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Moduli of Vector Bundles on Curves with Parabolic Structures

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Introduction

Let H be the upper half plane and Γ a discrete subgroup of $\operatorname{Aut} H$. When $H \bmod \Gamma$ is *compact*, one knows that the moduli space of unitary representations of Γ has an algebraic interpretation (cf. [7] and [10]); for example, if moreover Γ acts *freely* on H, the set of isomorphism classes of unitary representations of Γ can be identified with the set of equivalence classes of *semi-stable* vector bundles of degree zero on the smooth projective curve $H \bmod \Gamma$, under a certain equivalence relation. The initial motivation for this work was to extend these considerations to the case when $H \bmod \Gamma$ has *finite measure.*

Suppose then that $H \bmod \Gamma$ has finite measure. Let X be the smooth projective curve containing $H \bmod \Gamma$ as an open subset and S the finite subset of X corresponding to parabolic and elliptic fixed points under Γ. Then to interpret algebraically the moduli of unitary representation of Γ, we find that the problem to be considered is the moduli of vector bundles V on X, endowed with additional structures, namely flags at the fibres of V at $P \in S$. We call these *quasi parabolic structures* of V at S and, if in addition we attach some weights to these flags, we call the resulting structures *parabolic structures* on V at S (cf. Definition 1.5). The importance of attaching weights is that this allows us to define the notion of a *parabolic degree* (generalizing the usual notion of the degree of a vector bundle) and consequently the concept of *parabolic semi-stable and stable vector bundles* (generalizing Mumford's definition of semi-stable and stable vector bundles). With these definitions one gets a complete generalization of the results of [7, 10, 12] and in particular an algebraic interpretation of unitary representations of Γ via parabolic semi-stable vector bundles on X with parabolic structures at S (cf. Theorem 4.1).

The basic outline of proof in this paper is exactly the same as in [12], however, we believe, that this work is not a routine generalization. There are some new aspects and the following are perhaps worth mentioning. One is of course the idea of parabolic structures; this was inspired by the work of Weil (cf. [16], p. 56). The second is a technical one but took some time to arrive at, namely when one

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reduces the problem of constructing the moduli space of parabolic semi-stable vector bundles to one of "geometric invariant theory", the choice of weights should correspond to the choice of a polarization. The moduli problem of parabolic vector bundles gives a natural example of how the same moduli problem can have many natural solutions (namely, corresponding to choice of different weights) and this is reflected in geometric invariant theory by the choice of different polarizations. (This feature also occurs in getting compactifications of generalized Jacobians associated to *reducible projective curves with ordinary double points* [8].) One is also obliged to give a proof, different from that of [15], for the fact that the moduli space of parabolic semi-stable vector bundles is *complete* (Theorem 3.1).

The notion of parabolic structures appears to be quite useful in many applications. It is very much related to that of "Hecke correspondences" appearing in the work of Narasimhan and Ramanan [6]. This has also been used to obtain a desingularization of the moduli variety of semi-stable vector bundles, of rank two and degree zero [14]. This was also responsible for suggesting in the general setting of geometric invariant theory, a result (cf. [11], Theorem 5.1), which states that if a reductive group G operates, say on a projective space $\mathbb{P}$ given by a linear representation of G, then there exists an ample line bundle on $\mathbb{P} \times G/B$ (G/B-flag variety associated to G) for which semi-stable points are in fact stable.

The results of this paper have been announced in [13].

Outline of the Paper

This paper is divided into 5 sections, we briefly describe their contents as follows:

Section 1 is devoted to the motivation for introducing the notion of parabolic structures on a vector bundle. If Γ is a discrete subgroup of $PSL(2, \mathbb{R})$ and $X = H^+/\Gamma$, we show how a unitary representation of Γ gives rise to a vector bundle on X with parabolic structures corresponding to the parabolic vertices of Γ. We also show that these unitary bundles are parabolic semi-stable, in a sense which is made precise. It is also shown that the category of these parabolic semi-stable bundles, with an appropriate notion of morphism, is an abelian category with every object having a Jordan-Holder series.

Section 2 is quite short and is devoted to proving that for the moduli of parabolic bundles, it is sufficient to cover those cases where the weights of the parabolic structure are non-zero and rational.

Section 3 is devoted to proving that the parabolic semi-stable functor is "complete" in the sense of Langton [3]. In other words, if we are given a family of vector bundles on X with parabolic structure and the general fibre is semi-stable, then the special fibre, if unstable, can be modified so as to become semi-stable. This theorem follows very closely the proof given by Langton for proving the properness of the functor of semi-stable sheaves in the higher dimensional case.

Section 4 gives the statement and method of proof for the existence of the moduli space. We imbed the set of all parabolic semi-stable bundles in an open subset of a suitable Hilbert Scheme, which has the usual properties, i.e. non-singular, irreducible and of a given dimension. We map this open set to a product of Grassmannians and Flag varieties. It is shown that the image of a stable bundle is a stable point (in the sense of Mumford [5]) and that a similiar property holds for semi-stable bundles. Hence by the methods of Geometric Invariant Theory and the properness theorem of Sect. 3 we get the existence of a normal moduli scheme which is projective. However as the "covariant" is not necessarily a closed immersion, we do not know the identifications on the parabolic bundles, i.e. we still have to show that two bundles have the same image in the moduli space if and only if they have the same associated graded.

In Sect. 4 the above problem is reduced to the existence of stable parabolic bundles by the device of introducing parabolic structures at extra points. This existence theorem, in turn, depends on computing the dimension of the moduli space in question.

In order to compute the dimension, we proceed as follows: We construct the moduli variety over a discrete valuation ring and show that the fibres are equidimensional. The dimension of the general fibre is then computed by first identifying the parabolic semi-stable bundles on X with the unitary representations of Γ and then computing, by hand, the dimension of the space of these unitary representation. This is done in Sect. 5.

1. Representations of Γ and Parabolic Structures

Let Γ be a discrete subgroup of $PSL(2,\mathbb{R})$, acting on the upper half plane H such that $H \bmod \Gamma$ has finite measure and Γ acts freely on H. Let H^+ denote the union of H and the parabolic cusps of Γ. Then $X = H^+ \bmod \Gamma$ is a compact Riemann surface, containing $Y = H \bmod \Gamma$. If $\sigma: \Gamma \to \mathrm{GL}(n, \mathbb{C})$ is a representation of Γ in a complex vector space E, the vector bundle $H \times E$ on E has the structure of a Γ-vector bundle the action given by $\gamma(z,v) = (z, \sigma(\gamma)v)$ for $\gamma \in \Gamma$, $z \in H$, and $v \in E$. The quotient of $H \times E$ by the action of Γ is a vector bundle of rank n over Y, whose sections are in one-one correspondence with the Γ-invariant sections of $H \times E$. The representation σ also defines a Γ-vector bundle structure on $H^+ \times E$, which we shall see also yields a vector bundle on $X = H^+ \bmod \Gamma$. These Γ-invariant sections can be interpreted in terms of the geometry of the upper half plane by studying their behaviour explicitly at the parabolic cusps. We are interested in the case where the representation σ is unitary.

Let $P \in X - Y$ be a *parabolic cusp*. (We shall call a point of X a parabolic cusp if it corresponds to a parabolic cusp of H^+.) By supposing P to be the point at ∞, (which we can do without loss of generality) we may represent a suitable neighbourhood of P by a set of the form U/Γ_∞, where $U = \{z = x + iy \mid y \geqq \delta, \delta\alpha + ve$ constant$\}$ and Γ_∞ is generated by an element of the form $z \to z + \alpha$, α real, we may take $\alpha = 1$ for simplicity. We then make the following:

Definition 1.1. Let $\sigma: \Gamma \to \mathrm{GL}(E)$ be a unitary representation and $\mathbf{V}$ the Γ-vector bundle $H \times E$ on H (we shall call such a bundle a *unitary Γ-bundle*). Then the map $F: H \to H \times E$ given by $F(z) = (z, f(z))$ is said to be a *holomorphic* Γ-invariant section in H^+ if

a) F is holomorphic and Γ-invariant in H, and

b) representing a parabolic vertex P as above. f is bounded in every region of the form $z = x + iy$, $|x| \leqq \alpha$, $y \geqq \delta > 0$ for all α and for all strictly positive δ. (We note that since f is Γ-invariant, it suffices to assume the boundedness of f for some α such that $\alpha \geqq 1$, or the boundedness of f in the whole of the region $\{z=x+iy \mid y \geqq \delta > 0\}$; the conditions are clearly equivalent.)

To see that this definition is the right one for Γ-invariance of a section in H^+, we study it closely in terms of its representation on E. Let σ_∞ be the restriction of σ to the isotropy group Γ_∞ at ∞. Then σ_∞ is determined by its value on $A=$ the generator of Γ_∞ which is given by $z \to z+1$. By choosing a basis for E, we can write

$$\sigma_\infty(A) = \begin{pmatrix} \exp(2\pi i\alpha_1) & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi i\alpha_n) \end{pmatrix},$$

with $0 \leqq \alpha_i < 1$, $1 \leqq i \leqq n$.

Let $f(z) = (f_1(z), \ldots, f_n(z))$ be the representation of f with respect to this basis. Since F is Γ-invariant it is Γ_∞-invariant, and hence

$$\begin{pmatrix} f_1(z+1) \\ \vdots \\ f_n(z+1) \end{pmatrix} = \begin{pmatrix} \exp(2\pi i\alpha_1) & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi i\alpha_n) \end{pmatrix} \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix}$$

i.e.

$$f_j(z+1) = \exp(2\pi i\alpha_j) f_j(z), \qquad 1 \leqq j \leqq n, \quad \text{or}$$
$$f_j(z) = \exp(2\pi i\alpha_j z)\, g_j(\tau),$$

where $\tau = \exp(2\pi iz)$ and g_j is holomorphic in a punctured disc around P; τ is the local parameter at P on the compact Riemann surface X.

In terms of this representation of F, the condition b) in Definition 1.1 is equivalent to saying that g_j is bounded in a punctured disc C around P, which implies that g_j is holomorphic in C. Thus we have shown that the definition is equivalent to saying that F corresponds to a section of the vector bundle on X defined by the Γ-vector bundle $H^+ \times E$ on H^+.

This leads us to define the sheaf of sections of the vector bundle $p_*^\Gamma(\mathbf{V})$ on X as follows:

On $Y = H \bmod \Gamma$ the sheaf is given by $p_*^\Gamma(\mathbf{V})$, where $p: H \to Y$ is the canonical morphism, with the usual meaning for $p_*^\Gamma(\mathbf{V})$; i.e. the sections over Y of this sheaf are the Γ-invariant sections of $\mathbf{V} = H \times E$. For any parabolic vertex $P \in X$ and a neighbourhood U of P of the form H_δ/Γ_∞, where $H_\delta = \{z = x+iy \mid y \geqq \delta > 0\}$, we have $p_*^\Gamma(\mathbf{V})(U) =$ the "bounded" Γ_∞-invariant sections of $\mathbf{V}$ on H_δ. From the above considerations it follows that the Γ_∞-invariant sections $z \to \exp(2\pi i\alpha_i z)e_j$ form a basis for $p_*^\Gamma(\mathbf{V})_P$ as an $\mathcal{O}_{X,P}$ module; here e_j is a basis for E and hence also for the sections of $H \times E$, and σ, the generator of Γ_∞ acts on e_j by $\sigma(e_j) = \exp(2\pi i\alpha_i)e_j$. In fact these sections $z \to \exp(2\pi i\alpha_i z)e_j$ generate $p_*^\Gamma(\mathbf{V})(U)$ for any suitable neighbourhood U of P.

Let E^Γ denote the Γ-invariant points in E; then any point in E^Γ defines a Γ-invariant section of $\mathbf{V}$ in H^+, namely the "constant section" defined by this point.

Proposition 1.2. *Let $\mathbf{V}$ be a unitary Γ-vector bundle on H associated to a unitary Γ-module by $\sigma: \Gamma \to \mathrm{GL}(E)$. Then the canonical homomorphism $j: E^\Gamma \to H^0(p_*^\Gamma(\mathbf{V}), X)$*

which associates to any Γ-invariant point a Γ-invariant section of $\mathbf{V}$ *in* H^+ *i.e. a global section of* $p_*^\Gamma(\mathbf{V})$, *is an isomorphism.*

Proof. The map j is clearly a monomorphism. To show that it is surjective, we see easily that if $F: H \to H \times E$ given by $F(z) = (z, f(z))$ is Γ-invariant in H^+, then f is a constant i.e. $f(z)$ is independent of z. For, if we define $g(z)$ by

$$g(z) = \|f(z)\|^2 = \sum_k |f_k(z)|^2,$$

then g is a real-valued positive function on H and since F is Γ-invariant and E is a unitary Γ-module, we see that g is invariant under Γ. Hence g descends to a function $h: Y \to \mathbb{R}^+$. We see also that g is continuous at ∞ owing to its boundedness on any region of the form $\{z = x + iy \mid x \leqq \alpha,\ y \geqq \delta > 0\}$ which represents a neighbourhood of a parabolic cusp. This implies that h can be extended to a continuous function $X \to \mathbb{R}$. Being locally a sum of functions of the form $|f_k|^2$, for f_k holomorphic, h is subharmonic. Now since X is compact h reduces to a constant which in turn implies that f itself is constant. Hence F comes from a Γ-invariant point of E, namely the constant value of $f(z)$, showing the surjectivity of j.

Definition 1.3. Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be two Γ-vector bundles associated to unitary Γ-modules E_1 and E_2. We say that $F: \mathbf{V}_1 \to \mathbf{V}_2$ is a *Γ-homomorphism* in H^+ if

a) it is a holomorphic Γ-homomorphism in H, and

b) at every parabolic vertex assumed without loss of generality to be ∞, $F: H \times E_1 \to H \times E_2$ represented by $F(z) = (z, f(z))$ where each $f(z): E_1 \to E_2$ is a homomorphism of E_1 into E_2, satisfies the condition that f is bounded in the region $H_\delta = \{z = x + iy \mid y \geqq \delta > 0\}$ for every $\delta > 0$ (or even some $\delta > 0$).

