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The Compton Scattering and the New Statistics.

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1. *Introduction.*

Great success has been achieved by Sommerfeld in the electron theory of metals by assuming that there are free electrons in them which obey the Fermi-Dirac statistics. It has been assumed in the case of univalent metals that on the average one electron per atom is free. In general, however, the valency electrons can be considered as free.* These free electrons will take part in the Compton scattering. The analysis of such a Compton effect reduces to the analysis of the collisions between radiation quanta and an electron gas. The general features of such a scattering was first considered by Dirac.† But he has assumed a Maxwellian distribution for the electrons which will not be applicable to the case under consideration, because the electrons in a conductor being degenerate do not obey the Maxwell's law, but the Fermian distribution.

In considering such a process we take it that the conservation of momentum and energy principles are satisfied for each particular collision just as in Compton's theory—only we are here dealing with moving electrons instead of stationary electrons which Compton considers. Thus electrons of different momenta components will produce different Compton shifts, and the intensity of any particular shift will depend on the number of electrons in that state. Thus we have to average for the radiation falling on an assembly of electrons whose momenta are distributed according to the Fermi-Dirac law.

The above is just a natural extension of Compton's theory. In this connection mention should be made of Jauncey's‡ theory of bound electrons whose arguments are essentially what we have put forward in the previous paragraph. But his theory has not been quite satisfactory because he has not assumed any definite distribution of the electrons.

* Rosenfeld, 'Naturw.,' p. 49 (1929).
† 'M.N.R.A.S.,' vol. 85, p. 825 (1925).
‡ 'Phys. Rev.,' vol. 25, p. 723 (1925).

2. *Compton scattering with Moving Electrons.*

Let m_x, m_y, m_z be the momentum of the scattering electron and g_x, g_y, g_z those of the quantum, g_t, m_t represent the masses of the electron and the quantum multiplied by the velocity of light c. If we take polar co-ordinates

$$g_x = h\nu \cos \theta/c \,;\; g_y = h\nu \sin \theta \cos \phi/c \,;\; g_z = \sin \theta \sin \phi \;\; g_t = h\nu/c. \tag{1}$$

Then the conservation of momentum and energy gives

$$(m_u, g_u) - (m_u, g_u') = (g_u, g_u'). \tag{2}$$

The above equation gives the frequency of the scattered quantum in terms of the initial momentum of the electron and the incident quantum, and the directions of the incident and scattered quanta.

Equation (2) reduces to

$$m_t - m_x \cos \theta' - m_y \sin \theta' = \frac{\nu}{\nu'}(m_t - m_x) - \frac{h\nu}{c}(1 - \cos \theta'), \tag{3}$$

if we assume that the directions of the incident quantum is along the x axis and that of the scattered quantum in the xy plane. Here θ' is simply the angle of scattering.

3. *The Spectral-intensity Distribution Function.*

Before considering the case of scattering of monochromatic X-radiation, we will consider first the more general case when the incident radiation is continuous. Suppose we have such a pencil of radiation confined to a small solid angle $d\omega$ and let I_ν be the intensity per unit frequency range. Let this radiation be incident on an assembly of dn electrons of momentum m_x, m_y, m_z. Let the intensity of radiation scattered in the solid angle $d\omega'$ and frequency range ν' and $\nu' + d\nu'$ be given by

$$\mathrm{R}(\nu')\, d\nu'\, d\omega'. \tag{4}$$

Then it has been shown by Dirac (*loc. cit.*, equation (8)) that

$$\mathrm{R}(\nu') = \frac{h^2}{m^2c^3} \cdot dn \,.\, \mathrm{I}_\nu\, d\omega \frac{\nu' \mathrm{F}(a, b)}{\nu m_t}. \tag{5}$$

Here ν' is to be regarded as a function of g_x', g_y', g_z' and m_x, m_y, m_z, being that frequency of the incident quantum which will be scattered by an (m_x, m_y, m_z) electron into the frequency range ν' to $\nu' + d\nu'$.

In the above equation $\mathrm{F}(a, b)$ is a function which depends on the scattering law adopted and a and b the two invariants connected with the scattering process which as well as the initial momentum m_x, m_y, m_z of the electron and g_x, g_y, g_z of the quantum specify the collision.

