}\psi(\alpha_V y + m) - \frac{\xi_r}{r_0}\right]_{r=r_0}, \tag{116}$$

or, more explicitly,

$$\left[\frac{\alpha_P^2 B_\theta^2}{4\pi} + (\bar{S} + \bar{R}) P^0_\perp\right] k^2 \Gamma^2 I_m(\Gamma y) = -\mathrm{i}\frac{B_\theta^2}{4\pi}(\alpha_V y + m)[C_1 I_m(y) + C_2 K_m(y)] + \frac{B_\theta^2}{4\pi}\frac{k\Gamma}{r_0} I'_m(\Gamma y). \tag{117}$$

Finally, substituting for C_1 and C_2 from equations (106), we obtain

$$\left[\frac{\alpha_P^2 B_\theta^2}{4\pi} + (\bar{S} + \bar{R}) P^0_\perp\right] k^2 \Gamma^2 I_m(\Gamma y) = -\frac{B_\theta^2}{4\pi}(\alpha_V y + m)^2 \frac{\Gamma}{r_0^2}\frac{I'_m(\Gamma y)}{1 - G_{m,\zeta}(y)}\left\{G_{m,\zeta}(y)\frac{I_m(y)}{I'_m(y)} - \frac{K_m(y)}{K'_m(y)}\right\} + \frac{B_\theta^2}{4\pi}\frac{k\Gamma}{r_0} I'_m(\Gamma y). \tag{118}$$

Letting
$$P_m(x) = \frac{I_m(x)}{xI'_m(x)} \quad \text{and} \quad Q_m(x) = \frac{K_m(x)}{xK'_m(x)}, \tag{119†}$$

and making use of equation (88) we find that equation (118) can be reduced to the form
$$\left\{1 + 4\pi \frac{P^0_\perp - P^0_\parallel}{\alpha_P^2 B_\theta^2} + \frac{4\pi}{k^2}\frac{\rho^0\Omega^2}{\alpha_P^2 B_\theta^2}\right\}\alpha_P^2 y^2 P_m(\Gamma y) + (\alpha_V y \pm m)^2 \frac{G_{m,\zeta}(y)\,P_m(y) - Q_m(y)}{1 - G_{m,\zeta}(y)} = 1, \tag{120}$$

where we have replaced m by $\pm m$ to emphasize that m can be a positive or a negative integer (including, of course, the value zero).

Equation (120) is the required equation for determining Ω^2.

6. The marginal state $\Omega^2 = 0$

If one assumes that the principle of the exchange of stabilities is valid, then the marginal state separating the domains of stability and instability will be characterized by $\Omega^2 = 0$; and when $\Omega^2 = 0$, the characteristic equation (120) reduces to

$$\left\{1 + 4\pi \frac{P^0_\perp - P^0_\parallel}{\alpha_P^2 B_\theta^2}\right\}\alpha_P^2 y^2 P_m(\Gamma y) + (\alpha_V y \pm m)^2 \frac{G_{m,\zeta}(y)\,P_m(y) - Q_m(y)}{1 - G_{m,\zeta}(y)} = 1, \tag{121}$$

where it may be recalled that now (cf. equation (88))

$$\Gamma^2 = \frac{P^0_\perp - P^0_\parallel + \alpha_P^2 B_\theta^2/4\pi}{(\bar{S} + \bar{R})\,P^0_\perp + \alpha_P^2 B_\theta^2/4\pi}, \tag{122}$$

$$\bar{S}P^0_\perp = \sum_{+,-} S p^0_\perp,$$
$$\bar{R}P^0_\perp = \lim_{\Omega\to 0} -\frac{k^2}{\Omega^4}\frac{\{\sum_{+,-}(eJ_1/m)\}^2}{\sum_{+,-}(e^2N^0K/m)} \tag{123}$$

and
$$S\iint f^0 s^2\,\mathrm{d}q\,\mathrm{d}s^2 = \iint s^4\left(\frac{\partial f^0}{\partial q^2} - \frac{\partial f^0}{\partial s^2}\right)\mathrm{d}q\,\mathrm{d}s^2. \tag{124}$$

Except for the appearance of the term in $\bar{R}$, these equations are the same as those derived by Rosenbluth from an analysis of the particle orbits. The term in $\bar{R}$ arises from electric fields induced in the direction of $\mathbf{B}^0_P$; and the possibility of such electric fields are not allowed for in Rosenbluth's treatment. As we shall presently see, only under very special conditions does $\bar{R}$ vanish.