Looking closely at this definition, we see that the family $f(z): E_1 \to E_2$, $z \in H$, of homomorphisms satisfies (since F is a Γ-homomorphism) the following properties:

$$f(\gamma z)(\gamma v) = \gamma[f(z)v] \quad \text{for} \quad \gamma \in \Gamma, v \in E_1$$

or

$$f(\gamma z)(\gamma v) = [\gamma f(z) \gamma^{-1}](\gamma v), \gamma \in \Gamma, v \in E,$$

or

$$f(\gamma z) = \gamma f(z) \gamma^{-1}.$$

In terms of the actions of Γ on E_1 and E_2, we see that the action of Γ on $\mathrm{Hom}(E_1, E_2)$ is given by $\gamma(g) = \gamma(g)\gamma^{-1}$, so that the above condition is equivalent to the condition that $f: H \to \mathrm{Hom}(E_1, E_2)$ be a Γ-invariant map, i.e., $f(\gamma z) = \gamma f(z)$. This means that F defines a Γ-invariant section of Γ-vector bundle $(\mathbf{V}_1^* \otimes \mathbf{V}_2) = \mathrm{Hom}(\mathbf{V}_1, \mathbf{V}_2)$. Then the condition b) in Definition 1.3 states that the Γ-invariant section of $\mathbf{V}_1^* \otimes \mathbf{V}_2$ defined by F is in fact bounded at ∞, and is hence a Γ-invariant section of $\mathbf{V}_1^* \otimes \mathbf{V}_2$ on H^+. By Proposition 1.1 this reduces to a constant. Hence we deduce:

Corollary 1.4. *If* $\mathbf{V}_1$ *and* $\mathbf{V}_2$ *are two unitary Γ-vector bundles associated to unitary Γ-modules* E_1 *and* E_2, *then the map* $j: \mathrm{Hom}(E_1, E_2) \to \mathrm{Hom}(\mathbf{V}_1, \mathbf{V}_2)$ *is an isomor-*

phism. Thus the functor (Unitary Γ-modules)$\to$(Unitary Γ-bundles) given by $E \to \mathbf{V}=$ the Γ-vector bundle $H^+ \times E$ on H^+, is full and faithful.

Looking at Γ-homomorphisms on H^+ of $\mathbf{V}_1$ into $\mathbf{V}_2$ in terms of the vector bundles $p_*^\Gamma(\mathbf{V}_1)$ and $p_*^\Gamma(\mathbf{V}_2)$ defined on X, we consider their behaviour in a neighbourhood of $P \in X-Y$ coming from a parabolic vertex, say ∞. Let Γ_∞ be generated by $\gamma: z \to z+1$, $U_0 = H_\delta \bmod \Gamma_\infty = a$ punctured disc around 0, $U = U_0 \cup (0)$; U represents a neighbourhood of P in X. We choose bases $(e_1 \dots e_m)$ and $(d_1 \dots d_n)$ of E_1 and E_2 respectively, and set

$$\gamma(e_j) = \exp(e^{2\pi i \alpha_j}) e_j, \qquad 1 \leqq j \leqq m,$$

$$\gamma(d_k) = \exp(e^{2\pi i \beta_k}) d_k, \qquad 1 \leqq k \leqq n.$$

We order the α_j and β_k in ascending order,

$$0 \leqq \alpha_1 \leqq \alpha_2 \dots \leqq \alpha_m < 1, \qquad 0 \leqq \beta_1 \leqq \beta_2 \dots \leqq \beta_n < 1.$$

If U is sufficiently small, we have natural bases $\theta_j = z \to \exp(2\pi i \alpha_j z) e_j$, $1 \leqq j \leqq m$ and $\Psi_k = z \to \exp(2\pi i \beta_k z) d_k$, $1 \leqq k \leqq n$ for $p_*^\Gamma(\mathbf{V}_1)$ and $p_*^\Gamma(\mathbf{V}_2)$ in U respectively. Representing the Γ-homomorphism $F: H \times E_1 \to H \times E_2$ by $F(z) = (z, f(z))$, $f(z): E_1 \to E_2$ with respect to e_j and d_k is given by

$$f(z)(e_j) = \sum_{k=1}^{n} f_{jk}(z) d_k,$$

the $f_{jk}(z)$ being a $m \times n$ matrix. The Γ-homomorphism condition on F requires

$$f(\gamma z) = \gamma f(z) \gamma^{-1}$$

or

$$f(\gamma z) \exp(2\pi i \alpha_j) e_j = \gamma \left(\sum_{k=1}^{n} f_{jk}(z) \exp(2\pi i \beta_k) d_k \right)$$

that is,

$$f_{jk}(\gamma z) = \exp(-2\pi i [\alpha_j - \beta_k]) f_{jk}(z)$$

or

$$f_{jk}(z+1) = \begin{pmatrix} \exp(-2\pi i \alpha_1) & & 0 \\ & \ddots & \\ 0 & & \exp(-2\pi i \alpha_m) \end{pmatrix} (f_{jk}(z)) \begin{pmatrix} \exp(2\pi i \beta_1) & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi i \beta_n) \end{pmatrix}.$$

It follows that $f_{jk}(z) = \exp(-2\pi i [\alpha_j - \beta_k] z) g_{jk}(\tau)$, where $\tau = \exp(2\pi i z)$. The fact that f_{jk} is bounded in $H_\delta = \{z = x + iy \mid y \geqq \delta > 0\}$ implies that g_{jk} *is holomorphic at* $\tau = 0$ and in fact, if $-\alpha_j + \beta_k < 0$, then $g_{jk}(0) = 0$. Thus $g_{jk}(\tau)$ represents the Γ-homomorphism of $p_*^\Gamma(\mathbf{V}_1)$ into $p_*^\Gamma(\mathbf{V}_2)$ at P.

We see thus that a Γ-homomorphism on H^+ of $\mathbf{V}_1$ into $\mathbf{V}_2$ is one which defines a homomorphism of $p_*^\Gamma(\mathbf{V}_1)$ into $p_*^\Gamma(\mathbf{V}_2)$ whose matrix $g_{jk}(\tau)$ in a neighbourhood of a parabolic vertex $P \in X - Y$ satisfies $g_{jk}(0) = 0$ whenever $-\alpha_j + \beta_k < 0$, where the α_j, β_k, and g_{jk} arise from the local representation of the bundles and the homomorphism at P. Conversely given a homomorphism of $p_*^\Gamma(\mathbf{V}_1)$ into $p_*^\Gamma(\mathbf{V}_2)$ which satisfies the condition $g_{jk}(0) = 0$ whenever $-\alpha_j + \beta_k < 0$, at parabolic cusps as above, we get a Γ-homomorphism of $\mathbf{V}_1$ into $\mathbf{V}_2$ on H^+.

We can interpret the condition for a Γ-homomorphism on H^+ [or for a homomorphism of $p_*^\Gamma(\mathbf{V}_1)$ into $p_*^\Gamma(\mathbf{V}_2)$] geometrically as follows. Take first the case of a Γ-isomorphism. Since F is a Γ-isomorphism, the modules E_1 and E_2 must be equivalent. It then follows that $m=n$ and $\alpha_1=\beta_1$, $\alpha_2=\beta_2 \ldots \alpha_n=\beta_n$, once we have assumed the normalization $0 \leqq \alpha_1 \leqq \alpha_2 \ldots < 1$ and $0 \leqq \beta_1 \leqq \beta_2 \ldots < 1$. Writing $\mathbf{W}_1 = p_*^\Gamma(\mathbf{V}_1)$ and $\mathbf{W}_2 = p_*^\Gamma(\mathbf{V}_2)$ as product bundles $U \times W_1$ and $U \times W_2$ in a neighbourhood of P, F induces the linear map:

$$g(0):(W_1)_P \to (W_2)_P \quad \text{at } P,$$

which we have written as the matrix $(g_{jk}(0))$ in terms of the basis chosen for W_1 and W_2 at P. The $n \times n$ matrix $(g_{jk}(0))$ has the property that $g_{jk}(0)=0$ whenever $-\alpha_j+\alpha_k$ 0 i.e. $\alpha_j > \alpha_k$. Thus $g_{jk}(0)=0$ whenever $\alpha_j > \alpha_k$.

Let the (α_j) have distinct elements:

$$\alpha_1 = \alpha_2 \ldots = \alpha_{k_1}$$

$$\alpha_{k_1+1} = \alpha_{k_1+2} \cdots = \alpha_{k_2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots = \alpha_{k_r}.$$

Consider the decreasing flag in $(W_1)_P$ defined by

$$F_1(W_1)_P = (W_1)_P$$

$$F_2(W_1)_P = \text{subspace spanned by } \theta_{k_1+1}, \ldots, \theta_n$$

$$F_3(W_1)_P = \text{subspace spanned by } \theta_{k_2+1}, \ldots, \theta_n \text{ etc.}.$$

We see that the condition $g_{jk}(0)=0$ whenever $\alpha_j > \alpha_k$ means that

$$g(0)[F_s(W_1)_P] \subset F_s(W_2)_P,$$

or that $g(0)$ preserves the flag structure at P. Hence a Γ-isomorphism of $\mathbf{V}_1$ to $\mathbf{V}_2$ on H^+ is the same as an isomorphism of $\mathbf{W}_1$ to $\mathbf{W}_2$ which preserves the flag structure at each parabolic cusp. We note that the flag structure is given by the weights α_j normalised in the order

$$0 \leqq \alpha_1 = \ldots = \alpha_{k_1} < \alpha_{k_1+1} = \ldots = \alpha_n < 1.$$

Now in the general case of a Γ-homomorphism, the α_j and β_k will differ. Introducing the flag structures on $\mathbf{W}_1$ and $\mathbf{W}_2$ at P, we see that a Γ-homomorphism is equivalent to a homomorphism G of $\mathbf{W}_1$ into $\mathbf{W}_2$ which has the property that at P, denoting by g_P the map $(\mathbf{W}_1)_P \to (\mathbf{W}_2)_P$, we have $g_P[F_j(W_1)_P] \subset F_{k+1}(W_2)_P$ whenever $\alpha_j > \beta_k$, where α_j are defined as

$$\alpha_1 = \alpha_2 \ldots = \alpha_{k_1} = \alpha_1$$

$$\alpha_{k_1+1} = \ldots = \alpha_{k_2} = \alpha_2 \quad \text{etc.}$$

and similarly for the β_k. We have $k_1 = \dim F_1(W_1)_P - \dim F_2(W_1)_P, \ldots, k_r = \dim F_r(W_1)_P$. This leads us to define parabolic structures:

Definition 1.5. Let X be a compact Riemann surface with a finite set of points $P_1, \ldots, P_n$ and $\mathbf{W}$ a vector bundle on X.

I) A *parabolic structure* on $\mathbf{W}$ is, giving at each P_i,

a) a flag $\mathbf{W}_P = F_1\mathbf{W}_P \supset F_2\mathbf{W}_P \ldots \supset F_r\mathbf{W}_P$,

b) weights $\alpha_1, \ldots, \alpha_r$ attached to $F_1\mathbf{W}_P, \ldots, F_r\mathbf{W}_P$ such that $0 \leqq \alpha_1 < \alpha_2 \ldots < \alpha_r < 1$.

We call $k_1 = \dim F_1\mathbf{W}_P - \dim F_2\mathbf{W}_P, \ldots, k_r = \dim F_r\mathbf{W}_P$ the *multiplicities* of $\alpha_1, \ldots, \alpha_r$.

II) A *morphism* $G: \mathbf{W}_1 \to \mathbf{W}_2$ of *parabolic vector bundles* (i.e. vector bundles with parabolic structures) is a homomorphism of $\mathbf{W}_1$ into $\mathbf{W}_2$ such that for any P, denoting G on the fibre at P by g_P, we have $g_P[F_i(\mathbf{W}_1)_P] \subset F_{j+1}(\mathbf{W}_2)_P$ whenever $\alpha_i > \beta_j$.

III) A *quasi-parabolic structure* is just condition a) above.

We note that since the definition of a morphism of parabolic vector bundles depends on the weights, only isomorphisms of quasi-parabolic bundles can be defined.

We also remark that the above definitions can be given for a smooth projective algebraic curve over an arbitrary field k. However if k is not algebraically closed, then quasi-parabolic structures on vector bundles on X will always be assumed to be concentrated at k-rational points of X. Weights and multiplicities for a quasi-parabolic structures are defined as in I b) above.

Sub and Quotient Γ-Bundles

Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be unitary Γ-bundles, given by unitary Γ-modules E_1 and E_2.

Definition 1.6. A Γ-homomorphism $F: V_1 \to V_2$ identifies $\mathbf{V}_1$ as a *Γ-sub bundle* of $\mathbf{V}_2$ if F is injective in H and $F(\infty)$, its value over any parabolic cusp, is also injective.

Thus $\mathbf{V}_1$ is a Γ-sub-bundle of $\mathbf{V}_2$ if it is a sub-bundle of $\mathbf{V}_2$ through a Γ-homomorphism F. Representing $F(\infty)$ at a parabolic cusp on H^+ as a matrix $f_{jk}(\infty)$ and applying the Γ_∞-invariance criterion we get,

$$f_{jk}(\infty) = \begin{pmatrix} \exp(2\pi i\beta_1) & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi i\beta_n) \end{pmatrix} (f_{jk}(\infty)) \begin{pmatrix} \exp(-2\pi i\alpha_1) & & 0 \\ & \ddots & \\ 0 & & \exp(-2\pi i\alpha_m) \end{pmatrix}.$$

Thus the $(m \times n)$ matrix $f_{jk}(\infty)$ must have rank m, and from the above equation we deduce that $f_{jk}(\infty) = 0$ whenever $\alpha_j = \beta_k$. This means that $\{\alpha_j\}$ is a subset of $\{\beta_k\}$. Let $G: \mathbf{W}_1 \to \mathbf{W}_2$ be the homomorphism of bundles on X given by $F: \mathbf{V}_1 \to \mathbf{V}_2$. As F is a Γ-sub bundle homomorphism, we see that

a) G is a sub bundle homomorphism,

b) given j_0 and taking the greatest k_0 such that $G[F_{j_0}(\mathbf{W}_1)_P] \subset F_{k_0}(\mathbf{W}_2)_P$, we have the weight of the flag $F_{k_0}(\mathbf{W}_2)_P =$ the weight of the flag $F_{j_0}(\mathbf{W}_1)_P$ i.e. $\alpha_{j_0} = \beta_{k_0}$.

Hence we define the notion of parabolic subbundle;

Definition 1.7. A parabolic vector bundle $\mathbf{W}_1$ on X is a *parabolic sub bundle* of a parabolic vector bundle $\mathbf{W}_2$ on X, if

a) $\mathbf{W}_1$ is a sub bundle of $\mathbf{W}_2$ and

b) at each parabolic vertex P, the weights of $\mathbf{W}_1$ are a proper subset of those of $\mathbf{W}_2$. Further, given $1 \leqq j_0 \leqq m$, and taking the greatest k_0 such that $F_{j_0}(\mathbf{W}_1)_P \subset F_{k_0}(\mathbf{W}_2)_P$, the weight $\alpha_{j_0} =$ the weight β_{k_0}.

We have a similar description, leading to a similar definition for *quotient parabolic bundles.*

Definition 1.8. A homomorphism $G:\mathbf{W}_1\to\mathbf{W}_2$ of parabolic bundles makes $\mathbf{W}_2$ a *quotient parabolic bundle* of $\mathbf{W}_1$ if

a) $\mathbf{W}_2$ is a quotient bundle of $\mathbf{W}_1$ under the homomorphism G, and

b) at every parabolic vertex P, for $1\leqq k_0\leqq n$, let j_0 be the largest j such that $G[F_{j_0}(\mathbf{W}_1)_P]=F_{k_0}(\mathbf{W}_2)_P$ [that is $G(F_{j_0+1}(\mathbf{W}_1)_P)\neq F_{k_0}(\mathbf{W}_2)_P$]. Then the weight α_{j_0} of $F_{j_0}(\mathbf{W}_1)_P=$ the weight β_{k_0} of $F_{k_0}(\mathbf{W}_2)_P$.