Now for dn in equation (5), we have to put the Fermi-expression

$$dn = \frac{V}{h^3} \cdot G \cdot \frac{dm_x \, dm_y \, dm_z}{\exp\left(\Sigma m_x^2/2mkT\right)/A + 1}, \tag{6}$$

and integrate with respect to m_x, m_y, m_z. In the above equation A is the constant appearing in the Fermi-Dirac statistics. It has different values according as we consider a degenerate or a non-degenerate gas. When the system is non-degenerate A is a small positive quantity and then has the value

$$A = nh^3 \,.\, (2\pi mkT)^{-3/2}/G. \tag{7}$$

A degenerate system corresponds to A being a large quantity and in that case

$$\log A = \left(\frac{3n}{4\pi G}\right)^{2/3} \cdot \frac{h^2}{2mkT}. \tag{8}$$

Then by equation (6)

$$R(\nu') = \frac{h^2}{m^2c^3} \cdot \frac{V}{h^3} \cdot G \iiint_{-\infty}^{\infty} \frac{I_\nu d\omega \nu' F}{\nu m_t} \cdot \frac{dm_x \, dm_y \, dm_z}{\exp\left(\Sigma m_x^2/2mkT\right)/A + 1} \tag{9}$$

$$= \frac{h^2}{m^2c^3} \cdot \frac{V}{h^3} \cdot G \int_0^{\infty} I_\nu \, d\omega \, \psi(\nu, \nu') \, d\nu. \tag{10}$$

Where

$$\psi(\nu, \nu') = \iint_{-\infty}^{\infty} \frac{\nu' F}{\nu m_t} \cdot \frac{dm_y \, dm_z}{\exp\left(\Sigma m_x^2/2mkT\right)/A + 1} \Big/ \frac{\partial \nu}{\partial m_x}. \tag{11}$$

Where m_x and $\partial \nu/\partial m_x$ are to be evaluated in terms of m_y, m_z and ν by means of equation (3)

$$(6) \quad \frac{\partial m_x}{\partial \nu} = \frac{mc}{\nu'(1 - \cos\theta')} - \frac{h}{c}, \tag{12}$$

and

$$m_x^2 + m_y^2 = \frac{\beta}{\gamma^2}\left[m_y - \frac{K \sin\theta'}{\beta}\right]^2 + \frac{K^2}{\gamma^2}\left(1 - \frac{\sin^2\theta'}{\beta}\right), \tag{13}$$

where

$$\beta = 1 - 2\nu\cos\theta'/\nu' + (\nu/\nu')^2, \quad \gamma = \nu/\nu' - \cos\theta',$$

$$K = -mc(\nu/\nu' - 1) + h\nu(1 - \cos\theta')/c. \tag{14}$$

Then

$$\psi(\nu, \nu') = \iint_{-\infty}^{\infty} \frac{F\left[\dfrac{1}{\nu(1 - \cos\theta')} - \dfrac{h\nu'}{mc^2\nu}\right] dm_y \, dm_z}{\dfrac{1}{A}\exp\left\{\dfrac{\dfrac{\beta}{\gamma^2}\left[m_y - \dfrac{K\sin\theta'}{\beta}\right]^2 + \dfrac{K^2}{\gamma^2}\cdot\left[1 - \dfrac{\sin^2\theta'}{\beta}\right] + m_z^2}{2mkT}\right\} + 1}. \tag{15}$$

Suppose now that the radiation is monochromatic, then the number of quanta scattered between the frequency range ν' and $\nu' + d\nu'$ into the solid angle $d\omega'$ is given by

$$\mathrm{R}(\nu') = \frac{h^2}{m^2c^3} \cdot \frac{\mathrm{V}}{h^3} \cdot \mathrm{G} \cdot \mathrm{I}_\nu \, d\nu \, dw \, \psi(\nu, \nu'). \tag{16}$$

So that the (spectral) distribution of intensity about the primary frequency ν is given by $\psi(\nu, \nu')$, which we shall now evaluate.

$$\psi(\nu, \nu') = 4\mathrm{F}_0 \left[\frac{1}{\nu(1 - \cos\theta')} - \frac{h\nu'}{mc^2\nu}\right] \iint_0^\infty \frac{dm_y \, dm_z}{\frac{1}{\mathrm{B}} \exp\left(\frac{\beta}{\gamma^2} \cdot \frac{m'^2_y + m_z^2}{2mk\mathrm{T}}\right) + 1}, \tag{17}$$

where

$$\left.\begin{aligned} m'_y &= m_y - \frac{\mathrm{K}\sin\theta'}{\beta} \\ \frac{1}{\mathrm{B}} &= \frac{1}{\mathrm{A}} \cdot \exp\left\{\frac{\mathrm{K}^2}{\gamma^2 \cdot 2mk\mathrm{T}} \cdot \left(1 - \frac{\sin^2\theta'}{\beta}\right)\right\} \end{aligned}\right\}. \tag{18}$$