If the distributions of q and $\mathbf{s}$ are both Gaussian but with different dispersions ('temperatures') then it can be readily shown from the foregoing definitions that

$$S = 2(1-\eta) \quad \text{and} \quad Rp^0_\perp = \frac{eN^0\eta\sum_{+,-} eN^0\eta}{\sum_{+,-}[e^2(N^0)^2\eta/p^0_\perp]}, \tag{125}$$

where
$$\eta = p^0_\perp/p^0_\parallel. \tag{126}$$

† These are the functions denoted by L_m and K_m by Kruskal & Tuck (1958).

If, in addition, the ratio of the temperatures in the longitudinal and the transverse directions are the same for ions and electrons, then

$$\bar{S} = S = 2(1-\eta) \quad \text{and} \quad \bar{R} = R = 0, \tag{127}$$

since $N^{0,+} = N^{0,-}$. In our further discussion we shall assume the validity of equations (127).

In discussing the implications of equation (121) (under the circumstances leading to (127)) it is convenient to introduce the parameter

$$\beta = \frac{4\pi}{B_\theta^2} P_\perp^0; \tag{128}$$

defined in this manner, it has (apart from a factor) the usual meaning of 'β'. In terms of η and β

$$\Gamma^2 = \frac{\alpha_P^2 + \beta(1-\eta^{-1})}{\alpha_P^2 + 2\beta(1-\eta)} \tag{129}$$

and equation (121) becomes

$$\left\{\alpha_P^2 + \beta\left(1 - \frac{1}{\eta}\right)\right\} y^2 P_m(\Gamma y) + (\alpha_V y \pm m)^2 \frac{G_{m,\zeta}(y)\, P_m(y) - Q_m(y)}{1 - G_{m,\zeta}(y)} = 1. \tag{130}$$

In equation (130), α_P^2, α_V^2 and β are not all independent; for, according to equation (111),

$$2\beta = 1 + \alpha_V^2 - \alpha_P^2; \tag{131}$$

in particular,

$$\alpha_P^2 \leqslant (1 + \alpha_V^2), \tag{132}$$

and the maximum value of α_P^2 (namely, $1 + \alpha_V^2$) corresponds to $\beta = 0$ and, therefore, to a vanishing plasma.

The case $\eta = 1$ has been discussed quite completely by Rosenbluth. The case $\eta \neq 1$ can be discussed in a very similar manner, though there is one important limitation which does not arise when $\eta = 1$. For, if η is sufficiently large (or small) Γ^2 can be negative; and one can convince oneself that if Γ^2 should be negative then the pinch would be definitely unstable for certain values of y. This arises from the fact that if $\Gamma^2 < 0$, then $P_m(\Gamma y)$ becomes

$$P_m(|\Gamma| y) = \frac{J_m(|\Gamma| y)}{|\Gamma| y J_m'(|\Gamma| y)}, \tag{133}$$

where $J_m(x)$ is the ordinary Bessel function. The function $P_m(|\Gamma| y)$ thus defined has poles for infinitely many values of y; in the neighbourhood of these poles, the right-hand side of equation (130) will become negatively infinite; and this clearly implies instability. Therefore, a supplementary condition for stability is

$$\left.\begin{aligned} &\alpha_P^2 + 2\beta(1-\eta) > 0 \quad \text{when} \quad \eta > 1 \\ \text{and} \qquad &\alpha_P^2 + \beta(1-\eta^{-1}) > 0 \quad \text{when} \quad \eta < 1. \end{aligned}\right\} \tag{134}$$

On eliminating β by making use of equation (131), we find that these conditions are

$$\left.\begin{aligned} &\alpha_P^2 > (1+\alpha_V^2)(1-\eta^{-1}) \qquad \text{when} \quad \eta > 1 \\ \text{and} \qquad &\alpha_P^2 > (1+\alpha_V^2)(1-\eta)/(1+\eta) \quad \text{when} \quad \eta < 1. \end{aligned}\right\} \tag{135}$$

