

Brane Constructions, Conifolds and M-Theory

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ABSTRACT

We show that a set of parallel 3-brane probes near a conifold singularity can be mapped onto a configuration of intersecting branes in type IIA string theory. The field theory on the probes can be explicitly derived from this formulation. The intersecting-brane metric for our model is obtained using various dualities and related directly to the conifold metric. The M-theory limit of this model is derived and turns out to be remarkably simple. The global symmetries and counting of moduli are interpreted in the M-theory picture.

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1. Introduction

Recently, a system of parallel 3-branes in the presence of a conifold singularity has received some attention[1]. The $\mathcal{N} = 1$ supersymmetric gauge theory on the 3-branes has been deduced indirectly from properties of the conifold. For a large number N of 3-branes, the system is conjectured to be dual to a certain IIB string compactification, extending the AdS-CFT correspondence[2] in an interesting way. One of the notable features of this system is that the compact space which occurs on the string theory side of the duality is not even locally S^5 .

This system has been investigated further in Refs.[3,4], where baryon-like chiral operators built out of products of N chiral superfields were identified with D3 branes wrapped over the three cycle of the compact space $T^{1,1}$. Also a D5 brane wrapped over a two cycle of $T^{1,1}$ was identified with a domain wall in AdS_5 . Upon crossing it, the gauge group is argued to change from $SU(N) \times SU(N)$ to $SU(N) \times SU(N + 1)$.

Our goal in what follows will be to derive the conformal field theory on 3-branes at the conifold singularity using a version of the brane construction pioneered by Hanany and Witten[5]. This construction will enable us to explicitly read off the spectrum and other properties of the conformal field theory on 3-branes at a conifold.

It has been argued some time ago that the conifold singularity (in the absence of a transverse 3-brane) is dual to a system of perpendicular NS 5-branes intersecting over a $3 + 1$ dimensional world volume[6]. In this formulation, a 3-brane wrapped over a 3-cycle that shrinks as one approaches the conifold limit from the “deformation” side is replaced by an open D-string connecting the two NS branes. By an S-duality, this can be replaced by a fundamental open string connecting perpendicular D 5-branes, and one can now see in perturbation theory the famous massless hypermultiplet whose presence “cures” the conifold singularity[7]. However this picture, though it provided initial inspiration for the present work, will not be directly useful for us.

Starting from a slightly different viewpoint, we will argue that the conifold singularity is represented by a configuration of two type IIA NS 5-branes that are rotated with respect to one another, and located on a circle, with D 4-branes stretched between them from both sides. The result is an elliptic version of a model studied in Refs.[8,9,10], where it was used to analyse pure $\mathcal{N} = 1$ supersymmetric QCD. In particular, this clarifies the relationship between the AdS duals to branes at quotient singularities and branes at conifolds, explaining why the latter is a relevant deformation of the former.

After deriving our model, we will use the supergravity solution for intersecting branes to give a heuristic but more explicit map from brane configurations to conifolds. This will allow us to make many identifications more precise, and also to argue that the separation of the NS and NS' 5-branes along one of the common transverse directions can be interpreted as turning on a constant flux of the NS-NS B -field on the type IIB side.

Finally, we investigate the M-theory limit of our model[11]. This has been useful in the past in studying the solution of two kinds of nontrivial models, those with nonzero beta function and those which are conformally invariant. Our model falls in the latter class, but unlike its $\mathcal{N} = 2$ supersymmetric counterpart, it has a surprisingly simple lift to M-theory as we will show. Some of the continuous and discrete global symmetries of the model, and the counting of moduli, come out naturally in the M-theory picture. In particular the RR B -field will make its appearance symmetrically with the NS-NS B -field, describing separation of branes along the x^{10} direction.

This construction allows several interesting generalizations, which will not be analysed here. A class of conifold-like singularities parametrized by two integers (n, n') was investigated, for example, in Ref.[6]. In this case we expect to find several rotated NS 5 branes arranged around a circle, a model recently investigated by Uranga[12]. More general conical and other singularities have been addressed from the AdS viewpoint in Ref.[13] and more recently in Ref.[14].

2. Conifolds and Intersecting Branes

Let us review the basic idea in Ref.[6] to map a conifold or generalization thereof to a set of intersecting NS and NS' 5-branes. The equation of a conifold,

$$(z_1)^2 + (z_2)^2 + (z_3)^2 + (z_4)^2 = 0 \tag{2.1}$$

can be rewritten

$$\begin{aligned} (z_1)^2 + (z_2)^2 &= \zeta \\ (z_3)^2 + (z_4)^2 &= -\zeta \end{aligned} \tag{2.2}$$

which describes two degenerating tori varying over a P^1 base. By performing two T-dualities, one over a cycle of each torus, one ends up with a pair of NS 5-branes which are locally along the directions x^1, x^2, x^3, x^4, x^5 and x^1, x^2, x^3, x^8, x^9 respectively, where x^4 and x^8 are the directions along which T-duality was performed. The D3-brane that shrinks

at the conifold singularity becomes a D-string stretching between these NS 5-branes in this language.

If we place N D3-branes transverse to the conifold singularity (i.e., along the (x^1, x^2, x^3) directions), then after these dualities they turn into D5-branes covering the 2-torus along the 4 – 6 directions. The result is identical to a “brane box” [15,16], but unfortunately this does not seem to be a useful description of the model in which we are interested.

According to Ref.[16], such a model should have gauge group $U(N)$ and $\mathcal{N} = 4$ supersymmetry, unless the brane box is “twisted”. In the latter case it would have the desired gauge group $U(N) \times U(N)$ but still $\mathcal{N} = 2$ supersymmetry rather than $\mathcal{N} = 1$ which we expect. We will return to this point in a subsequent section. Presumably this model is not actually incorrect, but rather the standard techniques to analyse brane configurations are less useful in this description. The model that we construct in the next section will turn out to be easy to analyse and to describe all qualitative features of branes at conifolds rather well.

3. Branes at Conifolds and Fibred Brane Configurations

Let us write Eq.(2.1) above as

$$(z_1)^2 + (z_2)^2 + (z_3)^2 = -(z_4)^2 \tag{3.1}$$

In this form it describes the Z_2 ALE space R^4/Z_2 blown up by a P^1 of size $|z_4|$. Thus it can be thought of as a fibration where the base is the z_4 plane and the fibre is an ALE (Eguchi-Hanson) space of linearly varying scale size.

Let us choose conventions in which $z_4 = x^4 + ix^5$ and the directions x^6, x^7, x^8, x^9 describe the ALE space embedded in z_1, z_2, z_3 . The ALE space is centred at $(x^6, x^7, x^8, x^9) = (0, 0, 0, 0)$. Moreover, at $(x^4, x^5) = (0, 0)$ the scale size shrinks to zero and there is a singularity, the node of the conifold.

Suppose we are very close to this singularity. Then the ALE space can be replaced with a positive-mass two-centre Taub-NUT space. The scale size of the ALE space is traded for the distance separating the two centres in the Taub-NUT space, hence this distance also varies linearly as a function of x^4, x^5 .

This means that we have a pair of 5-brane Kaluza-Klein monopoles filling the x^1, x^2, x^3 directions and separating from each other linearly along x^6, x^7, x^8, x^9 as a function of

x^4, x^5 . This function must be holomorphic in suitable complex coordinates in order to produce a supersymmetric model. Hence we can choose the two KK 5-branes to lie at $(x^6, x^7, x^8, x^9) = \alpha(0, 0, x^4, x^5)$. As a result, they intersect over a 3-brane in the x^1, x^2, x^3 directions, at the point $x^4 = x^5 = 0$. Thus we will replace the conifold by this configuration of intersecting KK monopoles.

A T-duality along the x^6 direction converts these KK monopoles to a pair of NS 5-branes aligned in the same way. More generally, we can separate the two NS5-branes along the x^6 and x^7 directions. The x^6 coordinates of these branes are actually determined by the background B -field if there is one. Turning on such a B -field causes the x^6 separation to be proportional to the integral of the B -field over the (vanishing) 2-cycle of the original ALE space. Hence the branes are also separated in the x^6 direction if the B field is nonzero. This will be confirmed in a subsequent section.

Now, if N D3-branes are placed transverse to the conifold then upon performing the T-duality described above, they turn into D4-branes stretched along the compact x^6 direction. These 4-branes are “broken” twice, once on each NS 5-brane, so there are really two independent segments for each 4-brane on the x^6 circle.

Let us now look at the limit in which the number N of D3-branes (which are now D4-branes) becomes large. For parallel NS5-branes, the spacetime becomes $AdS_5 \times S^5/Z_2$ with a total of 16 supersymmetries. For our rotated branes, the transverse space is similar except that the singular circle on S^5 is blown up by a P^1 . This P^1 is in fact present all over the 4, 5 plane, but its size is varying with distance away from the centre. However, since the fixed circle of S^5/Z_2 arises when

$$(x^6, x^7, x^8, x^9) \rightarrow -(x^6, x^7, x^8, x^9) \quad (3.2)$$

it lies at the origin of 6, 7, 8, 9 and along a circle in the 4, 5 plane. Along this circle, the P^1 has a constant size.