If we introduce the new variables

$$y = m'^2_y \, \beta/\gamma^2 \cdot 2mk\mathrm{T}, \quad z = m_z^2/2mk\mathrm{T}.$$

Then

$$\begin{aligned} \psi(\nu, \nu') &= 2\mathrm{F}_0 \left[\frac{1}{\nu(1 - \cos\theta')} - \frac{h\nu'}{mc^2\nu}\right] \frac{\gamma \cdot mk\mathrm{T}}{\beta^{\frac{1}{2}}} \cdot \iint_0^\infty \frac{y^{-\frac{1}{2}} \cdot z^{-\frac{1}{2}} \cdot dy \, dz}{e^{y+z}/\mathrm{B} + 1} \\ &= 2\mathrm{F}_0 \left[\frac{1}{\nu(1 - \cos\theta')} - \frac{h\nu'}{mc^2\nu}\right] \cdot \frac{\gamma \cdot mk\mathrm{T}}{\beta^{\frac{1}{2}}} \cdot \mathrm{U}_0, \end{aligned} \tag{19}$$

where U_0 is the special case of the general Sommerfeld integral

$$\mathrm{U}_\rho = \frac{1}{\Gamma(\rho + 1)} \cdot \int_0^\infty \frac{u^\rho \, du}{e^u/\mathrm{B} + 1}, \tag{20}$$

which gives for $\rho = 0$*

$$\mathrm{U}_0 = \pi \log(\mathrm{B} + 1). \tag{21}$$

Hence we get our intensity distribution function

$$\psi(\nu, \nu') = 2\mathrm{F}_0 \left[\frac{1}{\nu(1 - \cos\theta')} - \frac{h\nu'}{mc^2\nu}\right] \cdot \frac{\gamma \cdot mk\mathrm{T}\pi}{\beta^{\frac{1}{2}}} \cdot \log(\mathrm{B} + 1). \tag{22}$$

Case I.—If B is large and positive we get

$$\begin{aligned} \psi(\nu, \nu') = 2\mathrm{F}_0 &\left[\frac{1}{\nu(1 - \cos\theta')} \cdot \frac{h\nu'}{mc^2\nu}\right] \cdot \frac{\gamma \cdot mk\mathrm{T}\pi}{\beta^{\frac{1}{2}}} \\ &\times \left[\log \mathrm{A} - \frac{\mathrm{K}^2}{\gamma^2 \cdot 2mk\mathrm{T}}\left(1 - \frac{\sin^2\theta'}{\beta}\right)\right], \end{aligned} \tag{23}$$

the value of log A being given by (8).

* Sommerfeld, 'Z. Physik,' vol. 47, p. 1 (1928), equation (31A).

An approximation of the above equation to an order of accuracy where the Compton-shift is neglected is

$$\psi(\nu, \nu') = \frac{\sqrt{2}\,\mathrm{F}_0\pi mk\mathrm{T}}{\nu(1-\cos\theta')^{\frac{1}{2}}} \cdot \left[\log \mathrm{A} - \frac{(\nu'-\nu)^2}{a\nu^2}\right]. \tag{24}$$

where $a = 4k\mathrm{T}(1-\cos\theta')/mc^2$.

Case II.—If B is small due to the smallness of A we get, to the same order of accuracy as (24), the equation

$$\psi(\nu, \nu') = \frac{n}{h^3 \,.\, \mathrm{G}} \cdot (2\pi mk\mathrm{T})^{-3/2} \cdot \frac{\sqrt{2}\mathrm{F}_0\pi mk\mathrm{T}}{\nu(1-\cos\theta')^{\frac{1}{2}}} \cdot e^{-(\nu'-\nu)^2/a\nu^2}, \tag{25}$$

the one given by Dirac (*loc. cit.*, equation (13)).

Equation (25) gives an exponential distribution of intensity about the primary frequency for the scattered radiation. But equations (23) and (24) indicate that the distribution of intensity of the radiation scattered by a degenerate electron gas does not follow an exponential law but gives a parabolic distribution. This perhaps explains the rather broad structure of the Compton modified radiation.*

4. *The Compton effect.*

It is natural that the distribution of intensity predicted by equation (23) places the maximum peak of intensity at a place where the Compton's theory for a free-stationary electron predicts a line. Remembering that in any case $\nu/\nu' = 1$ the maximum frequency will be at a modified frequency where $\mathrm{K} = 0$ where

$$\mathrm{K} = -mc(\nu/\nu' - 1) + h\nu(1-\cos\theta')/c = 0,$$

i.e., where

$$\lambda' - \lambda = h(1-\cos\theta')/mc, \tag{26}$$

i.e., on an intensity-frequency graph the maximum occurs at a place corresponding to the Compton shift.