We remark that if $\mathbf{W}_2$ is a parabolic vector bundle or X and $\mathbf{W}_1$ a sub-bundle of $\mathbf{W}_2$ in the usual sense, then we can define a canonical parabolic structure on $\mathbf{W}_1$ which makes it a patabolic sub-bundle of $\mathbf{W}_2$. This is possible because we get a canonical "induced" flag in $(\mathbf{W}_1)_P$ from that of $(\mathbf{W}_2)_P$ and we attach the same weights to the induced flag. We proceed similarly in the quotient situation. Thus, given an exact sequence of vector bundles $0\to\mathbf{W}_1\to\mathbf{W}\to\mathbf{W}_2\to 0$, with $\mathbf{W}$ parabolic, we get a parabolic structure on $\mathbf{W}_1$ and $\mathbf{W}_2$ which makes it an *exact sequence of parabolic vector bundles.* (We call $0\to\mathbf{W}_1\to\mathbf{W}\to\mathbf{W}_2\to 0$ an *exact sequence of parabolic vector bundles* if it is an exact sequence of vector bundles, $\mathbf{W}_1$ is a parabolic sub-bundle of $\mathbf{W}$ and $\mathbf{W}_2$ a parabolic quotient bundle of $\mathbf{W}$.)

Parabolic Degree

Let L be a complex line bundle on X. Its first Chern class determines an element of $H^2(X;\mathbb{Z})$ which is canonically isomorphic to $\mathbb{Z}$. The integer associated to L in this fashion is called the *degree of* L. The degree of L can be computed by taking a non zero meromorphic section s of L and taking the algebraic sum $\sum_{P\in X}\mathrm{Ord}_P s$ of the orders of s at point of X.

It is well known that this sum is independent of the meromorphic section s. Now if $\mathbf{V}$ is a vector bundle on X of rank n, we define $\deg\mathbf{V}=\deg\left(\bigwedge^n\mathbf{V}\right)$, where $\bigwedge^n V$ is the nth exterior power of $\mathbf{V}$.

Let now $\mathbf{V}$ be a unitary Γ-line bundle on H^+ defined by $\sigma:\Gamma\to\mathrm{GL}(1)$ for a character σ. Let β_Q be the generator of the isotopy group Γ_Q at a parabolic cusp Q. Then $\sigma(\beta_Q)=\exp(2\pi i\theta_Q)$ for some θ_Q, $0\leqq\theta_Q<1$. Put $W=p_*^\Gamma(\mathbf{V})$.

Proposition 1.9. $\mathrm{Deg}\,\mathbf{W}=-\sum\theta_Q$, *where the summation is taken over all the parabolic cusps Q in X.*

Proof. Let f be a meromorphic section of $\mathbf{V}$ in H^+. (Such a section exists because a meromorphic section of $\mathbf{W}$ exists.) Then f is a meromorphic function such that $f(sz)=\sigma(s)f(z)$ for $s\in\Gamma$. Then df/f is a 1-form which is Γ invariant and hence defines a meromorphic 1-form on X. Now for any point $Q\in H$,

$$\mathrm{Res}_P\frac{df}{f}=\mathrm{Ord}_Q f,\quad\text{where}\quad Q\to P\in X.$$

For any $Q \in H^+ - H$, $f = \tau^\theta g(\theta)$ where $\tau = \exp(2\pi i z)$ is the local parameter at $P \in X$ corresponding to Q. Hence we get

$$\mathrm{Res}_P\left(\frac{df}{f}\right) = d[\tau^\theta g(\theta)]/g(\tau) = \mathrm{Ord}_P g(\tau) + \theta.$$

At a point $P \in X$ such that P comes from $Q \in H$, the section g of $\mathbf{W}$ on X is the same as f and $\mathrm{Ord}_Q f = \mathrm{Ord}_P g$, whereas at a parabolic cusp P, we have

$$\mathrm{Ord}_Q f = \mathrm{Ord}_P g + \theta.$$

Now $\sum_{P \in X} \mathrm{Res}_P(df/f) = 0$. Hence $\sum_{Q \to P, Q \text{ parabolic}} \theta_Q + \sum_{P \in X} \mathrm{Ord}_P g = 0$. But $\sum_{P \in X} \mathrm{Ord}_P g$ is the deg of $\mathbf{W}$, which proves the proposition.

Corollary 1.10. *Let $\mathbf{V}$ be a unitary Γ-bundle, $W = p_*^\Gamma(\mathbf{V})$ whose weights at a parabolic vertex P are $\alpha_1, \ldots, \alpha_r$ with multiplicities $k_1, \ldots, k_r$. Then*

$$\deg \mathbf{W} + \sum_{P \text{ parabolic}} (k_1\alpha_1 + \ldots + k_r\alpha_r) = 0.$$

Proof. This is immediate from Proposition 1.9 and the definition of $\deg \mathbf{W}$ as $\deg\left(\bigwedge^n \mathbf{W}\right)$.

The above corollary leads us to define the parabolic degree of a parabolic vector bundle.

Definition 1.11. Let $\mathbf{W}$ be a parabolic vector bundle on X, with weights at a parabolic vertex $P \in X$ given by $\alpha_1 \ldots \alpha_r$, whose multiplicities are $k_1 \ldots k_r$. Then the *parabolic degree of W* is defined by

$$\operatorname{par\,deg} \mathbf{W} = \deg W + \sum_P \left(\sum_i k_i \alpha_i\right).$$

Thus if $\mathbf{W}$ comes from a unitary Γ-bundle, then $\operatorname{par\,deg} \mathbf{W} = 0$. Note that parabolic degree is defined even if the curve X is defined over a field which is not algebraically closed.

Proposition 1.12. *Let $\mathbf{V}$ be a Γ-vector bundle on H^+ associated to a unitary Γ-module E. If $\mathbf{W}$ is a Γ-subbundle of $\mathbf{V}$ on H^+ which is locally unitary (cf. Definition 1.13), then we have* $\operatorname{par\,deg} p_*^\Gamma(\mathbf{W}) \leqq 0$. *If E is an irreducible unitary Γ-module, then* $\operatorname{par\,deg} p_*^\Gamma(W) < 0$.

Proof. By taking $\bigwedge^k \mathbf{W} \subset \bigwedge^k \mathbf{V}$, where $k = \operatorname{rk} \mathbf{W}$, we can assume that $\operatorname{rk} \mathbf{W} = 1$. Now $\bigotimes^m \mathbf{W} \subset \bigotimes^m \mathbf{V}$ and $\bigotimes^m \mathbf{V}$ is again unitary. Assume that $\operatorname{par\,deg} p_*^I(W) > 0$. If f is a Γ-invariant meromorphic section of $\mathbf{W}$, then $\operatorname{par\,deg} p_*^\Gamma(\mathbf{W}) = \sum_{P \in R} \mathrm{Ord}_P f + \sum_{P \text{ parabolic}} \theta_P$, where R is any subset of H^+ mapping bijectively onto X and θ_P is defined at each parabolic vertex P by $f(z) = \tau^{\theta_P} g(\tau)$, τ being the local parameter for $P \in X$. It follows that as f^m is a Γ-invariant meromorphic section of $\bigotimes^m \mathbf{W}$, we have $\operatorname{par\,deg} p_*^\Gamma\left(\bigotimes^m W\right) = m \operatorname{par\,deg} p_*^\Gamma(\mathbf{W})$ for all positive m. So replacing $\mathbf{V}$ and $\mathbf{W}$ by $\bigotimes^m V$ and $\bigotimes^m \mathbf{W}$, we may assume that $\operatorname{par}\deg p_*^\Gamma(\mathbf{W}) \gg 0$. Hence $\deg p_*^\Gamma(\mathbf{W}) \gg 0$ and consequently $p_*^\Gamma(\mathbf{W})$ has a non constant section s which vanishes at least at one point of X. But then s is a global section of $p_*^\Gamma(\mathbf{V})$ which is identified with E, and hence S is a constant, which is a contradiction. Hence $\operatorname{par}\deg p_*^\Gamma(\mathbf{W}) \leqq 0$.

We now show that $\operatorname{par}\deg p_*^\Gamma(W) < 0$ if E is irreducible as a Γ-module. Again we may assume that W is a Γ-line bundle on X. If $\operatorname{par}\deg p_*^\Gamma(\mathbf{W}) = 0$, then a generalisation of Abel's theorem enables us to conclude that $p_*^\Gamma(\mathbf{W})$ is obtained from a unitary character of Γ. Without loss of generality we may assume $\mathbf{W}$ is the trivial Γ-line bundle on H^+. This implies that we have a Γ-invariant section s of $\bigwedge^k \mathbf{W}$ which is *non zero every where*, which can be identified with an every where non zero Γ-invariant section s of $\bigwedge^k \mathbf{V}$. As $\bigwedge^k \mathbf{V}$ is unitary, s is given by a Γ-invariant element of $\bigwedge^k E$. It is easy to see that s is a decomposable element of $\bigwedge^k E$ because at each point P of H, $s(P)$ is a decomposable element of $\left(\bigwedge^k \mathbf{W}\right)_P$ as all elements of $\left(\bigwedge^k \mathbf{W}\right)$ are decomposable. Hence $s(P)$ is a decomposable element in $\left(\bigwedge^k \mathbf{V}\right)_P = \bigwedge^k E$ for each P. Let $s = s \wedge \ldots \wedge s_k$, with $s_i \in E$. Now s being Γ-invariant, the subspace F of E spanned by the s_i is s stable under Γ. This contradicts the irreducibility of E as a Γ-module, unless $F = E$. Hence $\operatorname{par}\deg p_*^\Gamma(\mathbf{W}) < 0$. This proves Proposition 1.12.

As a consequence of Proposition 1.12 and following Mumford (cf. [4]), we make the following

Definition 1.13. i) Let $\mathbf{V}$ be a *locally unitary* Γ-bundle on H^+ (i.e. V is a Γ-vector bundle on H^+ defined at each parabolic cusp P by a unitary representation of Γ_P). Then $\mathbf{V}$ is *Γ-stable (Γ-semi stable)* if $\forall \Gamma$-sub bundle $\mathbf{W}$ of $\mathbf{V}$ in H^+ we have

$$\frac{\operatorname{par}\deg p_*^\Gamma(\mathbf{W})}{\operatorname{rk} p_*^\Gamma(\mathbf{W})} < \frac{\operatorname{par}\deg p_*^\Gamma(\mathbf{V})}{\operatorname{rk} p_*^\Gamma(\mathbf{V})} \text{(resp. } \leqq \text{)}.$$

ii) Equivalently, a parabolic vector bundle $\mathbf{V}$ on X is said to be *parabolic stable (parabolic semi-stable)* if for every parabolic sub-bundle W of V we have:

$$\frac{\operatorname{par}\deg \mathbf{W}}{\operatorname{rk} \mathbf{W}} < \frac{\operatorname{par}\deg \mathbf{V}}{\operatorname{rk} \mathbf{V}} \text{(resp. } \leqq \text{)}.$$

Proposition 1.12 shows that a unitary Γ-bundle $\mathbf{V}$ is parabolic semi-stable and is in fact parabolic stable if the Γ-module E is irreducible.

Remark 1.14. If V is a parabolic vector bundle on a curve X which is defined over an arbitrary field k, then V is defined to be stable (semi-stable) if for some algebraically closed overfield Ω of k, V_Ω is stable (semi-stable) over X_Ω where $V_\Omega = V \times_k \Omega$ and $X_\Omega = X \times_k \Omega$. Note that the parabolic structure on V extends uniquely to a parabolic structure on V_Ω. Note also that the definition of stability (semi-stability) given above is independent of the field Ω (cf. Proposition 3, Sect. 2, [3]).

We shall prove in Sect. 5 that a stable parabolic vector bundle on $X = H^+/\Gamma$ comes from an irreducible unitary representation of Γ. The category of unitary Γ-vector bundles is isomorphic to the category of unitary Γ-modules, and is hence abelian. The following proposition confirms this result.

Proposition 1.15. *Let* $\mathbf{S}$ *be the category of all parabolic semi-stable vector bundles on* X *of parabolic degree* 0. *Then* $\mathbf{S}$ *is abelian. Over* $\mathbb{C}$, *the category* $\mathbf{S}$ *of* Γ*-semi stable vector bundles on* H^+ *is abelian and every homomorphism* $f: \mathbf{V}_1 \to \mathbf{V}_2$ *in* $\mathbf{S}$ *is of constant rank at every point of* H^+.

Proof. We shall only prove the first statement. Let $f: \mathbf{V} \to \mathbf{W}$ be a morphism in $\mathbf{S}$. Then f can be factored as

$$\begin{array}{c} 0 \to \mathbf{V}_1 \to \mathbf{V} \to \mathbf{V}_2 \to 0 \\ \qquad\qquad\quad \downarrow j \\ 0 \leftarrow \mathbf{W}_2 \leftarrow \mathbf{W} \leftarrow \mathbf{W}_1 \quad 0 \end{array} \tag{1}$$

where j is of maximal rank, i.e. $\bigwedge^k j \neq 0$ where $k = \operatorname{rk} \mathbf{V}_2 = \operatorname{rk} \mathbf{W}_1$.

Now as $\operatorname{par deg} \mathbf{V} = 0$ and $\mathbf{V}$ is semi-stable, we have $\operatorname{par deg} \mathbf{V}_1 \leqq 0$ and hence $\operatorname{par deg} \mathbf{V}_2 \geqq 0$. Similarly, $\operatorname{par deg} \mathbf{W}_1 \leqq 0$. Now the weights of $\mathbf{V}_2$ = weights of $\mathbf{W}_1$ and hence $\operatorname{par deg} \mathbf{V}_2 = \operatorname{par deg} \mathbf{W}_1$, which forces $\operatorname{par deg} \mathbf{V}_2 = \operatorname{par deg} \mathbf{W}_1 = 0$ and $\deg \mathbf{V}_2 = \deg \mathbf{W}_1$. It easily follows now that j is an isomorphism. In particular f is of constant rank at every point of X and f has both a kernel and cokernel. It also follows from (1) that $\mathbf{V}_1 = 0$ if $\mathbf{V}$ is stable and that $\mathbf{W}_1 = \mathbf{W}$ if $\mathbf{W}$ is stable. In particular f is an isomorphism if both $\mathbf{V}$ and $\mathbf{W}$ are stable and every non zero endomorphism of a stable object in $\mathbf{S}$ is an automorphism.