(Note that by definition η is positive.) For stability, the possible range of α_P^2 is, therefore, limited by

$$\left.\begin{aligned} &(1+\alpha_V^2) \geqslant \alpha_P^2 > (1+\alpha_V^2)(1-\eta^{-1}) \quad (\eta > 1)\\ \text{and}\quad &(1+\alpha_V^2) \geqslant \alpha_P^2 > (1+\alpha_V^2)(1-\eta)/(1+\eta) \quad (\eta < 1).\end{aligned}\right\} \tag{136}$$

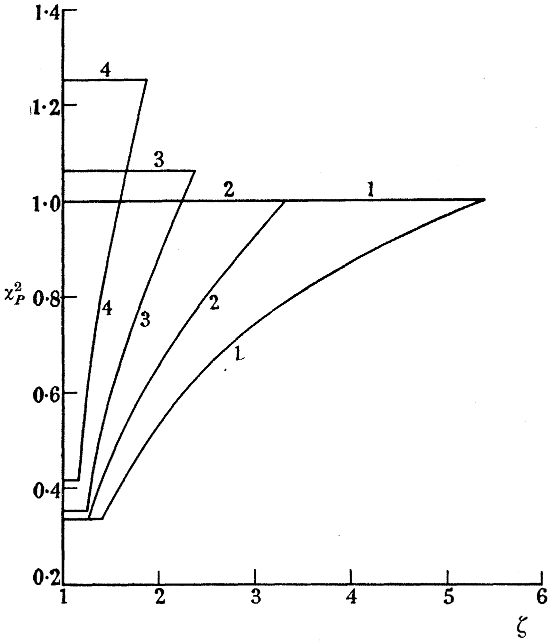

FIGURE 1

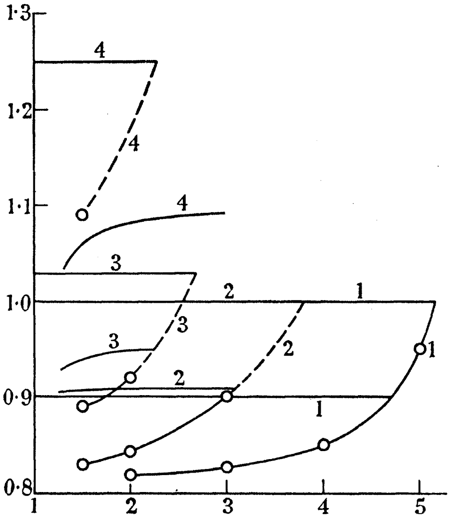

FIGURE 2

FIGURE 1. The regions of stability in the (α_P^2, ζ)-plane for the case $\eta = \frac{1}{2}$ and $\alpha_V = 0$, 0·1, 0·25 and 0·5 (distinguished by the numbers 1, 2, 3 and 4, respectively). For a given α_V, a stable pinch occurs in a region bounded on the left by the α_P^2-axis, on the right by the marginal curve for $m = -1$, above the equilibrium condition $\alpha_P^2 < (1+\alpha_V^2)$ and below by the condition $\alpha_P^2 > \frac{1}{3}(1+\alpha_V^2)$ (see (136)).

FIGURE 2. The regions of stability in the (α_P^2, ζ)-plane for the case $\eta = 5$ and $\alpha_V = 0$, 0·1, 0·25 and 0·5 (distinguished by the numbers 1, 2, 3 and 4, respectively). For a given α_V, a stable pinch occurs in a region bounded on the left by the α_P^2-axis, on the right by the marginal curve for $m = -1$, below by the condition (138) and above by the equilibrium condition $\alpha_P^2 < (1+\alpha_V^2)$. The circles are the calculated points on the marginal curve for $m = -1$; and the extrapolated parts of these curves are shown dashed.