It has been noted that precisely this blowup of S^5/Z_2 gives rise to the smooth Einstein manifold $T_{1,1}$. This blowup has some unusual properties relative to conventional blowups of ALE spaces: (i) it is a relevant and not a marginal deformation in the brane field theory, (ii) it breaks 16 supersymmetries down to 8. At this stage we can see roughly how these arise. The P^1 actually varies in size over the full spacetime, hence one may expect that it is not just a marginal perturbation, even though it is a constant-size blowup of the fixed locus in S^5/Z_2 . The branes are not parallel, but rotated in a definite way. This makes the

adjoint fields massive, inducing precisely the correct relevant perturbation to break $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$. We will see that the induced mass terms in the superpotential are antisymmetric under exchange of gauge groups, as expected from Ref.[1].

4. Analysis of the Model

In order to draw a figure of the model that we will be discussing, it is convenient to suppress certain directions. The x^1, x^2, x^3 directions are always suppressed as they correspond to the noncompact dimensions in which the field theory lives. It is convenient also to think of the coordinates $x^4 + ix^5$ and $x^8 + ix^9$ as representing one complex dimension each. We also suppress the x^7 direction. The configuration relevant to the conifold is then as in Fig. 1.

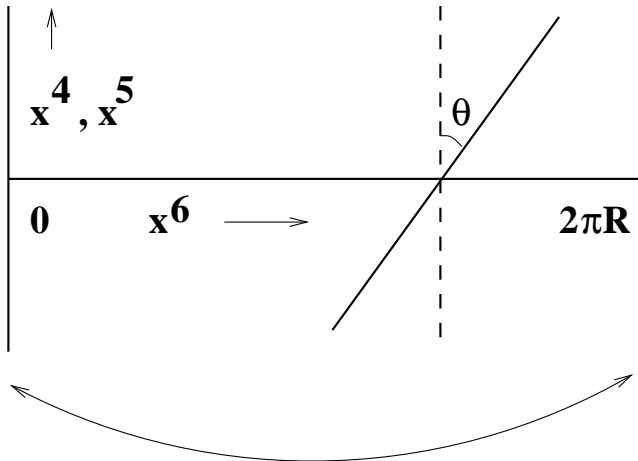


Fig. 1: Brane configuration for the conifold theory

The gauge group of this configuration is straightforward to read off. To start off, we had N D3-branes transverse to the conifold. By T-duality these have turned into N D4-branes, and moreover they stretch from the first NS5-brane to the second and back around the x^6 circle to the first. Thus they have “broken” into two segments, and from standard arguments we expect one $U(N)$ gauge group from each segment. This naive $U(N) \times U(N)$ gauge group contains two $U(1)$ factors, one of which is the pure centre of mass motion and the other decouples by standard reasoning as in Ref.[11]. Thus the gauge group is $SU(N) \times SU(N) \times U(1)$, exactly as desired.

Next let us look for the matter multiplets. Open strings stretching across the two points where the D4-brane is split by the NS5-branes correspond as usual to bi-fundamentals. In the language of $\mathcal{N} = 2$ supersymmetry we get two bi-fundamental

hypermultiplets, which decompose under $\mathcal{N} = 1$ supersymmetry as two chiral multiplets A_1, A_2 in the (N, \bar{N}) of $SU(N) \times SU(N)$ and two more chiral multiplets B_1, B_2 in the (\bar{N}, N) .

This is precisely the postulated field content of the model of N D3-branes at a conifold. However, so far we have just reproduced fields which arose already in the $\mathcal{N} = 2$ model. One difference now becomes apparent: there are no moduli for the centre of mass of the D4-branes to move in the 4, 5 or 8, 9 directions. This is well-known to imply that the adjoint has acquired a mass. Moreover, there is no way to separate the N D4-branes from each other and go to the Coulomb branch of $SU(N) \times SU(N)$. But this is exactly what we expect from the analysis of Ref.[1].

Moreover, because of their immobility, the D4-branes on opposite sides of each NS5-brane cannot split off from each other. This means that the bi-fundamental matter multiplets cannot acquire a mass, so they must always be massless.

The final aspect of the theory that we need to reproduce is the superpotential. Before the twisting, the $\mathcal{N} = 2$ theory had, in $\mathcal{N} = 1$ language, two pairs of chiral bi-fundamentals A_i, B_i and two adjoints $\Phi, \tilde{\Phi}$, one for each factor of the gauge group. They were coupled by a standard cubic superpotential as dictated by $\mathcal{N} = 2$ supersymmetry:

$$W(A_i, B_i, \Phi, \tilde{\Phi}) = g_1 \text{tr} \Phi (A_1 B_1 + A_2 B_2) + g_2 \text{tr} \tilde{\Phi} (B_1 A_1 + B_2 A_2) \quad (4.1)$$

Now the twisting clearly assigns a mass to the adjoints. The mass parameter m is actually known[8] to be proportional to $\mu = \tan \theta$ where θ is the rotation angle. Because the model is compactified on x^6 , it is clear that if the relative rotation angle between NS 5-brane 1 and 2 is θ , then the angle between 2 and 1 (going the other way around the circle) is $-\theta$. Hence the mass parameter is equal and opposite for the two adjoints, therefore it is antisymmetric under exchange of the two gauge groups.

This needs a slight qualification. Physically, the amount of twisting experienced by 4-branes which are not at the origin will depend inversely on the separation between the 5-branes, which in turn is inversely proportional to the square of the gauge coupling[11]. Hence the mass perturbation actually takes the form

$$W_m(\Phi, \tilde{\Phi}) = \frac{1}{2} m \left((g_1)^2 \text{tr} \Phi^2 - (g_2)^2 \text{tr} \tilde{\Phi}^2 \right) \quad (4.2)$$

Integrating out the adjoints then gives a quartic superpotential:

$$W(A_i, B_i) = \frac{1}{2m} \text{tr} \left((A_1 B_1 A_2 B_2) - \text{tr} (B_1 A_1 B_2 A_2) \right) \quad (4.3)$$

Note that the superpotential is finite as long as $\theta \neq \pi/2$, it goes smoothly to zero as the branes become exactly orthogonal¹. In what follows, we will mainly analyse the case of orthogonal branes as it is pictorially simpler.

This superpotential has a nonabelian global symmetry which becomes visible if we write it as:

$$W(A_i, B_i) = \frac{1}{2m} \epsilon^{ij} \epsilon^{kl} \text{tr} (A_i B_k A_j B_l) \quad (4.4)$$

This has manifest global $SU(2) \times SU(2)$ symmetry, under which the A_i and the B_j transform separately as doublets under the first and second factors respectively. We will identify parts of this symmetry after obtaining the M-theory limit of the model.

5. Brane Configurations and Geometric Singularities in various Dimensions

We now give an alternative derivation of the relationship between branes at conifolds and intersecting brane configurations, which will help us to see more clearly how the various geometrical data of the conifold fit together in the brane picture. Although the discussion will be somewhat qualitative and some constants have to be fixed by hand, this will provide us a definite map between coordinates in the brane configuration and in the conifold.

We will proceed in the reverse direction to the previous section, in the sense that we will start with particular brane configurations and use U-duality transformations to map them to D3 branes near geometric singularities.

Let us consider two NS5 branes in type IIA oriented along some directions. There are three interesting cases:

- (i) The 5-branes have all five directions common,
- (ii) The 5-branes have three of their directions common, and
- (iii) The 5-branes have only one common direction.

The first one gives rise to ALE spaces with A_{k-1} singularity (here we will focus mainly on A_1 singularities, the general case arises from having more than two NS5 branes). The second case is the subject of this paper. It gives rise to conifold singularities. The third case will give rise to toric Hyper-Kähler manifolds[18,19]. We will see that a certain set of U-duality transformations relate the various cases.

¹ Even though the identification $\mu = \tan\theta$ gives correct answer for many cases, it is actually valid only for *small* θ [17]. So for our case at $\theta = \pi/2$ we can still have some superpotential. We are grateful to A. Uranga for pointing this out.

ALE space with A_{k-1} singularity

Let us start with a configuration of a D3-brane suspended between two parallel D5-branes. The situation can be represented as follows:

$$\begin{array}{l} D5 : \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad - \quad - \quad - \quad - \\ D3 : \quad 1 \quad 2 \quad - \quad - \quad - \quad 6 \quad - \quad - \quad - \end{array}$$

This will be used, as in Ref.[20], to obtain the metric for D4-branes suspended between NS 5-branes.