5. *The Effect of Temperature.*

If we consider the Compton scattering by an electron-gas, the distribution function of which depends on temperature, it would naturally be expected

* Mr. J. W. Du Mond in a private communication to the author from the California Institute of Technology, Pasadena, has kindly pointed out that the above is the characteristic of the Compton-radiation from conductors. His paper in the May issue of the 'Physical Review' (vol. 33, p. 643) gives experimental details and theoretical calculations as well. He has independently derived the parabolic structure.

that the spectral intensity distribution function in the Compton scattering would also depend on temperature and Dirac's classical expression (25) does indicate this by the explicit appearance of the temperature factor in $\psi(\nu, \nu')$. But if we substitute the value of log A given by (8) in (23) we get

$$\psi(\nu, \nu') = 2F_0\left[\frac{1}{\nu(1-\cos\theta')} - \frac{h\nu'}{mc^2\nu}\right] \cdot \frac{\gamma . \pi}{\beta^{\frac{1}{2}}} \times \left[\tfrac{1}{8}\left(\frac{6}{\pi}n\right)^{2/3} h^2 - \frac{K^2}{\gamma^2} \cdot \left(1 - \frac{\sin^2\theta'}{\beta}\right)\right], \tag{27}$$

where all the temperature factors have cancelled out. Thus Compton-scattering by an *electron-gas* will not be influenced by temperature. Further the Compton scattering by the bound electrons will also not be influenced by the ranges of temperature available in the laboratory. Thus it appears that the *total* Compton scattering will not be affected by temperature.*

6. *The Effect of a Magnetic-field.*

We will consider the scattering by the conduction electrons only. When the scatterer is placed in a magnetic-field the distribution function for the electrons changes, and in that case the number of electrons in the momentum range m_x, m_y, m_z and $m_x + dm_x$, $m_y + dm_y$, $m_z + dm_z$ is given by the Pauli's expression†

$$dn = \frac{V}{h^3} \cdot \frac{dm_x \,.\, dm_y \,.\, dm_z}{\exp\left(\dfrac{\varepsilon_m}{kT} + \dfrac{\Sigma m_x^2}{2m_0kT}\right)\Big/A + 1}, \tag{28}$$

where $\varepsilon_m = mg\,\mu_0 H$; where $\mu_0 = -eh/4\pi m_0 c$ = a Bohr magneton, g = the Lande factor, H = the field strength.

For the summation over all the values of the quantum number m from $-j$ to $+j$ we have the relations

$$\left.\begin{aligned} \sum_{m=-j}^{+j} \varepsilon_m &= 0 \\ \sum_{m=-j}^{+j} \varepsilon_m^2 &= \tfrac{1}{3}G\mu^2H^2 \end{aligned}\right\}. \tag{29}$$

To derive the spectral-intensity distribution function we have to substitute

* Since the writing of the above a report by Jauncey and Bowers has appeared ('Bull. Amer. Phys. Soc.,' vol. 4, p. 26 (1929)) giving experimental observations which support the above conclusions.

† 'Z. Physik,' vol. 41, p. 81 (1927).

(28) instead of (6) in equation (5) and carry out the integration as before. The final result as one can easily see is

$$\psi(\nu, \nu') = 2F_0 \left[\frac{1}{\nu(1-\cos\theta')} - \frac{h\nu'}{mc^2\nu}\right] . \frac{\gamma \, . \, mkT \, , \pi}{\beta^{\frac{1}{2}}}$$
$$\times \left[\log A + \sum_{m=-j}^{+j} \frac{-\varepsilon_m}{kT} - \frac{K^2}{2\gamma^2 \, . \, mkT}\left(1 - \frac{\sin^2\theta'}{\beta}\right)\right], \quad (30)$$

which on account of (29) becomes identical with (23). Thus it appears on Pauli's theory of the paramagnetism of an electron-gas that the scattering of such an assembly should not be influenced by the presence of a magnetic field. In this connection mention should be made of an experimental observation of Bothe* where he tried the influence of a magnetic field. The scatterer he used was paraffin, and he tried up the field strengths of the order of 20,000 Γ. But he could detect no influence.

Summary.

In this paper the Compton scattering by an electron-gas on the Fermi-Dirac statistics is considered. The theory predicts a distribution of spectral intensity not exponentially falling off about the maximum but *parabolically*. It places the peak of maximum intensity at a place where the Compton relation $\lambda' - \lambda = h(1 - \cos\theta')/mc$ is satisfied. Further, the theory indicates that there should be no influence of temperature or magnetic field in Compton scattering.

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* 'Z. Physik,' vol. 41, p. 872 (1927).