Returning to equations (130) and (131) and considering first the case $m = 0$, we can show, as in Rosenbluth's discussion for the case $\eta = 1$, that a sufficient condition for the stability of the mode $m = 0$, for all y, is that it obtains in the limit $y = 0$. This yields the condition

$$\alpha_P^2 + 2\beta(1-\eta) + \frac{\alpha_V^2}{\zeta^2-1} > \tfrac{1}{2}; \tag{137}$$

or, eliminating β by means of equation (131), we obtain

$$\eta\alpha_P^2 + (1-\eta)(1+\alpha_V^2) + \frac{\alpha_V^2}{\zeta^2-1} > \tfrac{1}{2}. \tag{138}$$

This condition is always satisfied for $\eta < \frac{1}{2}$; and for $\eta > \frac{1}{2}$, equation (138) defines, for each prescribed η, a locus in the (α_P^2, ζ)-plane which, together with (136), delimits a region of stability (for the mode $m = 0$) in this plane. These regions for different values of η and α_V^2 are shown in figures 1 and 2.

When $|m| = 1$, a similar locus can be defined in the (α_P^2, ζ)-plane. However, the specification of this locus requires a careful numerical examination of equations (130) and (131). (It is sufficient to consider $m = -1$ as being more unfavourable for stability.) Precisely, it requires the determination (by trial and error) of the maximum value of α_P^2 which will solve equation (130) (for some y) for assigned values of ζ, η and α_V. Mrs Josephine Powers has carried out the necessary calculations and her results are included in figures 1 and 2.

Finally, considering the modes $|m| \geqslant 2$, we can again show, following Rosenbluth, that the pinch will be stable for all y and $|m| > 2$ if it is stable for $|m| = 2$, all y and $\zeta = \infty$. The case $|m| = 2$ and $\zeta = \infty$ needs a separate discussion. The most unfavourable circumstances for stability, in case $|m| = 2$, arise from

$$\{\alpha_P^2 + \beta(1-\eta^{-1})\}\, y^2 P_2(\Gamma y) + (\alpha_V y - 2)^2 Q_2(y) = 1. \tag{139}$$

For a given α_V^2 this equation determines a maximum value for α_P^2 for which it allows a solution for some y. These values are listed in table 1. From an examination of this table it appears that if the modes $m = 0$ and -1 are stable and the supplementary conditions (136) are satisfied then all the remaining modes are stable.

Table 1. The maximum value of α_P^2 for which equation (139) allows a solution for assigned values of α_V and η

$\eta = 0\cdot5$			$\eta = 5\cdot0$		
α_V	α_P^2	y	α_V	α_P^2	y
0·10	0·383	3·52	0·10	0·307	5·00
0·25	0·464	2·20	0·25	0·348	3·55
0·50	0·688	1·50	0·50	0·422	1·95
1·00	1·482	1·00	1·00	0·970	1·00

7. The time-dependent case

When $\Omega^2 \neq 0$, we must go back to equation (120) in which the definition of Γ^2 also involves Ω^2; thus,

$$\Gamma^2 = \frac{P_\perp^0 - P_\parallel^0 + \alpha_P^2 B_\theta^2/4\pi + \rho\Omega^2/k^2}{(\bar{S}+\bar{R})\, P_\perp^0 + \alpha_P^2 B_\theta^2/4\pi}, \tag{140}$$

where the 'shape factor' $(\bar{S}+\bar{R})$ also depends on Ω^2.

If as in § 6 we consider Gaussian distributions of q and $\mathbf{s}$ (but with different temperatures) then it can be shown that (cf. equation (125))

$$\bar{S}P_\perp^0 = 2\sum_{+,-}(1-\tilde{\eta})\, p_\perp^0, \quad \bar{R}P_\perp^0 = \frac{\{\sum_{+,-}(eN\tilde{\eta})\}^2}{\sum_{+,-}(e^2N^2\tilde{\eta}/p_\perp^0)}, \tag{141}$$

where
$$\tilde{\eta} = \eta[1-H(\sigma)] = \frac{p_\perp^0}{p_\parallel^0}[1-H(\sigma)], \tag{142}$$

$$\sigma = \frac{\Omega}{k}\left(\frac{m}{2\boldsymbol{k}T_\parallel}\right)^{\frac{1}{2}} \quad (\boldsymbol{k} = \text{Boltzmann constant}) \tag{143}$$

and
$$H(x) = \pi^{\frac{1}{2}} x\, e^{x^2}\left[1-\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy\right]. \tag{144}$$