The first step is easy since it is known[21] how to write down metrics for general configurations of intersecting D-branes² For NS-branes, certain scalings need to be done as we will see.

Let $H_i = 1 + Q_i/r_i^d$ be the relevant harmonic functions for the D3 and D5-branes respectively. Q_i is proportional to the charge of the brane, and r is the transverse distance. The metric for the above configuration is:

$$\begin{aligned} ds^2 = & (H_5 H_3)^{-1/2} ds_{012}^2 + (H_3/H_5)^{1/2} ds_{345}^2 + \\ & + (H_5/H_3)^{1/2} ds_6^2 + (H_3 H_5)^{1/2} ds_{789}^2 \end{aligned} \quad (5.1)$$

In this case, the H_i are harmonic functions in the three directions (x^7, x^8, x^9) since these are the ‘‘overall transverse’’ directions in the problem. Hence each of them is of the form $1 + 1/r$ where $r = ((x^7)^2 + (x^8)^2 + (x^9)^2)^{1/2}$.

² Consider a system of N intersecting Dp branes with harmonic functions H_i for each of them. The metric for such a system follows a general formula. Choose the maximal set of common directions, say n_1 , and write the metric for that part with a factor $(H_1 H_2 \dots H_m)^{-1}$. m is the number of D branes which have n_1 common directions. Now choose the next set. And so on. In the end the directions along which no branes lie appear in the metric without a prefactor. Finally the whole metric is multiplied with $(H_1 H_2 H_3 \dots H_N)^{1/2}$. As an example let n_1, n_2, n_3 be the set of common directions and m_1, m_2, m_3 be the number of D branes with those common directions, then the metric will be

$$\begin{aligned} ds^2 = & (H_1 \dots H_N)^{1/2} [(H_1 \dots H_{m_1})^{-1} ds_{012 \dots n_1}^2 + (H_1 \dots H_{m_2})^{-1} ds_{n_1+1, \dots, n_1+n_2}^2 \\ & + \dots + ds_{no \text{ common directions}}^2] \end{aligned}$$

For reviews, see for example Refs.[22,23].

Under a S-duality transformation the system becomes a D3 brane between two NS5 branes. The metric for this configuration is just the previous one multiplied by a factor of $H_5^{1/2}$:

$$ds^2 = (H_3)^{-1/2} ds_{012}^2 + (H_3)^{1/2} ds_{345}^2 + H_5 H_3^{-1/2} ds_6^2 + H_5 H_3^{1/2} ds_{789}^2 \quad (5.2)$$

A T-duality along x^3 will now bring the theory to IIA with a configuration of a D4 brane between two NS5 branes. The metric for this configuration will be:

$$ds^2 = (H_3)^{-1/2} ds_{0123}^2 + (H_3)^{1/2} ds_{45}^2 + H_5 H_3^{-1/2} ds_6^2 + H_5 H_3^{1/2} ds_{789}^2 \quad (5.3)$$

At this point we go to IIB via T-duality along x^6 . The resulting configuration turns out to be a bunch of D3 branes on a geometric singularity. This geometry is basically the T-dual manifestation of the NS5-branes.

The duality relations that we need can be found, for example, in Ref.[24]. We quote the relevant formulae below (g and B are the metric and the antisymmetric fields of type IIA and G is the metric of type IIB, x^6 is the compact direction).

$$G_{mn} = g_{mn} - (g_{6m}g_{6n} - B_{6m}B_{6n})/g_{66}, \quad G_{66} = 1/g_{66}, \quad G_{6m} = B_{6m}/g_{66} \quad (5.4)$$

Here m, n take all values from 0 to 9 except 6. For the IIA metric in Eq.(5.3), we have the following metric components:

$$\begin{aligned} g_{\mu\nu} &= H_3^{-1/2} \eta_{\mu\nu}, \quad g_{44} = g_{55} = H_3^{1/2}, \quad g_{66} = H_5 H_3^{-1/2}, \\ g_{77} &= g_{88} = g_{99} = H_5 H_3^{1/2} \end{aligned} \quad (5.5)$$

$\mu, \nu = 0, 1, 2, 3$ are the spacetime coordinates. The transformation formulae in eq.(5.4) will give the following metric components for the IIB case:

$$\begin{aligned} G_{\mu\nu} &= g_{\mu\nu}, \quad G_{44} = G_{55} = g_{44}, \quad G_{66} = (g_{66})^{-1}, \\ G_{6i} &= B_{6i}/g_{66}, \quad G_{ii} = g_{ii} + B_{6i}^2/g_{66} \end{aligned} \quad (5.6)$$

$i = 7, 8, 9$ and $B_{6i} = \omega_i$ is the antisymmetric background in the type IIA picture. ω_i solves the B-field equation

$$\vec{\nabla} \times \vec{\omega} = \vec{\nabla} H_5 \quad (5.7)$$

Therefore the IIB metric arising from T-duality on Eq.(5.3) will look like

$$ds^2 = G_{\mu\nu} ds_{0123}^2 + G_{ij} ds_{ij}^2, \quad i, j = 6, 7, 8, 9 \quad (5.8)$$

which after putting in all factors becomes:

$$ds^2 = H_3^{-1/2} ds_{0123}^2 + H_3^{1/2} [ds_{45}^2 + H_5^{-1} (dx_6 + \omega \cdot dx_{789})^2 + H_5 ds_{789}^2] \quad (5.9)$$

We observe that the metric describes a D3 brane with world-volume along (0123)-directions as well as a Taub-NUT space in the transverse directions (6789). The harmonic functions H_5 and H_3 are given by $1 + 1/r$. Here, however, we face a problem. H_5 (which is still $1 + 1/r$) is as desired for the Taub-NUT metric. But H_3 should have been the harmonic function of a 3-brane with 6 noncompact transverse directions, namely $H_3 = 1 + 1/r^4$ where $r = ((x^4)^2 + \dots + (x^9)^2)$ to get the correct behaviour. The problem occurs when we demand a “localised” 3-brane, as opposed to the “delocalised” one that occurs in Eq.(5.1). Therefore by following the standard T-duality rules (and assuming some delocalised directions) we *do not* recover the exact metric of a D3 brane near an A_{k-1} singularity. A more general analysis wherein no delocalisation is assumed may lead to a better result.

From Ref.[24] it is easy to check that in the IIB case, no other background will be excited under the above transformations.

At this point, as shown in [20], we can examine the near horizon geometry of this system, which turns out to describe type IIB theory on $AdS_5 \times S^5/Z_2$ (Z_k if there are k NS5 branes). To see this we first go near the center of the Taub-NUT space. For distances very close to this point, we can neglect the constant part in the H_5 harmonic function. This way the D3 brane is localised at the A_{k-1} singularity. Now in the near horizon region one can neglect the constant part of H_3 also. This way the geometry resembles S^5/Z_k .

The gauge theory on the D4 brane can be read off explicitly from this model. We have a configuration of k parallel NS5 branes arranged on a circle x^6 . The D4 brane is cut at k points to give a gauge group of $U(1)^k$ (or $U(N)^k$ if there are N D4 branes). The matter multiplets will come from strings joining two D4 branes across an NS5 brane. These will be hypermultiplets in bi-fundamental representations. We will return to this later.

Conifold singularity

Now we proceed in the same way but for the case relevant to the $\mathcal{N} = 1$ model that we have presented in the preceding sections. As before, we start with a configuration of two orthogonal D5 branes and a D3 brane between them. The configuration is

$$\begin{array}{lcl} D5 : & 1 & 2 & 3 & 4 & 5 & - & - & - & - \\ D5' : & 1 & 2 & 3 & - & - & - & - & 8 & 9 \\ D3 : & 1 & 2 & - & - & - & 6 & - & - & - \end{array}$$

Let H_i be the relevant harmonic functions for the D5, D5' and D3 branes. This time, the H_i are harmonic functions in one overall transverse direction, x^7 . Hence they behave as $H_i = 1 + r$ where $r = |x^7|$.

Under a S-duality this will turn into two NS5 branes and a D3 between them. A further T-duality along x^3 will give us our configuration, for the special case where the rotation parameter α discussed earlier is equal to 1. This situation is similar to the previous case of an ALE singularity. Therefore a further T-duality along x^6 should give us a bunch of D3 branes near a conifold singularity.