Remark 1.16. Let the category $\mathbf{S}$ be as above. If $\mathbf{V} \in \mathbf{S}$, then it is easily seen that there exists a *filtration* of $\mathbf{V}$,

$$\mathbf{V} = \mathbf{V}_n \supset \mathbf{V}_{n-1} \supset \dots \mathbf{V}_1 \supset \mathbf{V}_0 = 0,$$

where each $\mathbf{V}_i$ is a parabolic sub-bundle of V and each $\mathbf{V}_i/\mathbf{V}_{i-1}$ being a stable parabolic bundle on X of parabolic degree 0. Thus for $\mathbf{V} \in \mathbf{S}$, we can define $\operatorname{gr} \mathbf{V} = \bigoplus_i \mathbf{V}_i/\mathbf{V}_{i-1}$, where (V_i) is a filtration of $\mathbf{V}$ as above. It follows easily from the Jordan-Holder theorem that $\operatorname{gr} \mathbf{V}$ is unique upto isomorphism.

Remark 1.17. Let V be a parabolic vector bundle on X with parabolic structures at $P_1, \dots, P_n$. We call the parabolic structure at one of the points, say P_1, *special* if the flag at P_1 consists only of $F_1 V_{P_1} = V_{P_1}$. Let α be the weight of $F_1 V_{P_1}$. Let V' be the parabolic vector bundle obtained from V by forgetting the parabolic structure at P_1. Then the functor $V \to V'$ is fully faithful and $\operatorname{par deg} V' = \operatorname{par deg} V - \operatorname{rk} V \cdot \alpha$. This gives a method of altering the parabolic degree by introducing special parabolic structures at extra points.

2. Variation of Stability for Vector Bundles with Fixed Quasi-Parabolic Structures

Let $\mathscr{V}$ denote the category of all parabolic vector bundles of rank r with fixed quasi-parabolic structure at a point $P \in X$, fixed ordinary degree d_1, fixed parabolic degree 0 and varying weights $0 \leqq \alpha_1 < \alpha_2 \ldots < \alpha_r < 1$, with fixed multiplicities $m_1, \ldots, m_r$, so that we have

$$\sum_i m_i \alpha_i + d_1 = 0 .$$

We denote by Ω the subspace of $\mathbb{R}^r$ formed by these $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ and call it the weight space associated to $\mathscr{V}$. It is clear that Ω is a bounded convex subset of $\mathbb{R}^r$. We have

Proposition 2.1. *Given weights* $(\boldsymbol{\alpha}_0)$, *there exists a neighbourhood* U *of* $\boldsymbol{\alpha}_0$ *such that for all* $V \in \mathscr{V}$ *one has the following:*

V is $(\boldsymbol{\alpha}_0)$ stable (i.e. V is parabolic stable with respect to the weights $\boldsymbol{\alpha}_0$) $\Rightarrow V$ is $(\boldsymbol{\alpha})$ stable for all $\boldsymbol{\alpha} \in U$.

Proof. Let $V \in \mathscr{V}$. Then if $W \subset V$, the condition for $\boldsymbol{\alpha}$-stability is:

$$\frac{\deg W}{\operatorname{rk} W} + \frac{\sum_k m_{i_k} \alpha_{i_k}}{\operatorname{rk} W} < 0 = \operatorname{par} \deg V .$$

Let $\chi(V, W, \boldsymbol{\alpha})$ denote the negative of the left hand side above. Then there exist constants C_1 and C_2 such that

$$\deg W \leqq C_1 \Rightarrow \chi(V, W, \boldsymbol{\alpha}) < 0$$

and

$$\deg W \geqq C_2 \Rightarrow \chi(V, W, \boldsymbol{\alpha}) \geqq 0$$

for any $\boldsymbol{\alpha} \in \Omega$. Hence assume,

$$C_1 \leqq \deg W \leqq C_2 . \tag{A}$$

Then $\chi(V, W, \boldsymbol{\alpha})$ varies only over a finite number of linear forms in $\boldsymbol{\alpha}$. Hence there exist a constant $\delta > 0$ and a neighbourhood U' of $\boldsymbol{\alpha}_0$ such that for all $V \in \mathscr{V}$, V being $\boldsymbol{\alpha}_0$ stable and $W \subset V$ satisfying condition (A), we get:

$$\chi(V, W, \boldsymbol{\alpha}) \geqq \delta \quad \text{for all} \quad \alpha \in U' .$$

Now for any $V \in \mathscr{V}$ and $W \subset V$,

$$|\chi(V, W, \boldsymbol{\alpha}) - \chi(V, W, \boldsymbol{\alpha}')| \leqq \theta |\boldsymbol{\alpha} - \boldsymbol{\alpha}'|^1 ,$$

where θ is an absolute constant.

Hence $|\boldsymbol{\alpha} - \boldsymbol{\alpha}'| < \dfrac{\delta}{2\theta}$ implies that $\chi(V, W, \boldsymbol{\alpha}) \geqq \delta$, which in turn implies that $\chi(V, W, \boldsymbol{\alpha}') \geqq \dfrac{\delta}{2}$.

1 $|\,|$ denotes here the Euclidean norm in $\mathbb{R}^r$

Choose an open set $U \subset U'$ with $\boldsymbol{\alpha}_0 \in U$ and such that $|\boldsymbol{\alpha} - \boldsymbol{\alpha}_0| \leqq \delta/2\theta$ whenever $\alpha \in U$. Hence if $V \in \mathscr{V}$, the $(\boldsymbol{\alpha}_0)$ stability of V implies that $\chi(V, W, \boldsymbol{\alpha}) \geqq \frac{\delta}{2}$ for $W \subset V$ and W satisfying condition (A). If $\deg W \geqq C_2$ then V would not be $(\boldsymbol{\alpha}_0)$ stable. If $\deg W \leqq C_1$ then in any case V is $(\boldsymbol{\alpha})$ stable. The proposition follows.

Definition 2.2. Let k be a positive integer, $1 \leqq k \leqq r$ and choose integers $n_{i_1}, \ldots, n_{i_k}$ such that $n_{i_1} \leqq m_{i_1}, \ldots, n_{i_k} \leqq m_{i_k}$, where $m_1, \ldots, m_r$ are the multiplicities of the quasi-parabolic structure on $\mathscr{V}$. Choose a positive integer d with $d < -d_1$. Then consider the set of all $\boldsymbol{\alpha} \in \Omega$ with $\sum_k n_{i_k} \alpha_{i_k} = d$ and let D be the union taken over all possible integers d with $0 < d < -d_1$ and over all $\{n_{i_1}, \ldots, n_{i_k}\}$ with $n_{i_1} \leqq m_{i_1}, \ldots, n_{i_k} \leqq m_{i_k}$. We call D a *distinguished subset* of Ω.

Proposition 2.3. *Let K be a compact subset of $\Omega - D$. Then there exists $\delta > 0$ such that for any $V \in \mathscr{V}$, $W \subset V$ and $\boldsymbol{\alpha} \in K$, we have $|\chi(V, W, \boldsymbol{\alpha})| \geqq \delta$. Hence for any $V \in \mathscr{V}$, if V is $\boldsymbol{\alpha}$-stable of any $\boldsymbol{\alpha} \in K$ then there exists $\delta > 0$ such that $\chi(V, W, \boldsymbol{\alpha}) \geqq \delta > 0$ for all $\boldsymbol{\alpha} \in K \subset \Omega - D$.*

Proof. We have $\chi(V, W, \boldsymbol{\alpha}) = -\frac{1}{\operatorname{rk} W}\left(\deg W + \sum_k n_{i_k} \alpha_{i_k}\right)$. As $\alpha \in K$ and $K \subset \Omega - D$, the expression $\sum_k n_{i_k} \alpha_{i_k}$ is never an integer. Therefore the linear form represented by $\chi(V, W, \boldsymbol{\alpha}) \neq 0$ on K. Now there exists $C > 0$ such that:

$$|\deg W| \geqq C \Rightarrow |\chi(V, W, \boldsymbol{\alpha})| \geqq 1 .$$

Now as V ranges over $\mathscr{V}$ and W ranges over all parabolic sub-bundles of V with the condition that $|\deg W| \leqq C$, the set of linear forms in $\boldsymbol{\alpha}$ represented by $\chi(V, W, \boldsymbol{\alpha})$ is finite. Each such form does not vanish on K. This finishes the proof.

Proposition 2.4. *Let $(\Omega - D)_i$ be a connected component of $\Omega - D$. Then for any $V \in \mathscr{V}$, we have that the $\boldsymbol{\alpha}$ stability of V implies the $\boldsymbol{\alpha}_1$ stability of V whenever both $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_1$ both belong to $(\Omega - D)_i$. Moreover, if $V \in \mathscr{V}$ and $\boldsymbol{\alpha} \in \Omega - D$, then $\boldsymbol{\alpha}$ semi-stability of V implies the $\boldsymbol{\alpha}$ stability of V.*

Proof. We have seen that there exists an absolute constant θ such that

$$|\chi(V, W, \boldsymbol{\alpha}) - \chi(V, W, \boldsymbol{\alpha}_1)| \leqq \theta |\boldsymbol{\alpha} - \boldsymbol{\alpha}_1| .$$

Further, there exists $\delta > 0$ such that $\chi(V, W, \alpha) \geqq \delta > 0$ whenever $V \in \mathscr{V}$ is $\boldsymbol{\alpha}$ stable and $\boldsymbol{\alpha}$ belonging to any compact subset K of $\Omega - D$. Hence we deduce that if $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_1$ belong to K and $|\boldsymbol{\alpha} - \boldsymbol{\alpha}_1| \leqq \frac{\delta}{2\theta}$, then V is $\boldsymbol{\alpha}$ stable if and only if V is $\boldsymbol{\alpha}_1$ stable. Now if $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are two points of a connected component of $\Omega - D$, then we can join $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ by a finite sequence of open subsets V_j such that V_j is open in $(\Omega - D)_i$ and each V_j is relatively compact and is contained in a disc of radius $\frac{\delta}{2\theta}$. Hence it follows that V is $\boldsymbol{\alpha}$ stable if and only if V is $\boldsymbol{\alpha}'$ stable. The second part of the proposition is obvious from the definition of the set D.

Definition 2.5. Given weights $\boldsymbol{\alpha} \in \Omega$, denote by $\mathscr{F}_{\boldsymbol{\alpha}}$ the set of all couples (V, W) with $W \subset V$, $V \in \mathscr{V}$ and $\chi(V, W, \boldsymbol{\alpha}) \neq 0$, or equivalently parabolic degree of W with respect to $\boldsymbol{\alpha} \neq 0$.

Lemma 2.6. *Let K be a compact subset of Ω such that $\mathscr{F}_{\alpha}$ is independent of $\alpha \in K$. Then*

1) par deg W *with respect to* $\alpha = 0$ *if and only if* par deg $W = 0$ *with respect to* β *whenever* $\alpha, \beta \in K$ *and for any* $V \in \mathscr{V}$, $W \subset V$.

2) *There exists an absolute constant* $\delta > 0$ *such that* $|\chi(V, W, \alpha)| \geqq \delta$ *whenever* $\chi(V, W, \alpha \neq 0$, *for* $\alpha \in K$, $V \in \mathscr{V}$ *and* $W \subset V$.

Proof. 1) is clear from the assumption. Now $|\chi(V, W, \alpha)| \geqq 1$ for any $\alpha \in \Omega$ whenever $|\deg W| \gg 0$. Hence there exists $C > 0$ such that $|\chi(V, W, \alpha)| \geqq 1$ for any $\alpha \in \Omega$ whenever $|\deg W| \geqq C$. The set of all linear forms $\chi(V, W, \alpha)$ with $|\deg W| < C$ is finite, denote it by S. Hence there exists δ' such that $|\chi(V, W, \alpha)| \geqq \delta'$ whenever $\chi \in S$, $\alpha \in K$ and $(V, W) \in \mathscr{F}_{\alpha}$. Hence if $\delta = \inf(1, \delta')$, we get $|\chi(V, W, \alpha)| \geqq \delta$ whenever $(V, W) \in \mathscr{F}_{\alpha}$ for any $\alpha \in K$.

Corollary 2.7. *Let K be a compact subset of Ω. Then we have the following:*

1) *For any α in a connected component of K and for any $(V, W) \in \mathscr{F}_{\alpha}$, the sign of the form $\chi(V, W, \alpha)$ is the same.*

2) *Let K_0 be a connected component of K. Then for $V \in \mathscr{V}$, V is α-stable if and only if V is β-stable for $\alpha, \beta \in K_0$. Similarly, V is α semi-stable if and only if V is β semi-stable for $\alpha, \beta \in K_0$. Further, if V is α semi-stable, the family of sub-bundles W of V such that* par deg $W = 0$ *is independent of $\alpha \in K$, so that if $V \in \mathscr{V}$ and V is α semi-stable* gr V *is independent of $\alpha \in K$.*

The proofs are immediate.

Recall that if d is any positive integer less than $-d_1$ and $n_{i_1} \leqq m_{i_1}$, $n_{i_2} \leqq m_{i_2}, \ldots, n_{i_k} \leqq m_{i_k}$ are positive integers, where $m_1, \ldots, m_r$ are the weights of $\mathscr{V}$, then we have defined (Definition 2.2) a subset of Ω as consisting of those weights α with $\sum_k n_{i_k} \alpha_{i_k} = d$. Call this subset (distinguished subset) D. Now we have

Lemma 2.8. *Let $\alpha_0 \in \Omega$. Let E_0 be the intersection of all the distinguished subsets D_i passing through α_0. Then there exists a compact neighbourhood U of α_0 such that if $K = U \cap E_0$, then $\mathscr{F}_{\alpha}$ is independent of $\alpha \in K$.*

Proof. Since there exist only a finite number of distinguished subsets, we can find a compact neighbourhood U such that U does not meet any distinguished subset D_j with α_0 not in D_j. It follows that for $K = U \cap E_0$, $\alpha \in D_k \Leftrightarrow \beta \in D_k$ for $\alpha, \beta \in K$ and any distinguished subset D_k. It follows now that if we define G_{α} to be the family of distinguished subsets passing through α, for any $\alpha \in K$, then G_{α} is independent of α for $\alpha \in K$. Let now $V \in \mathscr{V}$ and $W \subset V$ be such that α_0 par deg $W = 0$ if and only if $\chi(V, W, \alpha_0) = 0$. But $\chi(V, W, \alpha) = 0$ defines a distinguished subset. Hence for any $\alpha \in K$, $\chi(V, W, \alpha) = 0$ implies that α par deg $W = 0$. Hence it follows that for $\alpha_0 \in K$, α_0 par deg $W = 0$ if and only if α par deg $W = 0$, for α_0, $\alpha \in K$. This proves the Lemma.