From the definition of $H(x)$ it follows that
$$H(x)\to 0 \text{ as } x\to 0 \quad\text{and}\quad H(x)\to 1 \text{ as } x\to\infty. \tag{145}$$

If $\Omega^2>0$, then the characteristic equation can be discussed without any difficulty of principle. If, however, we should be interested in stable oscillations, then we should exercise some care since the integrals such as I_1, J_1, etc., which we have to define, are divergent. Under these circumstances, we may suppose that we are solving the Boltzmann equation by considering its Laplace transform; thus
$$f(\mathbf{r},\Omega) = \int_0^\infty f(\mathbf{r},t)\, e^{-\Omega t}\,dt. \tag{146}$$

The solution we have found is formally $f(\mathbf{r},\Omega)$; and the required solution satisfying suitable boundary conditions can then be obtained by inversion. This will require us to go into the complex Ω-plane; and the solution of the problem can be completed as has been done in the case of plasma oscillations by Landau (1946; see also van Kampen 1955). We hope to return to these matters in greater detail on a later occasion.

Notes added in proof 10 *May* 1958

(*a*) *The non-occurrence of overstability in the pinch*

In discussing the stability of the pinch in § 6, we have assumed that $\Omega^2=0$ separates the domains of stability and instability. To complete the discussion, it is clearly necessary to examine whether overstability can occur. It can be shown that this is not possible. We had constructed a proof for this; at the same time Dr Marshall Rosenbluth communicated to us a somewhat different but very elegant proof. The following is an outline of Dr Rosenbluth's proof; we are grateful to him for allowing us to include it in this paper.

The basic equations are (107), (119), (120) and (140) to (144). These equations must be considered in the complex Ω-plane.

It is convenient to rewrite equation (120) in the form
$$\Phi(\Omega) = \left\{1+\frac{4\pi}{\alpha_P^2 B_\theta^2}\left(P_\perp^0-P_\parallel^0+\frac{\rho\Omega^2}{k^2}\right)\right\}\alpha_P^2 y^2 P_m(\Gamma y) - L = 0.$$

Here L is independent of Ω and all quantities except Γ^2 and Ω are real.

Considering first $H(\Omega)$ (as defined in equations (143) and (144)), we observe that this is analytic on the right half of the complex plane and on the imaginary axis; further the sign of its imaginary part is the same as that of Ω; and the imaginary part vanishes only for real Ω. From these properties of $H(\Omega)$ we deduce that: $\Gamma^2(\Omega)$ is analytic on the right half plane; it has zeros only at $\pm i\Omega_0$ where
$$\Omega_0 = k\sqrt{(P_\perp^0-P_\parallel^0+\alpha_P^2 B_\theta^2/4\pi)/\sigma};$$

it has a non-vanishing imaginary part except on the real axis and on the curves $L-1$ and $L-2$ passing through $\pm i\Omega_0$; and on these latter lines $\Gamma^2>0$.

Considering next $P_m(\Gamma y)$ (as defined in equation (119)) for complex Γ, we infer from the non-existence of complex zeros of $AJ_m(z)+BzJ'_m(z)$ for all real A and B (see Watson 1952, p. 482), that the imaginary part of $P_m(\Gamma y)$ is non-vanishing and that it has no poles except when Γ^2 is real and negative. Therefore $\Phi(\Omega)$ has no poles on the right half of the complex Ω-plane and is in fact analytic. Consequently, we may determine the number of zeros of $\Phi(\Omega)$ on this half of the complex plane by the principle of the argument, namely by evaluating the integral,

$$C = \frac{1}{2\pi \mathrm{i}}\int \frac{1}{\Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}\Omega}\mathrm{d}\Omega,$$