The metric for the above configurations of D branes can be written down following standard prescriptions as explained in the previous section. The result is:

$$ds^2 = (H_5 H'_5 H_3)^{-1/2} ds_{012}^2 + H_3^{1/2} (H_5 H'_5)^{-1/2} ds_3^2 + (H_3 H'_5)^{1/2} H_5^{-1/2} ds_{45}^2 + (5.10)$$

$$+ (H_3 H_5)^{1/2} H_5'^{-1/2} ds_{89}^2 + (H_5 H'_5)^{1/2} H_3^{-1/2} ds_6^2 + (H_5 H'_5 H_3)^{1/2} ds_7^2$$

The metric after a S and a T_3 duality will give a configuration of a D4 brane between two orthogonal NS5 branes. The metric for this configuration is easy to write down from eq.(5.10). The result is:

$$ds^2 = H_3^{-1/2} ds_{0123}^2 + H'_5 H_3^{1/2} ds_{45}^2 + H_5 H_3^{1/2} ds_{89}^2 + (5.11)$$

$$+ H_5 H'_5 H_3^{-1/2} ds_6^2 + H_5 H'_5 H_3^{1/2} ds_7^2$$

At this point we use the duality map Eq.(5.4) On the type IIA side there can be a nontrivial $B_{\mu\nu}$ background. We assume the non zero values are B_{46}, B_{68} and B_{ij} for $i, j \in (4, 5, 8, 9)$. x^6 is the compact circle³. We get the following metric components for the IIB case:

$$G_{\mu\nu} = g_{\mu\nu}, \quad G_{66} = g_{66}^{-1}, \quad G_{6i} = B_{6i}/g_{66}, (5.12)$$

$$G_{ii} = g_{ii} + B_{6i}^2/g_{66}, \quad G_{77} = g_{77}, \quad G_{48} = B_{64}B_{68}/g_{66}$$

where $i = 4, 8$ and $\mu, \nu = 0, 1, 2, 3$ (if we want to get the rotated case then $i = 4, 5, 8, 9$). The IIB metric therefore becomes:

$$ds^2 = H_3^{-1/2} ds_{0123}^2 + H_3^{1/2} [H'_5 ds_{45}^2 + H_5 ds_{89}^2 + H_5 H'_5 ds_7^2 + (5.13)$$

$$+ (H_5 H'_5)^{-1} (ds_6 + B_{64} ds_4 + B_{68} ds_8)^2]$$

³ If we were doing the reverse procedure, starting with a conifold geometry and using the T-duality relations to get to our brane picture, then the configuration would come out as two intersecting branes at an angle to both 45 and 89 planes. However we can rotate the configuration so that the two branes are along 45 and 89 respectively, for $\alpha = 1$. At a general value of α we would keep the NS5-brane fixed along 45 and rotate the NS5' brane in the (45,89) planes

In addition to B_{46}, B_{68} , whose roles will be explained below, we have chosen to excite constant nonzero B-fields, B_{ij} where $i, j \in (4, 5, 8, 9)$, on the IIB side[24]. This will be the nontrivial B background on a conifold. As we will see, this will parametrise the separation of the NS5-branes, or equivalently the difference of gauge couplings in the two factors, a quantity that is not encoded in the geometry of the problem in the type IIB description.

Now we will argue that the above metric is locally that of a 3-brane at a conifold. First of all, as in the previous discussion, we must take H_3 to be the harmonic function of a 3-brane localised in 6 transverse directions, while H_5 and H'_5 continue to be $(1 + |x^7|)$. Now since H_5, H'_5 are nonsingular for all finite x^7 , we can absorb them into the coordinates x^4, x^5 and x^8, x^9 . In any case, near the point $|x^7| = 0$ we can take the harmonic functions to be effectively constant.

Now we see that Eq.(5.13) resembles the form of the metric for a 3-brane at a geometrical conifold singularity. However, in Eq.(5.13), the (4, 5) and (8, 9) directions are planar, while for a genuine conifold they need to combine into the direct product of round spheres with a definite radius. In other words, we need to make the replacement:

$$ds_{45}^2 + ds_{89}^2 \rightarrow C \sum_{i=1}^{i=2} (d\theta_i^2 + \sin^2\theta_i d\phi_i^2) \quad (5.14)$$

where C is a constant. This amounts to the substitution:

$$dx^4 \rightarrow \sqrt{C} \sin\theta_1 d\phi_1, \quad dx^5 \rightarrow \sqrt{C} d\theta_1 \quad (5.15)$$

and similarly for (x^8, x^9) and (θ_2, ϕ_2) . As we will see again in later sections, the fact that our procedure does not quite reproduce a conifold, but rather something similar where two 2-spheres are replaced by 2-planes, is responsible for the fact that only the $U(1)$ subgroups of the global $SU(2)$ symmetries are manifest.

Next, in analogy with the Taub-NUT case, we define $\omega_4 = B_{64}$ and determine it by solving the equation

$$\vec{\nabla} \times \vec{\omega} = \text{constant} \quad (5.16)$$

where $\vec{\omega} = (\omega_4, 0, 0)$, and the right side is constant (unlike for the Taub-NUT case) because of the fact that the harmonic function H_5 is linear in x^7 . In polar coordinates this becomes:

$$\frac{1}{\sin\theta_1} \frac{\partial}{\partial\theta_1} (\sin\theta_1 \omega_4) = \text{constant} \quad (5.17)$$

This equation is solved by $\omega_4 = \cot \theta_1$. The same procedure gives $\omega_8 = B_6 8 = \cot \theta_2$. With the replacement $x^6 \rightarrow \psi$, we have:

$$(dx^6 + B_{64}dx^4 + B_{68}dx^8)^2 = (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 \quad (5.18)$$

Finally we make the conformal transformation $x^7 = \log r$, after which (suitably rescaling coordinates and making an appropriate choice of the constant C) the term in square brackets in Eq.(5.13) becomes the conifold metric:

$$ds_{conifold}^2 = dr^2 + r^2 \left(\frac{1}{6} \sum_{i=1}^2 (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + \frac{1}{9} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 \right) \quad (5.19)$$

This metric still solves the supergravity equations of motion, as was shown in Refs.[25,26].

This completes our map from the metric of a configuration of intersecting branes to that of 3-branes at a conifold (modulo the heuristic step of replacing two 2-planes by 2-spheres):

$$ds^2 = (H_3)^{-1/2} ds_{0123}^2 + (H_3)^{1/2} [ds_{conifold}^2] \quad (5.20)$$

We make the following observations:

(i) From eq.(2.1) we can calculate the base of the conifold by intersecting the space of solutions of (2.1) with a sphere of radius r in C^4 ,

$$\sum_{i=1}^4 |z_i|^2 = r^2 \quad (5.21)$$

If we now break up z into its real and imaginary parts, $z_i = x_i + iy_i$, then we have from (2.1) and (5.21)

$$x_i x_i = r^2/2, \quad y_i y_i = r^2/2, \quad x_i y_i = 0 \quad (5.22)$$

The first of these defines an S^3 with radius $r/\sqrt{2}$. The other two equations define an S^2 bundles over S^3 . Since all such bundles are trivial the base has a topology of $S^2 \times S^3$ [27].

(ii) If the (4, 5) and (8, 9) directions were planar, then we would have a $U(1)$ symmetry associated to individual rotations of each of them. Taking them to be round spheres should enhance these symmetries to $SU(2) \times SU(2)$. It appears that the enhanced symmetries are not directly visible in the brane construction, but the above manipulations (and the discussion to follow) nevertheless illuminate what they should geometrically correspond to. The coordinates z_i transform as a vector of $SO(4)$, giving rise to a global symmetry

$SO(4) \sim SU(2) \times SU(2)$. Since the z_i also parametrises the $\mathcal{N} = 1$ chiral multiplets A_i, B_i ; $i = 1, 2$, these multiplets transform as a doublet of $SU(2)$. As we will see in the following section, the lift of this model to M-theory is consistent with the appearance of this global symmetry, in contrast to the non-elliptic case studied in Refs.[8,9,10] where these symmetries cannot appear.

(iii) This mapping of a brane configuration to a geometric manifold actually allows us to identify locally all the directions of the conifold in the brane picture. For the previous case of an ALE space we found that an S^2 and the transverse distance r lie in the (789) direction. In the present case, there are two S^2 factors which play a symmetrical role, these are associated to the (45) and (89) directions. Therefore, as we have seen above, they are two supersymmetric cycles in the two NS5 branes. The transverse distance is again x^7 . The $U(1)$ fibre of the conifold base is the x^6 direction. Finally, the S^2 factor in the direct product $S^2 \times S^3$, over which the $U(1)$ part does not vary, is parametrised locally by two combinations of the coordinates (x^4, x^5, x^8, x^9) that are symmetric under the exchange $(4, 5) \leftrightarrow (8, 9)$. The S^3 factor is therefore parametrised by the other two combinations along with x^6 .

(iv) From Eqs.(5.20) and (5.19) we have:

$$ds^2 = H_3^{-1/2} ds_{0123}^2 + H_3^{1/2} [dr^2 + G_{ij} ds_{ij}^2] \quad (5.23)$$

where $i, j = 4, 5, 6, 8, 9$ and $H_3 = 1 + L^4/r^4$ with $L^4 = 4\pi g_s N(\alpha')^2$. Following[1] we see that the near horizon limit ($r \rightarrow 0$) of the geometry is $AdS_5 \times T^{1,1}$. At this point we can make the connection between S^5/Z_2 and $T^{1,1}$ clear. From Ref.[1] we expect that when one blows up the fixed circle of S^5/Z_2 one gets the Einstein space $T^{1,1}$. This simply amounts to rotating the brane configurations from being parallel to orthogonal.