Corollary 2.9. *Let $\alpha \in \Omega$. Then there exists α' such that*

1) *α' is rational, i.e. has rational components.*

2) *For $V \in \mathscr{V}$, V is α stable (semi-stable) if and only if V is α' stable (semi-stable).*

3) *If V is $\boldsymbol{\alpha}$ semi-stable, then its subbundles of* par deg 0 *with reference to $\boldsymbol{\alpha}$ are the same as those for $\boldsymbol{\alpha}'$. In particular $\boldsymbol{\alpha}-\operatorname{gr} V=\boldsymbol{\alpha}'-\operatorname{gr} V$.*

Proof. This is a consequence of the fact that the distinguished subsets are defined over Q. If $\boldsymbol{\alpha}\in\Omega$ and $E_{\boldsymbol{\alpha}}$ is the intersection of all distinguished subsets passing through $\boldsymbol{\alpha}$, then there exists $\boldsymbol{\alpha}'$ in $E_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha}'$ rational. This $\boldsymbol{\alpha}'$ has the required properties.

Remark 2.10. By Corollary 2.9, for the construction of the moduli of parabolic vector bundles, we could have assumed that the components (α_i) of $\boldsymbol{\alpha}$ are all rational. We now show that the components can be taken to be *non-zero.* Assume we are given weights at a point $P\in X$,

$$0\leqq\alpha_1<\alpha_2\ldots<\alpha_r<1\,.$$

Choose a real positive constant β such that if $\alpha_i'=\alpha_i+\beta$, $1\leqq i\leqq r$, then $0<\alpha_1'<\alpha_2'\ldots<\alpha_r'$. Let S be the category of all parabolic semi-stable bundles of fixed rank and fixed ordinary degree and parabolic degree 0 with respect to $\boldsymbol{\alpha}$. Let S' denote the same category, but with respect to $\boldsymbol{\alpha}'$. Now $\operatorname{par deg} V>0$ for any $V\in S'$. We assume that β was chosen such that the pattern of stability and semi-stability is the same in S and S'. Now take a line bundle L such that $\deg(V\otimes L)<0$ for any $V\in S$. Let S_1 be the category $\{V\otimes L\}$, $V\in S$. Again S_1 and S' have the same pattern. Choose a point $Q\neq P$ and introduce a special (cf. Remark 1.17) parabolic structure at Q so that the new parabolic degree is zero, say for the new category S_2. Once again it follows that S_2 and S_1 and hence S_2 and S have the same pattern of stability and semi-stability.

3. Properness of the Variety of Parabolic Semi-Stable Bundles on X

Let X be a smooth projective curve over an algebraically closed field k. Let $P\in X$ be a fixed point. We consider the category of all parabolic semi-stable bundles on X with fixed rank, weights $0<\alpha_1<\alpha_2\ldots<\alpha_r<1$ and fixed parabolic degree. The quasi-parabolic structure at P is given by:

$$V_P=F_1(V_P)\supset F_2(V_P)\ldots\supset F_r(V_P)\,.$$

We put $k_i=\dim F_i(V_P)-\dim F_{i+1}(V_P)$, $1\leqq i\leqq r-1$ and $k_r=\dim F_r(V_P)$.

If W is a finite-dimensional vector space over k, we denote by $\mathscr{F}(W)$, or just $\mathscr{F}$, the set of all flags of type as above, i.e. the set of all flags $W=W_1\supset W_2\ldots\supset W_r$ with $\dim W_i-\dim W_{i+1}=k_i$, $1\leqq i\leqq r-1$ and $\dim W_r=k_r$.

By a *family of parabolic vector bundles* on X, parametrized by a k-scheme S, we mean:

i) a vector bundle V on $X\underset{k}{\times}S$, and

ii) a section of $\mathscr{F}(V)/P\times S$, where $\mathscr{F}(V)$ is the flag variety on $X\times S$ and $P\in X$ is the point where the parabolic structure is situated.

Let $R=k[[T]]$ with quotient field K. We shall prove:

Theorem 3.1. *Let V be a parabolic semi-stable bundle on $X_K=X\underset{k}{\times}K$. Then there exists a parabolic bundle W on $X_R=X\underset{k}{\times}R$ such that the generic fibre of W is isomorphic to V and the special fibre of W is semi-stable on $X_R\underset{R}{\times}k=X$.*

Before the proof, we recall the notions of "β sub-bundles" according to Langton [3] and "strongly contradicting semi-stability" (S.C.S.S.) according to Harder-Narasimhan [2]. Let E be a parabolic vector bundle on X of rank n and parabolic degree d. For any subbundle F of E, we define

$$\mu(F)=\frac{\operatorname{par}\deg F}{\operatorname{rk} F}.$$

Similarly, we define

$$\begin{aligned}\beta(F)&=(\operatorname{par}\deg F)(\operatorname{rk} E)-(\operatorname{par}\deg E)(\operatorname{rk} F)\\&=n\operatorname{par}\deg F-d\operatorname{rk} F.\end{aligned}$$

It is easily seen that E is semi-stable (stable) if and only if $\mu(F)\leqq\mu(E)$ ($\mu(F)<\mu(E)$) for all $F\subset E$. Equivalently E is semi-stable (stable) if and only if $\beta(F)\leqq 0$ ($\beta(F)<0$) for all $F\subset E$. Now assume E is unstable, i.e. not semi-stable. Let $\mu_0=\sup_{F\subset E}\mu(F)$ and $\beta_0=\sup_{F\subset E}\beta(F)$. Define

$$S=\{F\subset E\,|\,\mu(F)=\mu_0\}$$

and

$$T=\{F\subset E\,|\,\beta(F)=\beta_0\}.$$

It is clear that if F belongs to $S\cap T$, we have

i) F is semi-stable, and

ii) for every $F_1\subset E$ with $F\subsetneqq F_1\subset E$, we have $\mu(F)>\mu(F_1)$, or equivalently, for every $Q\subset\frac{E}{F}$, we have $\mu(Q)<\mu(F)$.

Now we claim that $S\cap T$ consists of a single element. To show that $S\cap T$ has at most 1 element, we quote

Lemma 3.2 (cf. Harder-Narasimhan [2], Lemma 1.3.5). *Let F_1 and F_2 be subbundles of E such that F_1 is semi-stable and $\mu(F_2)>\mu(G)$ for every G with $F_2\subsetneqq G\subset E$. Then if F_1 is not contained in F_2, we have $\mu(F_2)>\mu(F_1)$.*

Now any element of S of maximal rank belongs to T and any element of T of minimal rank belongs to S. Thus $S\cap T$ consists of only 1 element, say B. It follows that B is "S.C.S.S." in E according to Harder-Narasimhan [2] and that B is the "β-subbundle" of E according to Langton [3]. We list the properties of B that we shall need:

1) $\beta(G)<\beta(B)$ for all $G\subset E$ with $G\subsetneqq B$.

2) If $F\subset E$ and $\beta(F)=\beta(B)$, then $F\supset B$.

Now let X, k, and R be as above and let ξ be the generic point of X considered as a closed subscheme of X_R. Let z be the generic point of X_R. If E is a rank r vector bundle on X_R, then E_ξ is a free module over $\mathcal{O}_{X_R,\xi}$ of rank r and $E\subset E_z$, which is a vector space of rank r over $\mathcal{O}_{X_R,z}$.

We have

Proposition 3.3 (cf. Langton [2], Proposition 6). *Let E_K be a vector bundle on X_K and $M \subset E_{K,z}$ a free module over $\mathcal{O}_{X_R,\xi}$ of* rank = rank $E_K = r$ *say. Then there exists a unique torsion-free sheaf E on X_R such that the generic fibre of E is E_K and the special fibre of E is torsion-free on $X_k = X$ with $M = E_\xi$.*

Note that E and the special fibre of E are vector bundles on X_R and X respectively. Note also that if we are given a parabolic structure on E_K, we can extend it uniquely to a parabolic structure on E. Locally, E is obtained as follows; assume X affine, then E is defined to be the sheaf associated to the module $M \cap N$, where N is $\Gamma(X_K, E_K)$ and M is a free module of rank r over $\mathcal{O}_{X_R,\xi}$, and $M \subset N_z$.

Now let E be a vector bundle on X_R. Then E_ξ is a free module over $\mathcal{O}_{X_R,\xi}$. Let E_1 be a sub-sheaf of E, locally free and such that the sheaf $\bar{C}$, given by $0 \to E_1 \to E \to \bar{C} \to 0$, is a vector bundle on X. Then if $(e_1, \ldots, e_n)$ is a basis for E, it is easy to see that $(e_1, \ldots, e_r, \pi e_{r+1}, \ldots, \pi e_n)$ is a basis for $E_{1,\xi}$, where $r = \operatorname{rank} \bar{C}$ and π is a parameter for R. Conversely, starting from a basis $(e_1, \ldots, e_n)$ for E_ξ, let E_1 be the vector bundle on X_R determined by the free module which has for a basis $(e_1, \ldots, e_r, \pi e_{r+1}, \ldots, \pi e_n)$ over $\mathcal{O}_{X_R,\xi}$. Then E_1 is a subsheaf of E and the co-kernel of $E_1 \to E$ is a vector bundle on X of rank r. Thus given a vector bundle E on X_R, there is a canonical bijection between:

1) quotient sheaves $\bar{C}$ of E which are vector bundles on X, and

2) sub sheaves E_1 of E which are locally free and a basis for $E_{1,\xi}$ is given by $(e_1, \ldots, e_r, \pi e_{r+1}, \ldots, \pi e_n)$, where $(e_1, \ldots, e_n)$ is a basis for E_ξ.

Now suppose that E_K is a parabolic semi-stable vector bundle on X of rank n. Extend E_K to a parabolic vector bundle E on X_R by choosing any sub-module of $E_{K,z}$ which is free over $\mathcal{O}_{X_R,\xi}$. Assume that $\bar{E}$ = restriction of E to X is unstable. Let $\bar{B} \subset \bar{E}$ be the β-subbundle of $\bar{E}$ and look at the sequences

$$0 \to \bar{B} \to \bar{E} \to \bar{F}_1 \to 0, \tag{1}$$

$$0 \to E_1 \to E \to \bar{F}_1 \to 0. \tag{2}$$

Tensor (1) with $\mathcal{O}_X$ and as $\operatorname{Tor}_1^{\mathcal{O}_{X_R}}(\mathcal{O}_X, \bar{F}_1) = \bar{F}_1$, we get

$$0 \to \bar{F}_1 \to \bar{E}_1 \to \bar{E} \to \bar{F}_1 \to 0. \tag{3}$$

Split (3) into

$$0 \to \bar{F}_1 \to \bar{E}_1 \to \bar{B} \to 0 \quad \text{and} \quad 0 \to \bar{B} \to \bar{E} \to \bar{F}_1 \to 0.$$

Proposition 3.4. *If $\bar{G}_1 \subset \bar{E}_1$, then $\beta(\bar{G}_1) \leqq \beta(\bar{B})$, with equality holding only if $\bar{G}_1 \cup \bar{F}_1 = \bar{E}_1$.*

Proof. From $0 \to \bar{B} \to \bar{E} \to \bar{F}_1 \to 0$ we get $\beta(\bar{B}) + \beta(\bar{F}_1) = \beta(\bar{E})$ and as $\beta(\bar{E}) = 0$ and $\beta(\bar{B}) > 0$, we get $\beta(\bar{F}_1) < 0$. First assume that $\bar{G}_1 \subset \bar{F}_1$. Then as $\bar{E}/\bar{B} \approx \bar{F}_1$, we get that $\bar{G}_1$ corresponds to a bundle $\bar{G}$ with $\bar{B} \subset \bar{G} \subset \bar{E}$ and $\bar{G}/\bar{B} \approx \bar{G}_1$. Thus $\beta(\bar{G}) \leqq \beta(\bar{B})$. Hence $\beta(\bar{G}_1) = \beta(\bar{G}) - \beta(\bar{B}) \leqq 0 < \beta(\bar{B})$. So assume that $\bar{G}_1 \not\subset \bar{F}_1$. Put $\bar{J}_1 = \bar{G}_1 \cup \bar{F}_1$ and $\bar{I}_1 = \bar{G}_1 \cap \bar{F}_1$. Then $\bar{I}_1 \subset \bar{F}_1 \subset \bar{J}_1$. Since $\bar{I}_1 \subset \bar{F}_1$, $\beta(\bar{I}_1) \leqq 0$ by the above argument; hence $\beta(\bar{G}_1) \leqq \beta(\bar{G}_1) - \beta(\bar{I}_1)$. But $\beta(\bar{G}_1) - \beta(\bar{I}_1) \leqq \beta(\bar{J}_1) - \beta(\bar{F}_1)$. Now with $\bar{E}_1/F_1 \approx \bar{B}$,

$\bar{J}_1 \supset \bar{F}_1$ corresponds to a subbundle $\bar{J} \subset \bar{B}$ with $\bar{J} \approx \bar{J}_1/\bar{F}_1$. Consequently $\beta(\bar{J}_1) - \beta(\bar{F}_1) = \beta(\bar{J}) \leqq \beta(\bar{B})$, with equality holding only if $\bar{J} = \bar{B}$, i.e. only if $\bar{J}_1 = \bar{E}_1$.

Now we proceed to the proof of Theorem 3.1. Let E_K be a semi-stable parabolic bundle on X_K extended to a bundle E on X_R. If $\bar{E}$ = restriction of E to X, is semi-stable, we are through. If $\bar{E}$ is unstable, define $E^{(1)}$ to be the kernel of the map $E \to \bar{F}_1 \to 0$, where $\bar{F}_1$ is defined by the sequence $0 \to \bar{B} \to \bar{E} \to \bar{F}_1$, $\bar{B}$ being the β-subbundle of $\bar{E}$. Continue in this fashion to get a sequence of bundles $E^{(m)}$ on X_R, all of which are generically isomorphic. If $\overline{E^{(m)}}$ is semi-stable on X for some m, we are through. Assume that $\bar{E}^{(m)}$ is unstable on X for all m and we derive a contradiction. Denote by $\bar{B}^{(m)}$ the β-subbundle of $\bar{E}^{(m)}$ and put $\beta_m = \beta(\bar{B}^{(m)})$. We have $\beta_m > 0$ for all m and by Proposition 3.4, the $\{\beta_m\}$ form a strictly decreasing set. Since the set $\{\beta_m\}$ is discrete, we must have $\beta_m = \beta_{m+1} = \beta_{m+2}$ from some integer onwards. Thus $\bar{B}^{(m)} \cup \bar{K}^{(m)} = \bar{E}^{(m)}$, where $\bar{K}^{(m)} = \ker \bar{E}^{(m)} \to \bar{E}^{(m-1)}$. So $\operatorname{rank} \bar{B}^{(m)} + \operatorname{rank} \bar{K}^{(m)} \geqq r = \operatorname{rank} E$. But $\operatorname{rank} \bar{K}^{(m)} = r - \operatorname{rank} \bar{B}^{(m-1)}$, so $\operatorname{rank} \bar{B}^{(m)} \geqq \operatorname{rank} \bar{B}^{(m-1)}$ for all $m \gg 0$. Now as $\operatorname{rank} \bar{B}^{(m)} < r$ for all m, we get that $\operatorname{rank} \bar{B}^{(m)}$ stabilizes. Thus $\operatorname{rank} \bar{B}^{(m)} + \operatorname{rank} \bar{K}^{(m)} = r$ for all $m \gg 0$. Hence $\bar{B}^{(m)} \cap \bar{K}^{(m)} = 0$ for all $m \gg 0$, which means that the canonical map $\bar{E}^{(m)} \to \bar{E}^{(m-1)}$ maps $\bar{B}^{(m)}$ into $\bar{B}^{(m-1)}$ injectively. Further, as $\beta(\bar{B}^{(m)})$ and $\operatorname{rank} \bar{B}^{(m)}$ are constant, we get that degree $\bar{B}^{(m)}$ is constant for all $m \gg 0$. In particular the canonical map $\bar{B}^{(m)} \to \bar{B}^{(m-1)}$ is an isomorphism for all $m \gg 0$. Without loss of generality, we may assume that the isomorphism holds for all m.