along a contour consisting of a semi-circular arc of sufficiently large radius and the intercepted part of the imaginary axis; C is the total change in the argument of Φ around the contour. From the fact that $\Phi(\Omega)$ becomes proportional to Ω as $\Omega \to \infty$, it is clear that the contribution to C from the semi-circular arc is $\frac{1}{2}$. On the imaginary axis, the imaginary part of Φ is positive during the whole contour and the change in the argument is $\pm\frac{1}{2}\pi$ depending on whether $\Phi(0)$ is positive or negative. Hence $C = 1$ if $\Phi(0) < 0$ and $C = 0$ if $\Phi(0) > 0$. The conclusion then is that if $\Gamma^2(0) > 0$ and $\Phi(0) > 0$, the pinch is stable; otherwise it is unstable. Moreover, in the latter case, since on the real axis $\Phi(\Omega)$ increases from a negative number to infinity, the instability evidently occurs with a real frequency. In the case of a stable pinch no purely imaginary roots are possible, i.e. the oscillations are Landau-damped.

(*b*) *The induced electric field parallel to* **B**

By combining equations (19), (60) and (69) we clearly obtain an expression for $E'_\parallel$. This latter expression can be derived more directly as follows: If the displacement current is ignored

$$\operatorname{curl}\mathbf{B} = 4\pi\mathbf{j}/c. \tag{i}$$

Taking the divergence of this equation, we clearly obtain

$$\text{charge density} = \sum_{+,-} eN' = 0. \tag{ii}$$

On the other hand according to equation (7)

$$N' = \frac{1}{2}\iint A_1\,\mathrm{d}q\,\mathrm{d}s^2; \tag{iii}$$

and in limit $\Omega^2 = 0$ we have (cf. equations (19), (22) and (29))

$$A_1 = \frac{B'_\parallel}{B^0}s^2\left(\frac{\partial f^0}{\partial q^2}-\frac{\partial f^0}{\partial s^2}\right)-\frac{2e}{imk}\frac{\partial f^0}{\partial q^2}E'_\parallel. \tag{iv}$$

We thus and

$$2E'_\parallel\left\{\sum_{+,-}\frac{e^2}{m}\iint \mathrm{d}q\,\mathrm{d}s^2\frac{\partial f^0}{\partial q^2}\right\} = \mathrm{i}k\frac{B'_\parallel}{B^0}\sum_{+,-}\iint \mathrm{d}q\,\mathrm{d}s^2\left(\frac{\partial f^0}{\partial q^2}-\frac{\partial f^0}{\partial s^2}\right)s^2, \tag{v}$$

in agreement with the result given in the paper.

(*c*) *The case* $\eta = 1$

As we have stated, Rosenbluth (1958) has carried out the discussion of the stability for the case $\eta = 1$. Since his paper may not be readily available we reproduce, with his permission, his results in figure 3.

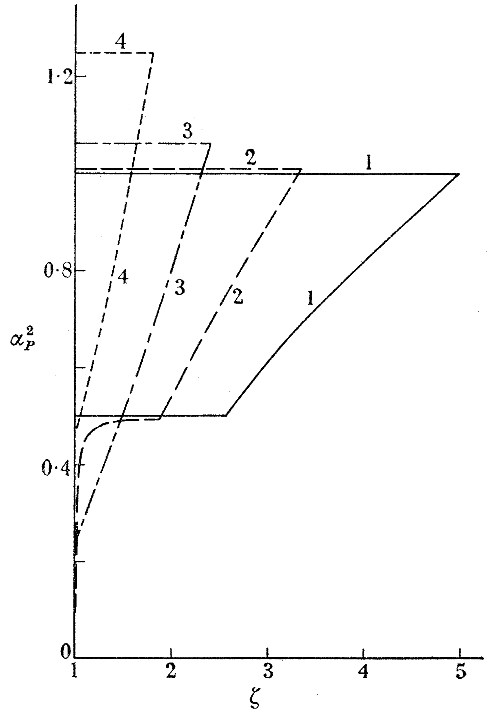

FIGURE 3. The regions of stability in the (α_P^2, ζ)-plane for the case $\eta = 1$ and $\alpha_V = 0$, 0·1, 0·25 and 0·5 (distinguished by the numbers 1, 2, 3 and 4, respectively). For a given α_V, a stable pinch occurs in a region bounded on the left by the α_P^2-axis, on the right by the marginal curve for $m = -1$, above by the equilibrium condition $\alpha_P^2 < (1 + \alpha_V^2)$ and below by the marginal curve for $m = 0$.