$AdS_5 \times T^{1,1}$ has the explicit form:

$$ds^2 = (r^2/L^2) ds_{0123}^2 + L^2 [dr^2 + G_{ij} ds_{ij}^2]/r^2 \quad (5.24)$$

The factor of L^2 implies that now the two S^2 do not shrink to zero size. In the brane picture, if the number of D4 branes is very large then in their near-horizon region, the S^2 's have a definite size.

(v) We have seen that a single T-duality along the isometry direction of a conifold (i.e. the x^6 direction) gives us a configuration of two orthogonal NS5 branes in type IIA theory having a common 3 + 1 dimensions. It is natural to ask how this picture is related

to the one proposed by BSV[6]. To make this connection we use the fact that the S^3 of the base can be written as a $U(1)$ fibration over S^2 . The $U(1)$ and the S^2 have already been identified from the brane picture.

Now, this S^2 can be thought of as a degenerating torus, one of whose cycles is shrinking to zero size. The point where the cycle degenerates can be removed and the manifold becomes topologically a sphere. One can similarly treat the other S^2 . One can T-dualise along the two directions of the degenerating torus. This will take the theory back to type IIB and the configuration will be two orthogonal NS5 branes having a common $3 + 1$ dimensions. This is the BSV picture. As we saw earlier, the brane construction of this maps to a “brane box” but as far as we can see, it does not provide the straightforward interpretation of the system that the type IIA picture gives.

(vi) We can also see the relation between ALE spaces and conifold a bit more clearly from the brane analysis. For the case of parallel branes (without the D4 in between) the orthogonal space is $ALE \times R^6$. As we rotate the branes from the parallel to the orthogonal configuration, we see that an R^2 in R^6 starts becoming a P^1 . This would imply that the orthogonal space is an ALE fibration over a P^1 which from eq.(3.1) is precisely a conifold.

Toric Hyper-Kähler manifolds

Although not directly related to the main theme of this paper, we digress briefly to discuss this case. Here it is more economical to study the reverse map, namely given a hyper-Kähler manifold, we can use eq.(5.4) to see what kind of brane construction this gives rise to. We will find that it is a set of intersecting NS5-branes along a string. This part will be mainly a review of Refs.[18,19].

The toric eightfold that we are interested in will have T^2 isometry. A generic toric eightfold has the following local form of the metric:

$$ds^2 = U_{ij} dx^i dx^j + U^{ij} (d\phi_i + A_i)(d\phi_j + A_j) \quad (5.25)$$

where U_{ij} are the entries of a positive definite symmetric $n \times n$ matrix function U of the n set of coordinates x^i . And $A_i = dx^j \omega_{ji}$ where ω is a $n \times n$ matrix. ϕ_i are the two isometry directions.

⁴ The two degenerating tori of Eq.(2.2) can be identified with the tori formed by taking one cycle from each of the S^2 . This is because from Ref.[6] we know that the S^3 of the base come from x^6 and one cycle from each of the two fibre torus.

Therefore we have a configuration of type IIA theory on a eightfold with a metric

$$ds^2 = ds_{01}^2 + U_{ij} dx^i dx^j + U^{ij} (d\phi_i + A_i)(d\phi_j + A_j) \quad (5.26)$$

If we T-dualise twice along the two isometry directions using the relations given in Eq.(5.4) we get a model back in IIA which can be interpreted as an arbitrary number of NS5-branes intersecting on a string[18] (for subsequent developments, see Refs.[28]).

Starting with M-theory on the toric fourfold, we compactify along one isometry direction and then T-dualise along other, which takes us to type IIB theory with a configuration of one NS5-brane and a D5 brane having $2 + 1$ common directions. A 2-brane probe in M-theory will now become a D3-brane suspended between the two 5-branes. This picture, as argued in[18], is not the Hanany-Witten model because we get $\mathcal{N} = 3$ in $d = 3$ from this model. A generalisation of this model and other questions related to the near horizon geometry etc. will be addressed in a future paper[29].

6. The M-theory Description

Remarkable insight can be gained into the dynamics of theories constructed via type IIA branes by taking the strong-coupling limit and going to M-theory. This approach was pioneered by Witten in Ref.[11] for $\mathcal{N} = 2$ theories, where it yields an amazing new derivation of the Seiberg-Witten curves and their generalizations for various gauge groups, and also gives rise to the solution found in Ref.[30] of the conformal $\mathcal{N} = 2$ models related to integrable systems. It was generalised to $N = 1$ models in Refs.[8,9,10], and subsequently studied by many other authors.

The common feature of these solutions is that a configuration of D4-branes ending on NS5-branes in type IIA string theory has to turn into a configuration purely made up of 5-branes in the M-theory limit, since there are no 4-branes in M-theory. Indeed, since M5-branes cannot actually “end” on each other because of charge conservation, so what actually happens is that there is a single smooth M5-brane at the end, whose weak-coupling (small M-circle radius) limit looks like branes ending on branes. On the other hand, the gauge theory that is being realised via brane configurations has coupling constants that depend only on the ratio of brane separations to the string coupling, so by scaling both of these up together, we can keep the gauge theory coupling finite and still justify the use of M-theory.

Clearly it is an interesting problem to understand what is the solution of our model using this M-theory limit. For clarity, let us list the four different models that we will be comparing in our discussion:

(i) Non-elliptic $\mathcal{N} = 2$: A model of 2 parallel NS5-branes with N D4-branes stretched between them (where the NS5-branes fill the directions $(x^1, x^2, x^3, x^4, x^5)$ and are separated along x^6 , while the D4-branes fill (x^1, x^2, x^3) and are stretched along x^6 , see Fig. 2),

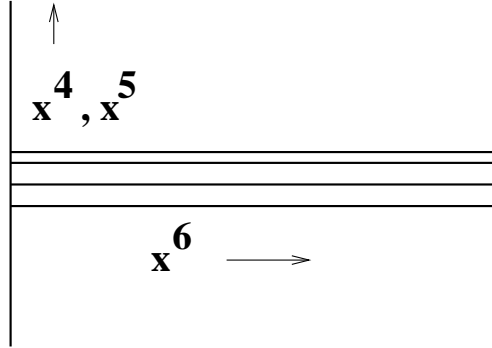


Fig. 2: Model (i): Non-elliptic $\mathcal{N} = 2$

(ii) Elliptic $\mathcal{N} = 2$: A model of 2 parallel NS5-branes located on a compact direction, with N D4-branes stretched between them from both sides (the directions are precisely as for model (i) but x^6 is compact, see Fig. 3),

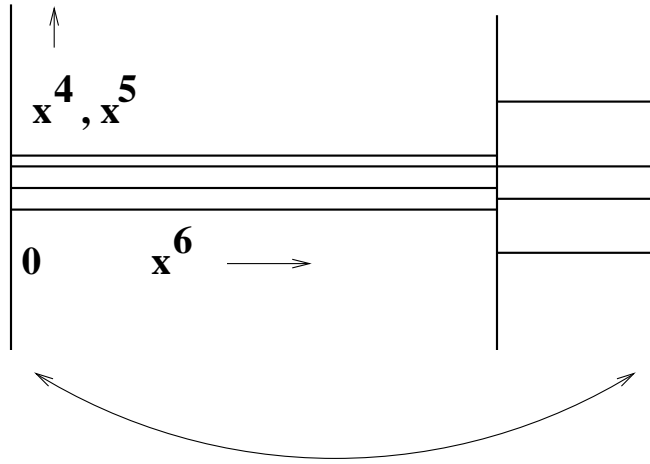


Fig. 3: Model (ii): Elliptic $\mathcal{N} = 2$

(iii) Non-elliptic $\mathcal{N} = 1$: A model of 2 orthogonal NS5-branes with N D4-branes stretched between them (where first NS5-branes fills the directions $(x^1, x^2, x^3, x^4, x^5)$ and is separated along x^6 from the second NS5'-brane, which fills $(x^1, x^2, x^3, x^8, x^9)$, while the D4-branes fill (x^1, x^2, x^3) and are stretched along x^6 , see Fig. 4),

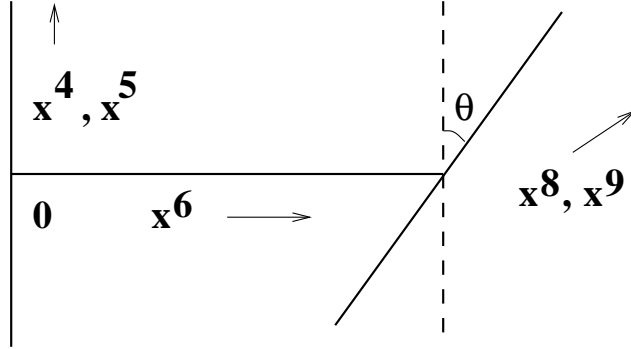


Fig. 4: Model (iii): Non-elliptic $\mathcal{N} = 1$

(iv) Elliptic $\mathcal{N} = 1$: A model of 2 orthogonal NS5-branes located on a compact direction, with N D4-branes stretched between them from both sides (the directions are precisely as for model (iii) but x^6 is compact, this model has already been illustrated in Fig. 1).