The proof of the next lemma is taken from Langton (cf. [3], Sect. 5, Lemma 2). We include the proof here for the sake of completeness. Note that although the discussion in [3] applies only to ordinary vector bundles, the extension to *parabolic* vector bundles is immediate.

Lemma 3.5. *Suppose we are given an infinite sequence of inclusions of bundles on* X_R*:*

$$E^{(m+1)} \to E^{(m)} \dots \to E^{(0)},$$

where each $E^{(m)}$ *has degree* 0 *and the maps induce isomorphisms generically. Assume that if* $E^{(0)}$ *is free with basis* $(e_1, \dots, e_r)$ *over the ring* $\mathcal{O}_{X_R, \xi}$, *then* $E^{(m)}$ *is free with basis* $(e_1, \dots, e_p, \pi^m e_{p+1}, \dots, \pi^m e_r)$ *over* $\mathcal{O}_{X_R, \xi}$. *Denoting by* $\bar{F}^{(m)}$ *the image of* $\bar{E}^{(m+1)}$ *in* $\bar{E}^{(m)}$, *assume further that the induced maps* $\bar{F}^{(m)} \to \bar{F}^{(m-1)}$ *are isomorphisms for all m. Then* $\beta(\bar{F}^{(0)}) \leqq 0$.

Note that all the assumption hold in our previous considerations.

Proof. For any integer m, denote by X_m the infinitesimal neighbourhood of X of order $(m-1)$ in X_R, i.e. $X_m = \dfrac{X_R}{(\pi^m)}$. If G is any sheaf on X_R, denote by G_m its reduction $\bmod \pi^m$. Denote by $(\bar{e}_1, \dots, \bar{e}_p)$ a basis for $\bar{F}^{(0)}_\xi$ and extend it to a basis $(\bar{e}_1, \dots, \bar{e}_p, \bar{e}_{p+1}, \dots, \bar{e}_r)$ for $\bar{E}_\xi$. This lifts to a basis $(e_1, \dots, e_r)$ for $E^{(0)}_\xi$. Hence $E^{(m)}_\xi$ is a free module on $(e_1, \dots, e_p, \pi^m e_{p+1}, \dots, \pi^m e_r)$. Reduce the inclusion $E^{(m)} \subset E^{(0)}$ to X_m to get a map $E^{(m)}_m \to E^{(0)}_m$. Let F_m denote the image of this map. In particular $F_1 = \bar{F}^{(0)}$. Now for any $l \geqq 0$, $E^{(l)}_m$ is a torsion free sheaf on X_m, i.e. for all U open in X_m, $E^{(l)}_m(U) \subset E^{(l)}_{m,\xi}$. This is obvious as both $E^{(l)}$ and $\bar{E}^{(l)}$ are torsion free on X_R and X respectively. Moreover, $F_m \subset E_m$ is a subbundle, i.e. $F_m(U) = E_m(U) \cap F_{m,\xi}$ for any

$U \subset X$ open. Now define sheaves Q_m by

$$E_m^{(m)} \to E_m \to Q_m \to 0.$$

It is easy to see that the inclusion $E^{(m)} \to E$ also has Q_m as its cokernel. Hence we get an exact sequence

$$0 \to E^{(m)} \to E \to Q_m \to 0. \tag{1}$$

Tensor (1) by $\dfrac{\mathcal{O}_{X_R}}{\pi^m \mathcal{O}_{X_R}}$ to get

$$0 \to Q_m \to E_m^{(m)} \to E_m \to Q_m \to 0, \tag{2}$$

which breaks up into

$$0 \to Q_m \to E_m^{(m)} \to F_m \to 0 \quad \text{and} \quad 0 \to F_m \to E_m \to Q_m \to 0.$$

Let $j_{m,m'}$ be the closed immersion $X_{m'} \hookrightarrow X_m$ for $m' \leqq m$. Pull the homomorphism $E_m^{(m)} \to F_m \to E_m$ to $X_{m'}$ to get

$$E_{m'}^{(m)} \to j_{m,m'}^*(F_m) \to E_{m'}. \tag{3}$$

The sequence (3) can be factored as follows:

$$\begin{tikzcd} E_{m'}^{(m)} \arrow[r] \arrow[dr] & E_{m'}^{(m')} \arrow[r] & E_{m'} \\ & j_{m,m'}^*(F_m) \arrow[ur] & \end{tikzcd}$$

So there exists a map:

$$j_{m,m'}^*(F_m) \to \operatorname{Image}(E_{m'}^{(m)} \to E_{m'}) \subset \operatorname{Image}(E_{m'}^{(m')} \to E_{m'}).$$

Putting $m' = 1$ and $j_{m,1}^* = j_m^*$, we get

$$j_m^*(F_m) \to \operatorname{Image}(\bar{E}^{(m)} \to \bar{E}) \subset \bar{F}.$$

Hence we have $\operatorname{Image}(\bar{E}^{(m)} \to \bar{E}) = \bar{F}$ and so $j_m^*(F_m) \to \bar{F} \to 0$. Further, in $E_m \to E_m^{(m)} \to Q_m \to 0$, compose $E_m^{(m)} \to F_m \to 0$ with $F_m \subset E_m$ to get the canonical map $E_m^{(m)} \to E_m$. Reducing to X we get

$$\begin{tikzcd} & & \bar{E} & \\ \bar{E} \arrow[r] & \bar{E}^{(m)} \arrow[r] & j_m^*(F_m) \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{4}$$

Denote by $\bar{K}^{(m)}$ the kernel of $\bar{E}^{(m)} \to \bar{E}^{(m-1)}$. As $\bar{F}^{(m-1)} \to \bar{F}^{(m-2)}$ is injective, $\bar{K}^{(m)}$ is also the kernel of $\bar{E}^{(m)} \to \bar{E}$. Going back to the inclusion $E^{(m)} \subset E^{(m-1)} \subset E^{(m-2)} \ldots$, we have

$$0 \to E^{(m)} \to E^{(m-1)} \to L^{(m)} \to 0 \quad \text{say}, \tag{5}$$

where $L^{(m)}$ is a line bundle over X. This results from the basis of $E^{(m)}$ and $E^{(m-1)}$ as modules over $\mathcal{O}_{X_R,\xi}$. Reduce (5) to X to get

$$0 \to L^{(m)} \to \bar{E}^{(m)} \to \bar{E}^{(m-1)} \to L^{(m)} \to 0.$$

Hence we get a backward map $\bar{E}^{(m-1)} \to \bar{E}^{(m)}$ whose kernel is precisely the image of $\bar{E}^{(m)} \to \bar{E}^{(m-1)}$, or $\bar{F}^{(m-1)}$. But, recalling that $\bar{K}^{(m-1)}$ is the kernel of $\bar{E}^{(m-1)} \to \bar{E}^{(m-2)}$ and that $\bar{F}^{(m-1)} \to \bar{F}^{(m-2)}$ is injective, we get $\bar{F}^{(m-1)} \cap \bar{K}^{(m-1)} = 0$ or that the map $\bar{K}^{(m-1)} \to \bar{K}^{(m)}$ is injective for all m. By degree and rank considerations, we get $\bar{K}^{(m)} \to \bar{K}^{(m+1)}$ is an isomorphism for all m. Hence, in the diagram (4), we get that the kernel of $\bar{E}^{(m)}$ to $\bar{E}$, which is $\bar{K}^{(m)}$ by the above argument is also the image of $\bar{E} \to \bar{E}^{(m)}$. Thus the map $j_m^*(F_m) \to \bar{F}$, which was surjective, is also injective for all m. Consider $j_{m,m'}^*(F_m) \to F_{m'}$, for any $m' \leqq m$. Let W be the cokernel, we get $j_{m,m'}^*(F_m) \to F_{m'} \to W \to 0$. Reduce to X to get $j_m^*(F_m) \to j_{m'}^*(F_{m'}) \to j_{m'}^*(W) \to 0$. But the first arrow being an isomorphism, we get that $j_{m'}^*(W) = 0$ and hence $W = 0$. We want to show now that $j_{m,m'}^*(F_m) \to F_{m'}$ is injective. We have $j_m^*(F_m) \subset \bar{E}$ for all m, hence for $x \in X_m$, $(F_m)_x \cap \pi(E_m)_x \subset \pi(F_m)_x$. We will show that $(F_m)_x \cap \pi^{m'}(E_m)_x \subset \pi^{m'}(F_m)_x$ for all $m' \leqq m$, which will prove that $j_{m,m'}^*(F_m) \to F_{m'}$ is injective. Assume by induction that $(F_m)_x \cap \pi^n(E_m)_x \subset \pi^n(F_m)_x$ for all n with $n < m'$. Let $a \in (F_m)_x \cap \pi^{m'}(E_m)_x$. Then $a \in \pi^{m'-1}(F_m)_x$ and consequently $a = \pi^{m'-1} b = \pi^{m'} c$, where $b \in (F_m)_x$ and $c \in (E_m)_x$. It follows that $\pi^{m'-1}(b - \pi c) = 0$ in $(E_m)_x = \dfrac{E_x}{\pi^m(E)_x}$, hence $b - \pi c \in \pi^{m-m'+1}(E)_x$, or $b - \pi c = \pi^{m-m'+1} g$, where $g \in (E_m)_x$. Thus

$$b = \pi(c + \pi^{m-m'} g) \in (F_m)_x \cap \pi(E_m)_x \subset \pi(F_m)_x .$$

So $b = \pi h$, with $h \in (F_m)_x$ and $a = \pi^{m'-1} b = \pi^{m'-1}(\pi h) = \pi^{m'} h$ with $h \in (F_m)_x$. Hence $a \in \pi^{m'}(F_m)_x$.

Thus we have a sequence of bundles F_m on X_m with $j_{m,m'}(F_m) \approx F_{m'}$ for any m, m' with $m' \leqq m$. By [E.G.A., III, 5.1.5 and 5.1.3] there exists a locally free subsheaf F of E with $F = \varprojlim F_m$, where $\lim F_m$ is the completion of F along X and $j^*(F) = \bar{F}$, where j is $X \hookrightarrow X_R$. Now F_K is a subsheaf of E_K and hence inherits a parabolic structure, which extends uniquely to a parabolic structure on F. Now as E_K is semi-stable, we must have par deg$F_K \leqq 0$, which implies that par deg$j^*(F) = \bar{F} \leqq 0$, or that $\beta(\bar{F}) \leqq 0$, which completes the proof of Lemma 3.5. But $j^*(F) = \bar{F}$ has positive parabolic degree, which is a contradiction. Hence E_K extends to a semi-stable parabolic vector bundle E on X_R, thus completing the proof of Theorem 3.1.

4. Existence of the Moduli Space

Let X, as usual, be a smooth projective curve of genus $g \geqq 2$ over an algebraically closed field k. Consider the set of all parabolic semi-stable bundles of rank k, fixed quasi-parabolic structure at a given point P, fixed weights $0 < \alpha_1 < \alpha_2 \ldots < \alpha_r < 1$ with all (α_i) rational, fixed degree d and parabolic degree 0. Denote this set by $S(k, \boldsymbol{\alpha}, d, 0)$ or just S. Recall that if $V \in S$, then gr(V) is defined to by $\bigoplus_i V_i/V_{i-1}$, where $V = V_n \supset V_{n-1} \ldots \supset V_0 = 0$ is a filtration (cf. Remark 1.15). Define V and V' to be equivalent if gr $V = $ gr V'.

We shall prove the following theorem:

Theorem 4.1. 1) *On the set of equivalence classes of S, there exists a natural structure of a normal projective variety of dimension $k^2(g-1)+1+\dim \mathscr{F}$, where $\mathscr{F}$ is the flag variety of type determined by the quasi-parabolic structure at $P \in X$.*

2) Let $X = H^+/\Gamma$ as in Sect. 1 with $Q \in H^+$ the parabolic vertex of H^+ corresponding to the point $P \in X$. Let Γ_Q be the stabilizer. Then the above variety is isomorphic to the equivalence classes of unitary representations of Γ with the image of the generator of Γ_Q being conjugate to the diagonal matrix ($\exp 2\pi i\alpha_1$, $\exp 2\pi i\alpha_2, \ldots, \exp 2\pi i\alpha_r$), *each α_i being repeated k_i times, where (k_i) are the multiplicities of (α_i).*

Further, in 2) *the parabolic stable bundles correspond precisely to the irreducible unitary representations of Γ.*

Proof. As the statement of the theorem refers to parabolic structures at only one point of X, at the end of the proof we shall indicate the changes that have to be made in order to handle the case of parabolic structures at more than one point.

We first note that the set S is *bounded*, i.e. there exists m_0 such that for all $m \geqq m_0$, we have $H'(V(m)) = 0$ and $H^0(V(m))$ generates $V(m)$ for all $V \in S$. This follows from observing that the degrees of all indecomposable components of $V \in S$ are bounded both above and below and then by applying a lemma of Atiyah [1]. Furthermore, by the same reasoning, it follows that for any real number θ, the set of all subbundles W of elements of S with degree $W \geqq \theta$ is also bounded.

Choose an integer m such that:

1) $m \geqq g+1$

2) $H^1(W(m)) = 0$ and $H^0(W(m))$ generates $W(m)$ whenever $W \in S(k, \boldsymbol{\alpha}, d, 0)$ or $W \subset V$, $V \in S(k, \boldsymbol{\alpha}, d, 0)$ and $\deg W \geqq (-g-8)k$.