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APPENDIX. THE PROPAGATION OF PLANE HYDROMAGNETIC WAVES IN AN INFINITE MEDIUM

In deriving the equation (§ 3, equation (75))

$$\nabla_\perp^2 \chi = k^2 \Gamma^2 \chi, \tag{A 1}$$

no assumptions (in addition to those underlying the general perturbation theory) were made except that there is a uniform field, $\mathbf{B}^0$, in the z-direction and that all quantities describing the perturbation have the dependence

$$e^{\Omega t+ikz} \tag{A 2}$$

on t and z. Accordingly, if the plasma should be of infinite extent, we may seek solutions of (A 1) whose dependence on x and y is also periodic with a (total) wave number $k_\perp$. For such plane waves (A 1) provides the dispersion relation

$$k_\perp^2 + k_\parallel^2 \Gamma^2 = 0, \tag{A 3}$$

where we have written $k_\parallel$ in place of k to emphasize that this represents the wave number of the disturbance in a direction parallel to $\mathbf{B}^0$. Substituting for Γ^2 from equation (88) in (A 3), we obtain

$$k_\perp^2 \left\{ \frac{|B^0|^2}{4\pi} + (\bar{S}+\bar{R})\, P_\perp^0 \right\} + k_\parallel^2 \left\{ P_\perp^0 - P_\parallel^0 + \frac{|B^0|^2}{4\pi} - \rho \frac{\omega^2}{k_\parallel^2} \right\} = 0, \tag{A 4}$$

where we have further written $i\omega$ in place of Ω; ω denotes, therefore, the frequency of the wave.

Letting (cf. equations (126) and (128))

$$\beta = \frac{4\pi}{|B^0|^2} P_\perp^0, \quad \eta = \frac{P_\perp^0}{P_\parallel^0} \tag{A 5}$$

and
$$k_\perp^2 = k^2 \cos^2 \vartheta, \quad k_\parallel^2 = k^2 \sin^2 \vartheta \quad (k^2 = k_\parallel^2 + k_\perp^2), \tag{A 6}$$

we can rewrite (A 4) in the form

$$\beta(\bar{S}+\bar{R}) \cos^2 \vartheta + \beta\left(1 - \frac{1}{\eta}\right) \sin^2 \vartheta + 1 = \frac{4\pi\rho\omega^2}{|B^0|^2 k^2}. \tag{A 7}$$

If the distributions of q and $\mathbf{s}$ are Gaussian (but with different dispersions) then $\bar{S}$ and $\bar{R}$ in (A 7) have the values given in § 7, equations (141) to (144).

From (A 7) it follows that we shall have *instability* if

$$\beta(\bar{S}+\bar{R}) \cos^2 \vartheta + \beta(1 - 1/\eta) \sin^2 \vartheta + 1 < 0, \tag{A 8}$$

where $\bar{S}$ and $\bar{R}$ are now to be evaluated for the limit $\omega = 0$.

If the particular assumptions of § 6 leading to equation (127) are made, then (A 8) becomes

$$2\beta(1-\eta) \cos^2 \vartheta + \beta(1 - 1/\eta) \sin^2 \vartheta + 1 < 0. \tag{A 9}$$

A case of special interest is when the waves are propagated in the direction of $\mathbf{B}^0$. Then $k_\perp = 0$ and the dispersion relation is (cf. (A 4))

$$P_\perp^0 - P_\parallel^0 + \frac{|B^0|^2}{4\pi} = \frac{\rho\omega^2}{k^2} \quad (k_\perp = 0;\ k = k_\parallel). \tag{A 10}$$

When $P_\perp^0 = P_\parallel^0$ this represents the usual Alfvén wave. The waves described by (A 8) represent unstable modes if

$$P_\perp^0 - P_\parallel^0 + \frac{|B^0|^2}{4\pi} < 0. \tag{A 11}$$

This special case of (A 9) has been stated by Longmire (1956, private communication); and it has been discussed more recently by Parker (1957, private communication). Also, it may be noted that (A 11) is equivalent to one of the conditions for stability considered in § 6 (cf. (134)).