Models (i) and (iii) are not conformally invariant – they can be thought of as the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetric versions of pure $SU(N)$ QCD. The elliptic models (ii) and (iv) are the ones dual to 3-branes at a Z_2 ALE singularity and a conifold, respectively, that have been the subject of previous sections.

Models (i) and (ii) were solved in the M-theory limit in Ref.[11]. The solutions can be summarised as follows: in model (i), the M-theory brane is wrapped on $R^4 \times \Sigma$ where R^4 is described by the coordinates (x^0, x^1, x^2, x^3) and Σ is a complex curve (Riemann surface) holomorphically embedded in $R^3 \times S^1$ parametrised by the complex coordinates (v, t) where $v = x^4 + ix^5$ and $t = \exp(-(x^6 + ix^{10}))$. The curve Σ is given by the equation

$$t^2 + t(v^N + u_2 v^{N-2} + \dots + u_N) + 1 = 0 \quad (6.1)$$

This curve has $(N - 1)$ complex parameters. This fits with the fact that it has Euler characteristic $\chi = 4 - 2N$ which, via $\chi = 2 - 2g$, determines the genus g to be $N - 1$. The Euler characteristic is computed by noting that within each of the NS5-branes we have a 2-sphere or P^1 (in the $(4, 5)$ directions) with N little tubes joining the two P^1 's. Each P^1 has $\chi = 2$ and each tube takes away 2 from χ . The situation is illustrated in Fig. 5.

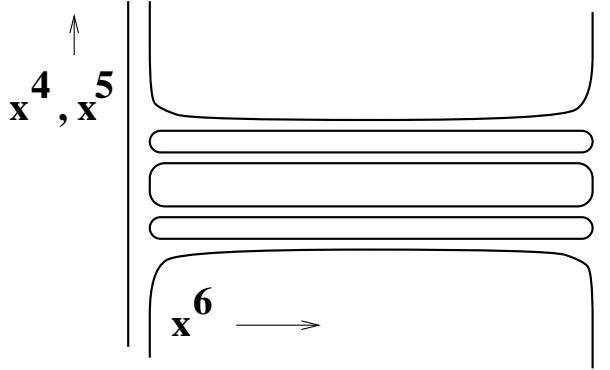


Fig. 5: Solution of model (i): genus = $N - 1$

The genus of the curve is the number of massless photons in the Coulomb branch of the field theory. We can check this by noting that in the present case, the gauge group is $SU(N)$ which has a Coulomb branch of dimension $N - 1$.

Model (ii) requires a slightly different approach. This time the spectrum is a $U(1) \times SU(N) \times SU(N)$ gauge theory with 2 bi-fundamental hypermultiplets (the $U(1)$ is decoupled). The theory is conformally invariant. The M5-brane to which the brane configuration tends in the limit of large type IIA coupling is again wrapped on $R^4 \times \Sigma$ where R^4 is the same as for model (i), but now Σ is a complex curve holomorphically embedded in $R^2 \times T^2$. The torus T^2 is described by the coordinates x^6, x^{10} , both of which are compact.

If the $(6, 10)$ torus is parametrised by two complex coordinates (x, y) satisfying a Weierstrass equation

$$y^2 = x^3 + fx + g \tag{6.2}$$

then the curve Σ is described by the equation

$$v^N + \sum_{i=1}^N f_i(x, y)v^{N-i} = 0 \tag{6.3}$$

where $f_i(x, y)$ are N meromorphic functions, each of which has a simple pole at two points (X, Y) and (X', Y') (with equal and opposite residues).

Each of the functions depends on two complex parameters, one being the common residue at the poles and the other a constant shift. Thus we have $2N$ real parameters, but one of them describes the masses of the two hypermultiplets, which are equal and opposite. This parameter has been identified in Ref.[11] to be the residue of f_1 . The remaining $2N - 1$

parameters describe the Coulomb branch of $U(1) \times SU(N) \times SU(N)$. The parameter for the decoupled $U(1)$ has also been identified – it is a constant shift in f_1 [11].

It follows that the genus of the curve Σ should be $2N - 1$, which can be confirmed by noticing that we now have two P^1 's (one on each NS5-brane) joined by N thin tubes from *both* sides along a circle, as in Fig. 6.

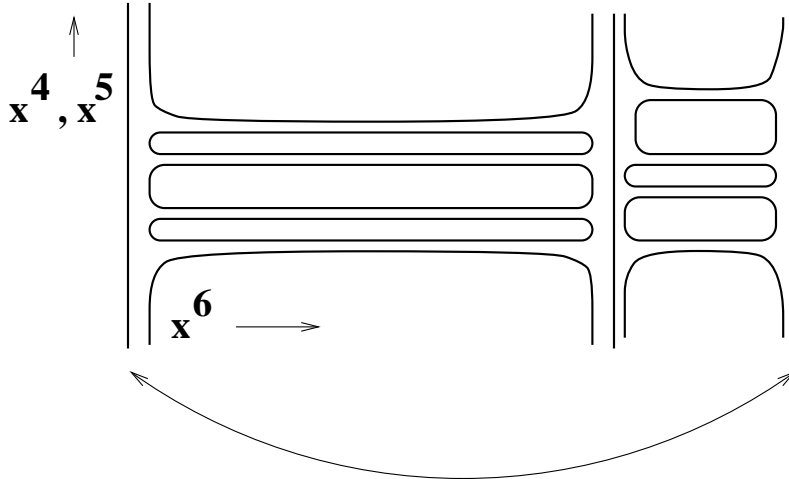


Fig. 6: Solution of model (ii): genus = $2N - 1$

Model (iii) has $\mathcal{N} = 1$ supersymmetry. The corresponding field theory is usually known as SQCD. The solution of this model can be found in Refs.[8,9]. Although it can be obtained directly, a very useful way to understand the solution is via a “rotation” of model (i). The second NS5-brane of model (i) is rotated in the $(4, 5), (8, 9)$ hyperplane until it fills the (x^8, x^9) directions (the w -plane, where $w = x^8 + ix^9$) and lies at a point in (x^4, x^5) , say $v = 0$. The effect of this rotation, as we have discussed for the analogous elliptic model in previous sections, is to induce a mass term for the adjoint, breaking $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$.

It turns out that one cannot perform this rotation at any arbitrary point of moduli space. This is the brane manifestation of a well-known phenomenon first noted in Ref.[31] for $SU(2)$ gauge theory, that an adjoint mass term can be turned on (partially breaking supersymmetry and leading to confinement) only at points on the u -plane where the Seiberg-Witten torus has a node, in other words where it degenerates to genus 0. In the present context, the D4-branes lying between the NS5-branes must first come together at the origin, or else they will get twisted by the rotation and all supersymmetry will be broken. (The origin is of course replaced non-perturbatively by a pair of points, where monopoles and dyons become massless, hence there are actually two points in moduli space

where rotation is allowed.) In short, rotation of an NS5-brane breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$ is possible only when the curve Σ has degenerated to genus 0.

The resulting model, in the M-theory limit, is described by an M5-brane wrapped over $R^4 \times \Sigma$ where again R^4 is spanned by (x^0, x^1, x^2, x^3) but this time Σ is a certain complex curve holomorphically embedded in $R^4 \times T^2$. Thus in the $\mathcal{N} = 1$ models, the curve is a holomorphic embedding in a non-compact complex Calabi-Yau 3-fold rather than a complex surface (2-fold) as was the case for $\mathcal{N} = 2$.

The 3-fold is parametrised by the complex coordinates v, w (for the R^4 part which is made up of the (4, 5) and (8, 9) directions) and (x, y) which describe the 2-torus T^2 . The fact that Σ in this model has genus 0 is evident both from the simple fact that the Coulomb branch has disappeared, and from the brane construction in which the two P^1 's (one in each 5-brane) are connected by a single “tube” consisting of all the 4-branes bunched together. The situation is depicted in Fig. 7.