Choose an integer N with $N \geqq \dfrac{2n\theta}{\alpha_1}$, where θ is a positive constant with $\deg W(m) \leqq \theta$ for any $W \subset V$, $V \in S(k, \boldsymbol{\alpha}, d, 0)$ and n is the common dimension of $H^0(V(m))$, $V \in S(k, \boldsymbol{\alpha}, d, 0)$. Also choose $\varepsilon = \dfrac{N}{m-g}$. Let E be a vector space over k of dimension n, also denote by E the trivial bundle over X of rank n. Denote by $Q(E/^{\infty})$, or just Q, the Hilbert Scheme of coherent sheaves over X which are quotients of E and whose Hilbert Polypomial is that of $V(m)$, $V \in S$. Denote by R the open subset of Q consisting of those points $q \in Q$ such that if $E \to \mathscr{F}_q \to 0$ is the corresponding quotient, then $H'(\mathscr{F}_q) = 0$, $H^0(\mathscr{F}_q) \approx E$ and $\mathscr{F}_q$ is locally free. It is known (cf. [10]) that R is a non-singular variety of dimension $k^2(g-1)+1+n^2-1$. If the sheaf G on $X \times Q$ is the universal quotient, denote its restriction to $X \times R$ by $\mathscr{V}$, which is a vector bundle on $X \times R$. Denote by $\mathscr{F}(\mathscr{V})$ the flag variety over $X \times R$ and use the same letter to denote the restriction of $\mathscr{F}(\mathscr{V})$ to $(P) \times R$. Call the total space of this flag bundle $\tilde{R}$, with projection $\pi: \tilde{R} \to R$.

We see that $\tilde{R}$ has the *local universal property* for parabolic bundles, i.e. if $\mathscr{W}$ is a parabolic vector bundle on $X \times T$ with $H'(\mathscr{W}_t) = 0$ and $H^0(\mathscr{W}_t)$ generating $\mathscr{W}_t$ for all $t \in T$, the Hilbert Polynomial of $\mathscr{W}_t = P$ for all $t \in T$, then for every $t_0 \in T$ there is a neighbourhood U of t_0 and a map $f: U \to R$ such that $\mathscr{W}$ is the pull-back via f of the universal bundle on $X \times \tilde{R}$. Further f is unique upto an action of $\mathrm{SL}[H^0(\mathscr{W}_t)]$.

Denote by $\tilde{R}^{ss}(\tilde{R}^s)$ the set of points $q \in \tilde{R}$ such that the corresponding parabolic bundle on X is semi-stable (stable). We shall prove that $\tilde{R}^{ss}$ and $\tilde{R}^s$ are open subsets of $\tilde{R}$.

Denote by G the group $\mathrm{SL}(E)$. Then G acts on Q and it is easy to see that R is a G-invariant subset of Q. G also acts on $\tilde{R}$.

If $P \in X$ is the point where every $V \in S(k, \boldsymbol{\alpha}, d, 0)$ has a flag

$$V_P = F_1 V_P \supset F_2(V_P) \ldots \supset F_r(V_P),$$

define $t_i = \dim F_1 V_P - \dim F_i V_P$, $2 \leqq i \leqq r$ and $t_1 = k$. Let $H_{n,k}(E)$, or just $H_{n,k}$ denote the Grassmannian of k dim quotients of E and denote by Z the product space

$$H_{n,k}^N \times \prod_{i=1}^{r} H_{n,t_i}.$$

In order to define a *linear action* of G on Z, we have to define a *polarization* of Z. In general if a variety X is a product $\prod_i X_i$ with each $\operatorname{Pic} X_i = \mathbb{Z}$, then by choosing an ample generator for $\operatorname{Pic} X_i$, we can write an ample line bundle on X by (a_i), where each a_i is a positive integer. Hence we can define a polarization on X by (q_i), where each q_i is a positive rational number. We give Z the polarization:

$$\{\underbrace{1, 1, 1, \ldots}_{N \text{ times}}\} \times \{\varepsilon(1-\alpha_r), \varepsilon(\alpha_2 - \alpha_1), \ldots, \varepsilon(\alpha_r - \alpha_{r-1})\}.$$

We require a characterisation of stable and semi-stable points of Z for this polarization, In fact, more generally if

$$W = \prod_{k=1}^{N} H_{n,k_i}(E)$$

and W carries the polarization $(\delta_1, \ldots, \delta_N)$, then we see easily from the computations of (§4, Chap. 4, [5]), that a point w of W represented by

$$\varphi_i : E \to V_i, \quad 1 \leqq i \leqq N, \quad \dim V_i = k_i$$

is *stable* (resp. *semi-stable*) if and only if for any *proper* linear subspace M of E, we have

$$(*) \quad \dim E \left\{ \sum_{i=1}^{N} \delta_i \cdot \dim \varphi_i(M) \right\} > \dim M \left\{ \sum_{i=1}^{N} \delta_i k_i \right\} (\text{resp. } \geqq).$$

We define a map $T : \tilde{R} \to Z$ as follows:

If $q \in R$, then by writing $E \to \mathscr{F}_q \to 0$, we get $E_{P_i} \to (\mathscr{F}_q)_{P_i} \to 0$, $1 \leqq i \leqq N$, where the P_i are N arbitrarily chosen points on X, different from P. This gives the "co-ordinates" of $T(q)$ in $H_{n,k}(E)^N$ and for the co-ordinate in $\prod_{i=1}^{r} H_{n,t_i}(E)$ we take the quotients

$$E \to V_P = F_1(V_P), \quad E \to \frac{F_1(V_P)}{F_2(V_P)}, \ldots, E \to \frac{F_1(V_P)}{F_r(V_P)}.$$

It is easy to see that T is a G-map from $\tilde{R}$ to Z. With this map T, we have

Proposition 4.2. 1) $q \in \tilde{R}^{ss} \Rightarrow T(q) \in Z^{ss}$.

2) $q \in \tilde{R}^{s} \Rightarrow T(q) \in Z^{s}$.

3) $q \in \tilde{R}, q \notin \tilde{R}^{ss} \Rightarrow T(q) \notin Z^{ss}$.

4) $q \in \tilde{R}^{ss}, q \notin \tilde{R}^{s} \Rightarrow T(q) \notin Z^{s}$.

Note that 3) and 4) will prove that $\tilde{R}^{ss}$ and $\tilde{R}^s$ are open subsets of $\tilde{R}$. Here Z^{ss} and Z^s are the semi-stable and stable points of Z for the action of G on Z and the polarization defined above.

Proof. For every subspace M of E, denote by M_i the images of M in $V(m)_{P_i}$, $1 \leqq i \leqq N$ and by N_i the images of M in $F_1(V_P)$, $\frac{F_1(V_P)}{F_2(V_P)}, \ldots, \frac{F_1(V_P}{F_r(V_P)}$.

If $\dim M = p$, we have to prove that [because of (*) preceding Proposition 4.2]

$$\text{(I)}\begin{cases} n\left(\sum_{i=1}^{N} \dim M_i + \varepsilon(1-\alpha_r)\dim N_1 + \varepsilon \sum_{i=2}^{r} (\alpha_i - \alpha_{i-1}) \dim N_i\right) \\ \quad \geqq p\left(Nk + \varepsilon\left[(1-\alpha_r)k + \sum_{i=2}^{r} t_i(\alpha_i - \alpha_{i-1})\right]\right). \\ \quad \text{and } > \text{ holds if } V \text{ is stable.} \end{cases}$$

Now for any V, $\operatorname{wt} V = \operatorname{wt} V_P$ is defined by

$$\begin{aligned} \operatorname{wt} V_P = {} & \alpha_1(\dim F_1 V_P - \dim F_2 V_P) + \alpha_2(\dim F_2 V_P - \dim F_3 V_P) \\ & + \alpha_r \dim F_r V_P. \end{aligned}$$

Hence

$$\begin{aligned} \dim V_P - \operatorname{wt} V_P = {} & (1-\alpha_1)(\dim F_1 - \dim F_2) + (1-\alpha_2)(\dim F_2 - \dim F_3) \\ & + (1-\alpha_r)\dim F_r \\ = {} & (1-\alpha_1)(\dim F_1 - \dim F_2) + (1-\alpha_2)[(\dim F_1 - \dim F_2) \\ & - (\dim F_1 - \dim F_3)] \\ & + (1-\alpha_{r-1})[(\dim F_1 - \dim F_2) - (\dim F_1 - \dim F_{r-1})] \\ & + (1-\alpha_r)(\dim F_r) \\ = {} & (1-\alpha_r)\dim F_1 + (\alpha_2 - \alpha_1)(\dim F_{1/F_2}) + \ldots + (\alpha_r - \alpha_{r-1})(\dim F_{1/F_r}) \end{aligned}$$

after re-arranging the terms.

Hence, we have

$$\varepsilon\left[(1-\alpha_r)\dim F_1 + \sum_{i=2}^{r} (\alpha_i - \alpha_{i-1}) t_i\right] = \varepsilon[\dim V_P - \operatorname{wt} V_P].$$

Similarly,

$$(1-\alpha_r)\dim N_1 + \sum_{i=2}^{r} (\alpha_i - \alpha_{i-1}) \dim N_i = \dim N_1 - \operatorname{wt} N_1.$$

Thus condition (I) of semi-stability (stability) becomes

$$n\left[\sum_{i=1}^{N} \dim M_i + \varepsilon(\dim N_1 - \operatorname{wt} N_1)\right] \geqq p[Nk + \varepsilon(k - \operatorname{wt} V_P)]$$

($>$ for stability).

Now for any subspace M of E, denote by $W(m)$ the vector bundle generated generically by M. We have $W(m)\subset V(m)$ for all $V\in S$ and $M\subset H^0W(m)$. For any $M\subset E$, define

$$\sigma_M = n\left[\sum_{i=1}^{N} \dim M_i + \varepsilon(\dim N_1 - \operatorname{wt} N_1)\right] - p[Nk+\varepsilon(k-\operatorname{wt} V)], \quad \text{where} \quad \dim M = p.$$

Define

$$\chi_M = n[N \operatorname{rk} W + \varepsilon(\operatorname{rk} W - \operatorname{wt} W)] - \dim H^0 W(m)\,[Nk+\varepsilon(\operatorname{rk} V - \operatorname{wt} V)].$$

We have to show $\sigma_m \geqq 0 (>0)$ for V semi-stable (stable).

Case 1. *Assume that $M = H^0W(m)$, M generates $W(m)$ and $H'W(m)=0$*

Then we have $\sigma_M = \chi_M$.
Further,

$$\frac{\chi_M}{k \operatorname{rk} W} = \frac{h^0V(m)}{k}\left[N+\varepsilon-\varepsilon\frac{\operatorname{wt} W}{\operatorname{rk} W}\right] - \frac{h^0W(m)}{\operatorname{rk} W}\left[N+\varepsilon-\varepsilon\frac{\operatorname{wt} V}{\operatorname{rk} V}\right].$$

As $\operatorname{par}\deg V=0$, we have $\deg V+\operatorname{wt} V=0$, and as $\operatorname{par}\deg W\leqq 0$, we have $\operatorname{wt} W\leqq -\deg W$.

Consequently,

$$\frac{\chi_M}{k\cdot \operatorname{rk} W} = \left(\frac{h^0V(m)}{\operatorname{rk} V} - \frac{h^0W(m)}{\operatorname{rk} W}\right)(N+\varepsilon) - \left(\frac{h^0V(m)}{\operatorname{rk} V}\cdot\frac{\operatorname{wt} W}{\operatorname{rk} W} - \frac{h^0W(m)}{\operatorname{rk} W}\cdot\frac{\operatorname{wt} V}{\operatorname{rk} V}\right)$$
$$\geqq \left(\frac{\deg V}{\operatorname{rk} V} - \frac{\deg W}{\operatorname{rk} W}\right)(N+\varepsilon) + \varepsilon\left(\frac{h^0V(m)}{\operatorname{rk} V} - \frac{\deg W}{\operatorname{rk} W}\right) - \varepsilon\left(\frac{h^0W(m)}{\operatorname{rk} W}\frac{\deg V}{\operatorname{rk} V}\right).$$

Now the R.H.S. above

$$=(N+\varepsilon)\left(\frac{\deg V}{\operatorname{rk} V} - \frac{\deg W}{\operatorname{rk} W}\right) + \varepsilon(m+1-g)\left(\frac{\deg W}{\operatorname{rk} W}\right) - \varepsilon(m+1-g)\left(\frac{\deg V}{\operatorname{rk} V}\right)$$
$$=(N+\varepsilon)\left(\frac{\deg V}{\operatorname{rk} V} - \frac{\deg W}{\operatorname{rk} W}\right) + \varepsilon(m+1-g)\left(\frac{\deg W}{\operatorname{rk} W} - \frac{\deg V}{\operatorname{rk} V}\right).$$

Hence,

$$\frac{\chi_M}{k\cdot \operatorname{rk} W}\left(\frac{\deg V}{\operatorname{rk} V} - \frac{\deg W}{\operatorname{rk} W}\right)(N-\varepsilon[m-g]).$$

But

$$\varepsilon = \frac{N}{m-g}, \quad \text{hence} \quad \frac{M}{k \operatorname{rk} W} \geqq 0. \tag{2}$$

Further, from the above calculation, it follows that V is stable, the inequality in (2) is strict. If V is semi-stable but not stable, then by taking $M = H^0(W(m))$, where

$W \subset V$ and $\operatorname{par}\deg W = 0$, we get that $\chi_M = 0$. And if V is unstalbe and $W \subset V$ with $\operatorname{par}\deg W > 0$, we get $\chi_M < 0$ where again $M = H^0(W(m))$.

Thus the proposition is proved in case (1) and the assertions about $\tilde{R}^{ss}$ and $\tilde{R}^s$ being open subsets of $\tilde{R}$ are established.

We continue with the proof in the other cases.

Lemma 4.3. $\dfrac{\chi_M}{k \cdot \operatorname{rk} W} \geqq 3N$ *if* $\deg W \leqq (-g-8)k$.

Proof. We have

$$\frac{\chi_M}{k \cdot \operatorname{rk} W} \geqq \frac{n}{k}\left(N + \frac{1}{m-g} - \left(\frac{\operatorname{wt} W}{\operatorname{rk} W}\frac{1}{m-g}\right)\right]$$

$$-\left(1 + \frac{\deg W(m)}{\operatorname{rk} W}\right)\left[N\left(1 + \frac{1}{m-g}\frac{\operatorname{wt} V}{\operatorname{rk} V}\frac{1}{m-g}\right)\right].$$

[For this inequality, we use $R-R$ and the fact that $\dim H^0 W(m) \leqq \deg W(m) + \operatorname{rk} W$.]