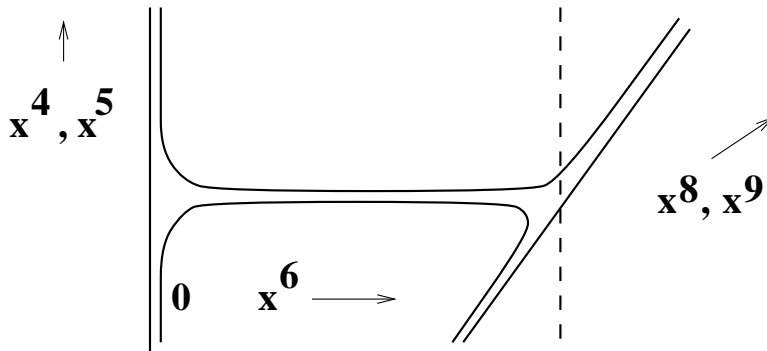


Fig. 7: Solution of model (iii): genus = 0

Now let us come to model (iv), the one which is the main subject of this paper. We can try to see it as a rotation of model (ii). Indeed, this rotation is just the relevant perturbation which in geometric language takes D3-branes transverse to $ALE \times R^2$ to D3-branes at a conifold. Let us ask now what is the condition for the model (ii) to admit a rotation breaking $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$. In this case it is easy to see that the Seiberg-Witten curve only has to degenerate to genus 1 to permit the rotation. However, there is another constraint, that the hypermultiplet mass parameter must also go to zero. This is shown as follows.

Geometrically, in order to rotate one 5-brane we need that the 4-branes connecting the 5-branes (from both sides) coalesce completely. This corresponds to going to the origin of the Coulomb branch for the $SU(N) \times SU(N)$ part of the gauge group (this collapses

each of the two bunches of 4-branes), and also making the single hypermultiplet mass parameter zero (this makes the two bunches of 4-branes coincide where they touch the 5-branes). This process freezes altogether $2N - 1$ parameters, leaving only the constant shift in f_1 . as a result, the curve Σ relevant to this model has genus 1.

This is of course confirmed by the fact that the Coulomb branch of this model just corresponds to the free decoupled $U(1)$. Moreover, geometrically, we have again N D4-branes joining the NS5-branes from both sides, as in model (ii), but now that they are all collapsed, there is only a single tube connecting a pair of P^1 's from both sides, with the result again that the genus is equal to 1.

Now we can ask, in the spirit of Refs.[11,8,9], what is the description of the genus 1 curve Σ on which the M 5-brane is wrapped. The answer is rather remarkable and is related to a phenomenon recently discussed by Dorey and Tong in Ref.[32]⁵. The situation considered in Ref.[32] is a pair of NS5-branes with some D4-branes stretched between them, and with additional semi-infinite 4-branes on either side. The particular configuration of interest arises when some of D4-branes “line up” on opposite sides of the NS5-brane (Fig. 8).

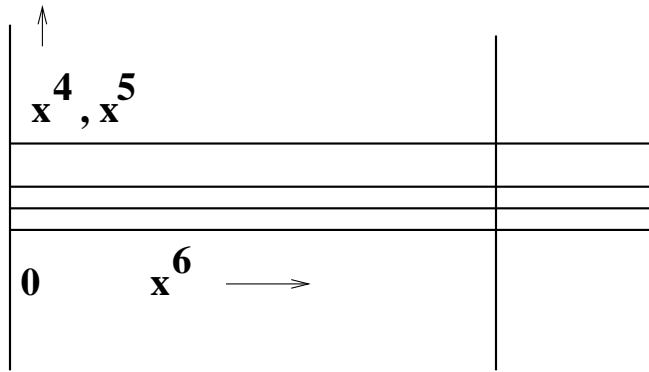


Fig. 8: 4-branes “lined up” across a 5-brane

One can ask what describes the M-theory limit of this configuration, which corresponds to special regions of moduli space. The answer turns out to be that while the NS5-brane turns into an M5-brane as usual, the D4-branes that are lined up on opposite sides of it turn into M5-branes that *go through* it. In other words, in the M-theory limit, the charge carried by the type IIA D4-branes does not flow onto the 5-brane on which they end from both sides, but goes right through. The situation is illustrated in Fig. 9.

⁵ We are grateful to David Tong for explaining these results to us before publication.

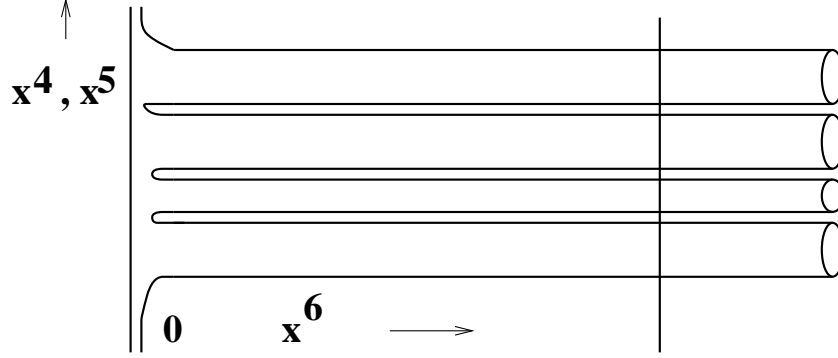


Fig. 9: Cylindrical M5-branes going through transverse M5-branes

In our model, called model (iv) above, this situation is indeed realised. As explained above, in order to allow rotation of an NS5-brane and break supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1$, the D4-branes of model (ii) must all collect at the origin. Hence by the above argument, in the M-theory limit we will have a bunch of N coincident toroidal M5-branes wrapped on the (x^6, x^{10}) , going through a pair of M5-branes: one aligned in the (x^4, x^5) direction and the other rotated in the $(x^4, x^5), (x^8, x^9)$ hyperplane (Fig. 10).

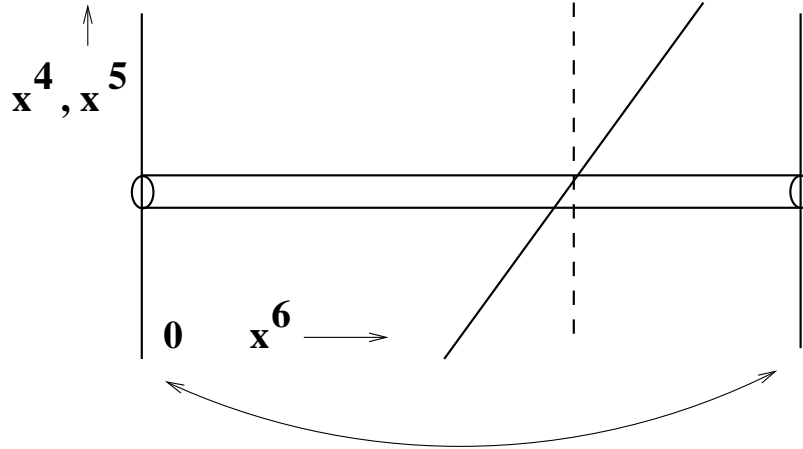


Fig. 10: Solution of model (iv): genus = 1

This can be seen starting from the M-theory curve for model (ii). Recall that this model is described by Eq.(6.3), specified by a set of N complex meromorphic functions $f_i(x, y)$ on the (x^6, x^{10}) torus. We want to consider the special point in moduli space where we are at the origin of the Coulomb branch and the hypermultiplet mass parameter is zero. From the discussion above, it follows that $f_1(x, y) = 0$ (the constant part is zero since the centre-of-mass $U(1)$ parameter is fixed at the origin, and the residue vanishes because the hypermultiplet mass is zero). Similarly, since the constant and residue parts of the other

$f_i(x, y)$ describe the VEV's associated to going to the Coulomb branch of the two $SU(N)$ factors of the gauge group (before rotation), to go to the origin of the Coulomb branch these must also be set to zero, as a result of which all the f_i 's vanish identically. Hence we get the curve

$$v^N = 0 \tag{6.4}$$

corresponding to N coincident M5-branes wrapped on the (x^6, x^{10}) torus. After rotation, the curve is no longer given by a single equation in $R^2 \times T^2$ but rather by two equations in $R^4 \times T^2$. By the symmetry that was present between the two NS5-branes before rotation, the equations must be

$$\begin{aligned} v^N &= 0 \\ w^N &= 0 \end{aligned} \tag{6.5}$$

where w labels the coordinate on the rotated brane. If the rotation is by 90 degrees then $w = x^8 + ix^9$, otherwise it is a suitable linear combination of the (x^4, x^5) and (x^8, x^9) coordinates.

These equations precisely describe the N toroidal branes, which are located at the origin of the v and w planes. One has to supplement these by hand with the requirement that the transverse planar branes are still present, at two points p_1, p_2 on the (x^6, x^{10}) torus. Hence the M-theory limit has three separate and disconnected 5-brane components, two of which are the original NS and NS' 5-branes and the third is the torus which has N coincident M5-branes wrapped on it, described by Eq.(6.5).

The above description has important implications for the global symmetries and other properties of the model, as we will now see.

7. Global Symmetries and Moduli

The fact that the stretched branes are decoupled from the transverse ones makes it clear that rotations of the v -plane and w -plane can be carried out independently of each other and of shifts in the x^6 and x^{10} directions. This is in contrast to model (iii), where the the v and w -plane rotations and the rotation along x^{10} were linked together into a single $U(1)$ symmetry by the constraint that all the branes merge into a genus-0 curve[9]. We will identify some of these transformations as factors of the global $SU(2) \times SU(2)$ symmetries associated to the conifold and discussed at length in Ref.[1], and another one as the $U(1)$ R-symmetry.

As we already briefly pointed out, the $U(1)$'s rotating the v and w planes will act on the chiral multiplets coming from “short strings” connecting the 4-branes across a 5-brane. Across the first 5-brane, we will get a pair of chiral fields, one in the (N, \overline{N}) representation of the gauge group and the other in the (\overline{N}, N) . Let us label these as A_1 and B_2 . Similarly the chiral fields arising across the second 5-brane are B_1, A_2 , transforming in the (\overline{N}, N) and (N, \overline{N}) representation respectively.

Now rotation of the 4, 5 plane by an angle α induces a transformation on the chiral fields as follows. The short strings living on the D4-brane would give rise to an $\mathcal{N} = 4$ multiplet if no NS5-branes were present. The NS5-branes break this to a hypermultiplet. From the geometry of the problem, it is clear that this hypermultiplet is the one which parametrised motion of the 4-brane in the (x^4, x^5, x^8, x^9) directions. It decomposes after breaking to $\mathcal{N} = 1$ (which is induced by a different NS5-brane located some distance away) into the two chiral multiplets A_1 and B_2 , hence these two fields can be thought of as being associated to the (x^4, x^5) and (x^8, x^9) directions respectively. It follows that under rotation of the (x^4, x^5) plane by an angle α , A_1 picks up a phase $e^{i\alpha}$ while B_2 is unchanged. On the NS5'-brane, the situation is reversed. There the chiral multiplets are B_1 and A_2 , but this brane is also rotated at 90 degrees to the previous one, hence we find that A_2 is charged under (x^4, x^5) rotations. Moreover, because A_2 comes from a “left-pointing” short string (this is clear since it is an (N, \overline{N}) of the gauge group), it picks up a phase $e^{-i\alpha}$ under this rotation.

Combining, we find that under rotations of the (x^4, x^5) plane, the chiral fields transform as

$$A_1 \rightarrow e^{i\alpha} A_1, \quad A_2 \rightarrow e^{-i\alpha} A_2, \quad B_1 \rightarrow B_1, \quad B_2 \rightarrow B_2 \quad (7.1)$$

This shows that these rotations define the $U(1)$ Cartan subalgebra of the global $SU(2)$ that is expected in this field theory. This in turn lends support to our earlier observation that $SU(2)$ arises by “enhancement” of this $U(1)$ rotation, if the (x^4, x^5) directions are compactified on a round 2-sphere.

Similar considerations can be used to show that rotations of the (x^8, x^9) plane generate a $U(1)$ lying in the second $SU(2)$.

Finally, a shift along x^6 generates an identical phase on all the chiral fields.

$$A_1 \rightarrow e^{i\gamma} A_1, \quad A_2 \rightarrow e^{i\gamma} A_2, \quad B_1 \rightarrow e^{i\gamma} B_1, \quad B_2 \rightarrow e^{i\gamma} B_2 \quad (7.2)$$

This must therefore be the $U(1)$ R-symmetry of the theory. Notice that we have already identified x^6 with the conifold coordinate that is usually called ψ , whose shift precisely generates the R-symmetry.

Next, let us consider the discrete symmetries. According to Ref.[1], there is supposed to be a global Z_2 symmetry that acts on the conifold as $z_4 \rightarrow -z_4$ and, in field theory language, exchanges the two $SU(N)$ factors in the gauge group. Hence it also exchanges the chiral fields A_i and B_i . This symmetry emerges very neatly from our construction. Recall that in the beginning of the argument, we had treated the conifold as a fibration of ALE spaces over a base, which was chosen to be parametrised by z_4 . The result of various dualities gave a pair of intersecting NS5-branes where the base z_4 was equal to the sum and difference of the brane coordinates. Thus, inversion of z_4 is nothing but exchange of the two NS5-branes. This obviously exchanges the factors of the gauge group and the chiral multiplets associated to each 5-brane.

Another interesting discrete symmetry is the element w of the centre of $SL(2, Z)$ in the type IIB description:

$$w = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad (7.3)$$

This was argued in Ref.[1] to exchange the two factors of the gauge group, while not exchanging A_i and B_i , but instead exchanging the fundamental representation N of the first $SU(N)$ with the anti-fundamental \bar{N} of the second, and vice-versa. In our picture this symmetry is just the geometrical inversion of both the torus directions:

$$x^6 \rightarrow -x^6, \quad x^{10} \rightarrow -x^{10} \quad (7.4)$$

which clearly performs the required tasks.

Finally, we turn to the moduli. The M-theory solution of our model has a set of N M5-branes wrapped on a 2-torus parametrised by x^6, x^{10} . We will now examine this point more closely. As is well-known, type IIB theory arises from M-theory by compactifying on a 2-torus and performing T-duality along one of the cycles. In the present case, x^6 is precisely the direction along which we performed T-duality to obtain our model from its type IIB description as 3-branes at a conifold. Hence this torus holds the key to understanding the relation to type IIB.

In the type IIA theory, the two NS5-branes are located at definite values of x^6 , which we may call 0 and a . The gauge coupling constants g_1 and g_2 of the two $SU(N)$ factors are then determined, by standard arguments[11] to be:

$$\begin{aligned}\frac{1}{(g_1)^2} &= \frac{R_6 - a}{g_{st}} \\ \frac{1}{(g_2)^2} &= \frac{a}{g_{st}}\end{aligned}\tag{7.5}$$

This can be recast in a useful form as

$$\begin{aligned}\frac{1}{(g_1)^2} + \frac{1}{(g_2)^2} &= \frac{R_6}{g_{st}} \\ \frac{1}{(g_1)^2} - \frac{1}{(g_2)^2} &= \frac{R_6 - 2a}{g_{st}}\end{aligned}\tag{7.6}$$

It follows that the difference in couplings is determined by a , the brane separation, while the sum of the couplings is determined only by the string coupling g_{st} (for fixed R_6).

In the M-theory limit, the two planar M5-branes are located at definite points on the (x^6, x^{10}) torus. Hence we have two complex parameters. One of these generalises a , and we will call it ϕ , while the other generalises $\frac{1}{g_{st}}$ and is in fact the complex structure parameter of the 2-torus, which in IIB becomes the dilaton-axion parameter τ_{IIB} . At the same time on the field theory side, g_1 and g_2 are part of complex parameters (by incorporating the θ -angles) which we will call τ_1^{YM} and τ_2^{YM} . The obvious holomorphic generalization of the above equations to the M-theory limit is:

$$\begin{aligned}\tau_1^{YM} + \tau_2^{YM} &\sim \tau_{IIB} \\ \tau_1^{YM} - \tau_2^{YM} &\sim \phi \tau_{IIB}\end{aligned}\tag{7.7}$$

It follows that the sum of the gauge theory parameters is determined by the (complex) type IIB string coupling, while the difference is determined by ϕ . This is identical to relations derived in Sec.3 of Ref.[1] (where they were derived using very different reasoning) if we make the identification:

$$\phi = 2 \int_{S^2} (B_{NS} + iB_{RR}) - 1\tag{7.8}$$

where the S^2 is one of the factors in the conifold base $S^2 \times S^3$. Here, B_{NS} and B_{RR} are the two 2-form fields in the type IIB theory, and their appearance in this context is related to considerations in Ref.[33].

This remarkable agreement moreover clarifies the meaning of the x^6 separation between the branes in conifold language, as was promised in Sec. 5 when we allowed constant B-fluxes in the (4, 5, 8, 9) directions. Indeed, we now see the more general result that ϕ , the complex distance between the two planar 5-branes, is determined by the value of the NS-NS and RR B-fields in the dual type IIB theory. Its real part $\int_{S^2} B_{NS}$ is the separation in x^6 . Apparently $\int B_{NS} = \frac{1}{2}$ corresponds to 5-branes located symmetrically at opposite points of the 2-torus, hence identical couplings and theta angles in the two gauge factors.

In the type IIA description, B_{NS} remains a 2-form while B_{RR} turns into the RR 3-form with one index along the 6 direction (we T-dualised along a direction orthogonal to the S^2 , namely x^6). In the M-theory limit, B_{NS} also becomes the M-theory 3-form, with one index in the “10” direction. So finally, in the M-theory picture there are two constant 3-form backgrounds, with the 3-form having one index in a toroidal direction and the other two indices outside. This is of course the familiar way in which the two B-fields of type IIB originate from M-theory.

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