Now the R.H.S. in the first sentence above

$$= N\left[1 + \frac{1}{m-g}\left(\frac{1-\operatorname{wt} V}{\operatorname{rk} V}\right)\right]\left[\frac{\deg V}{\operatorname{rk} V} + m - g + 1 - \frac{\deg W}{\operatorname{rk} W} - m - 1\right]$$

$$+ \frac{N}{m-g}\left(\frac{\deg V}{\operatorname{rk} V} + m - g + 1\right)\left(\frac{\operatorname{wt} V}{\operatorname{rk} V} - \frac{\operatorname{wt} W}{\operatorname{rk} W}\right).$$

$$= N\left[1 + \frac{1}{m-g}\left(1 - \frac{\operatorname{wt} V}{\operatorname{rk} V}\right)\right]\left[\frac{\deg V}{\operatorname{rk} V} - \frac{\deg W}{\operatorname{rk} W} - g\right]$$

$$+ \frac{N}{m-g}\left[(m-g) + 1 + \frac{\deg V}{\operatorname{rk} V}\right]\left[\frac{\operatorname{wt} V}{\operatorname{rk} V} - \frac{\operatorname{wt} W}{\operatorname{rk} W}\right].$$

As $\dfrac{\operatorname{wt} V}{\operatorname{rk} V} - \dfrac{\operatorname{wt} W}{\operatorname{rk} W} \geqq -2$, the second summand in the above expression is greater than or equal to $4N$. Likewise, the first summand is greater than or equal to

$$N\left(\frac{\deg V}{\operatorname{rk} V} - \frac{\deg W}{\operatorname{rk} W} - g\right).$$

Hence,

$$\frac{\chi_M}{k \operatorname{rk} W} \geqq N\left(\frac{\deg V}{\operatorname{rk} V} - \frac{\deg W}{\operatorname{rk} W} - g - 4\right).$$

As we have assumed that $\deg W \leqq (-g-8)k$, we have

$$\frac{\deg W}{\operatorname{rk} W} \leqq \frac{\deg V}{\operatorname{rk} V} - g - 7.$$

Hence we get the assertion of Lemma 4.3.

We now assume that $\deg W \leqq (-g-8)k$. Now $\sigma_M = \chi_M + (\sigma_M - \chi_M)$, and

$$\sigma_M - \chi_M = n\left[\sum_{i=1}^{N} \dim M_i + \varepsilon(\operatorname{rk} N_1 - \operatorname{wt} N_1)\right] - \dim M[Nk + \varepsilon(k - \operatorname{wt} V)]$$
$$- n[N \operatorname{rk} W + \varepsilon(\operatorname{rk} W - \operatorname{wt} W)] + \dim M'[Nk + \varepsilon(k - \operatorname{wt} V)],$$

where we have put $M' = H^0 W(m)$.

Hence

$$\sigma_M - \chi_M = n\left[\sum_{i=1}^{N} \dim M_i - N \operatorname{rk} W\right] + n[\varepsilon(\operatorname{rk} N_1 - \operatorname{wt} N_1)$$
$$- \varepsilon(\operatorname{rk} W - \operatorname{wt} W)] + (\dim M' - \dim M)\,[Nk + (k - \operatorname{wt} V)].$$

So we get

$$\sigma_M - \chi_M \geqq n\left[\sum_{i=1}^{N} \dim M_i - N \operatorname{rk} W\right] - n[\varepsilon(\operatorname{rk} W - \operatorname{wt} W)].$$

Hence

$$\frac{\sigma_M - \chi_M}{k \cdot \operatorname{rk} W} \geqq \frac{n}{k \cdot \operatorname{rk} W}\left[\sum_{i=1}^{N} \dim M_i - N \operatorname{rk} W\right] - \frac{n}{k}\left[\varepsilon\left(1 - \frac{\operatorname{wt} W}{\operatorname{rk} W}\right)\right].$$

Now, we have

$$0 < \left(N \operatorname{rk} W - \sum_{i=1}^{N} \dim M_i\right) \leqq \operatorname{rk} W$$

[number of points on X where M does not generate $W(m)$], which in turn is less than or equal to $\operatorname{rk} W \cdot \deg W(m)$.

Hence

$$\frac{\sigma_M - \chi_M}{k \cdot \operatorname{rk} W} \geqq -\frac{n}{k} \deg W(m) - \frac{n}{k}\frac{N}{m-g}.$$

As

$$\frac{\mathrm{n}}{\mathrm{k(m-g)}} = \frac{\dfrac{\deg V}{\operatorname{rk} V} + m + 1 - g}{m-g} = 1 + \frac{1 + \dfrac{\deg V}{\operatorname{rk} V}}{m-g} \leqq 2.$$

We get $\dfrac{\sigma_M - \chi_M}{k \cdot \operatorname{rk} W} \geqq -\dfrac{n}{k} \deg W(m) - 2N.$

Now by Lemma 4.3, we have $\dfrac{\chi_M}{k \cdot \operatorname{rk} W} \geqq 3N.$

So,

$$\frac{\sigma_M}{k \cdot \operatorname{rk} W} \geqq N - \frac{n}{k} \deg W(m)$$
$$\geqq \frac{kN - n \deg W(m)}{k}$$
$$> N - \frac{n \deg W(m)}{N}.$$

Now $\deg W(m) \leqq \theta$ and $N \geqq n\theta$, so $\dfrac{\sigma_M}{k \cdot \operatorname{rk} w} > 0$, which means that $\sigma_M > 0$.

So to finish the proof of Proposition 4.2, we have to treat the case where $\deg W > (-g-8)k$. But by our choice of m, we have $H'(W(m)) = 0$ and $H^0(W(m))$ generates $W(m)$. So if $M = H^0(W(m))$, then $\sigma_M = \chi_M$ and we are through by Case 1.

So, finally, assume that $M \subsetneqq H^0 W(m) = M'$.

Now $\sigma_M = (\sigma_M - \chi_M) + \chi_M$ and $\chi_M = \chi_{M'} > 0$ by Case 1. So it suffices to prove that $\sigma_M - \chi_M > 0$.

We have

$$\sigma_M - \chi_M = n\left[\sum_{i=1}^{N} (\dim M_i - \operatorname{rk} W) + \varepsilon(\operatorname{rk} N_1 - \operatorname{wt} N_1) - \varepsilon(\operatorname{rk} W - \operatorname{wt} W)\right]$$
$$- (\dim M - \dim M')(Nk + \varepsilon[k - \operatorname{wt} V]).$$

So

$$\frac{\sigma_M - \chi_M}{N} = (\dim M' - \dim M)\left(k + \frac{\varepsilon(k - \operatorname{wt} V)}{N}\right) + \frac{n}{N}\sum_{i=1}^{N} (\operatorname{rk} M_i - \operatorname{rk} W)$$
$$+ \frac{n}{m-g}(\operatorname{rk} N_1 - \operatorname{wt} N_1) - \frac{n}{m-g}(\operatorname{rk} W - \operatorname{wt} W).$$

Sub-case 1. Assume that $N_1 = W_P$. In this situation, we have $\operatorname{rk} N_1 = \operatorname{rk} W$ and $\operatorname{wt} N_1 = \operatorname{wt} W$.

Then

$$\frac{\sigma_M - \chi_M}{N} = (\dim M' - \dim M)\left(k + \frac{k - \operatorname{wt} W}{m-g}\right)$$
$$+ \frac{n}{N}\sum_{i=1}^{N} (\operatorname{rk} M_i - \operatorname{rk} W).$$

The absolute value of the second summand in the R.H.S. above is $\leqq \dfrac{n \deg W(m) \operatorname{rk} W}{N}$.

Hence

$$\left|\frac{n}{N \cdot \operatorname{rk} V}\sum_{i=1}^{N} (\operatorname{rk} M_i - \operatorname{rk} W)\right| \leqq \frac{n \deg W(m) \operatorname{rk} W}{N \cdot \operatorname{rk} V}$$

and

$$(\dim M' - \dim M)\left(k + \frac{k - \operatorname{wt} V}{m-g}\right) > k = \operatorname{rk} V.$$

Hence

$$\frac{\sigma_M - \chi_M}{k \cdot \operatorname{rk} V} \geqq \frac{N - n\theta}{n} > 0.$$

Sub-case 2. $N_1 \subsetneqq W_P$

Then

$$\frac{\sigma_M-\chi_M}{N}=(\dim M'-\dim M)\left(k+\frac{k-\operatorname{wt}V}{m-g}\right)+\frac{n}{m-g}(\operatorname{rk}N_1-\operatorname{wt}W)$$
$$+\frac{n}{m-g}(\operatorname{wt}W-\operatorname{wt}N_1)+\frac{n}{N}\sum_{i=1}^{N}(\operatorname{rk}M_i-\operatorname{rk}W).$$

Now $0<(\operatorname{rk}W-\operatorname{rk}N_1)<(\dim M'-\dim M)$, which follows from

$$\begin{array}{ccccc} H^0W(m)_P & \to & W(m)_P & \to & 0 \\ \cup & & \cup & & \\ M & \to & N_1 & \to & 0 \end{array}$$

Hence

$$\frac{\sigma_M-\chi_M}{N\cdot\operatorname{rk}V}\geqq(\dim M'-\dim M)\left(1+\frac{k-\operatorname{wt}V}{m-g}\frac{1}{k}\right)$$
$$-(\dim M'-\dim M)\left(\frac{n}{\operatorname{rk}V}\frac{1}{m-g}\right)$$
$$+\frac{n}{m-g}\left(\frac{\operatorname{wt}W-\operatorname{wt}N_1}{\operatorname{rk}V}\right)-\frac{n\theta}{N}.$$

Now,

$$\frac{n}{\operatorname{rk}V}=\frac{h^0V(m)}{\operatorname{rk}V}=\frac{\deg V}{\operatorname{rk}V}+(m+1-g),$$

hence $\displaystyle\frac{n}{k}\cdot\frac{1}{m-g}=\frac{1+\dfrac{1+\deg V}{\operatorname{rk}V}}{m-g}=\frac{1+\dfrac{1-\operatorname{wt}V}{k}}{m-g}.$

So,

$$\sigma_M-\chi_M\geqq(\operatorname{wt}W-\operatorname{wt}N_1)\frac{1+\dfrac{1-\operatorname{wt}V}{k}}{m-g}-\frac{n\theta}{N}.$$

As $\operatorname{wt}V\leqq k$, we have $1-\dfrac{\operatorname{wt}V}{k}\geqq 0$ and $(\operatorname{wt}W-\operatorname{wt}N_1)\geqq\alpha_1$. So $\sigma_M-\chi_M\geqq\alpha_1-\dfrac{n\theta}{N}$. But $N\geqq\dfrac{2n\theta}{\alpha_1}$, which implies that $\dfrac{\alpha_1}{2}\geqq\dfrac{n\theta}{N}$. So we get that $\sigma_M-\chi_M\,\alpha_1-\dfrac{\alpha_1}{2}=\dfrac{\alpha_1}{2}>0$, which proves Proposition 4.2.

Remark 4.3. We should like to point out the changes in the above proof if a parabolic structure is given at another point $Q\in X$. Similar considerations hold if parabolic structures are given at several points. So let the parabolic structure at Q be defined by

$$V_Q=F_1V_Q\supset F_2V_Q\cdots\supset F_sV_Q,$$

and the weights are given by $0<\beta_1<\beta_2\dots<\beta_j<1$.

In this case $\tilde{R}$ is a fiber-space over R of fiber-type $\mathscr{F} \times \bar{\mathscr{F}}$, where $\bar{\mathscr{F}}$ is the flag variety of type given by the quasi-parabolic structure at Q. The range variety for the map T is

$$H_{n,k}(E)^{2N} \times \prod_{i=1}^{r} H_{n,t_i} \times \prod_{j=1} H_{n,l_j},$$

where $l_1 = k$ and $l_j = \dim F_1 V_Q - F_j V_Q$, $2 \leqq j \leqq s$.

The constants are chosen as follows:

M has the properties 1) $m \geqq g+1$ and 2) $H'W(m)=0$ and $H^0(W(m))$ generates $W(m)$ whenever $W(m) \in S(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, d, 0)$ or $W \subset V$, $V \in S(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, d, 0)$ and $\deg W \geqq (-g-14)k$, θ is the same as before and N is chosen so that $N \geqq \dfrac{4n\theta}{\alpha_1 + \beta_1}$ and $\varepsilon = \dfrac{N}{m-g}$. Here $S(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, d, 0)$ denotes the set of parabolic semi-stable bundles of rank k, ordinary degree d, parabolid degree 0 and weights (α_i) and (β_j) at P and Q respectively.

We now continue with the proof of Theorem 4.1. From Proposition 4.2 we know now that T maps $\tilde{R}^{ss}$ into Z^{ss}. By standard methods in geometric invariant theory, we also know that a good quotient Z^{ss}/G exists. Let M be the image of $\tilde{R}^{ss}$ in Z^{ss}/G. By the completeness theorem (cf. Theorem 3.1), M is a closed sub variety of Z^{ss}/G. Let $\tilde{M}$ be its normalization. Then, we contend that $\tilde{M}$ is the variety we are looking for. We have only to prove that two points V and V' of $\tilde{R}^{ss}$ are equivalent if and only if they have the same image in $\tilde{M}$. For this it suffices to prove

Proposition 4.4. *If C_1 and C_2 are two closed disjoint G invariant subsets of $\tilde{R}^{ss}$, their images in $\tilde{M}$ are disjoint.*

Before the proof we need an auxiliary construction. Choose a point $Q \in X$, distinct from P and consider the set $S = S(k, \boldsymbol{\alpha}, d, 0)$ defined before. On each $V \in S$, define a parabolic structure by choosing a full flag at V_Q, i.e. a flag of the form $V_Q = F_1 V_Q \supset F_2 V_Q \ldots \supset F_k V_Q$. Choose weights $0 < \beta_1 < \beta_2 \ldots < \beta_k < 1$ so small such that the following properties are satisfied:

1) If $V \in S$ is semi-stable for the (α, β) structure, then it is stable.
2) If V is (α, β) semi-stable then V is (α) semi-stable.
3) V is (α) stable implies that V is (α, β) stable for any choice of full flag at Q.

We construct the Hilbert scheme Q_β for the new category of parabolic vector bundles on X with parabolic structures at P and Q and denote by $\tilde{R}^s_\beta$ the open subset of all stable parabolic bundles. We map $\tilde{R}^s_\beta$ into the corresponding product of Grassmannians and Flag varieties and compose it with the canonical map onto the quotient variety. Let the image of $\tilde{R}^s_\beta$ in the quotient variety be denoted by W. Now W is a set-theoretic orbit space for $\tilde{R}^s_\beta$, i.e. two bundles V_1 and V_2 in $\tilde{R}^s_\beta$ are G-equivalent if and only if they have the same image in W. By the results of [11, Proposition 6.1], a geometric quotient of $\tilde{R}^s_\beta \bmod G$ exists, say $\tilde{M}_\beta$. Although the "forgetful" map $\tilde{R}^s_\beta \to \tilde{R}^{ss}$ is not globally defined, by the local universality of $\tilde{R}^{ss}$, for all $t \in \tilde{R}^s_\beta$ there is a neighbourhood U of t and a map $f: U \to \tilde{R}^{ss}$, which is unique upto G-translation.

Now we can prove Proposition 4.4. Assume that C_1 and C_2 are two disjoint, closed, G-invariant subsets of $\tilde{R}^{ss}$ whose images in M intersect, say

$x \in \tilde{\pi}(C_1) \cap \tilde{\pi}(C_2)$, where $\pi : \tilde{R}^{ss} \to M$. Now the "inverse" images of C_1 and C_2 under the family of local maps $\tilde{R}^s_\beta \to \tilde{R}^{ss}$ are still closed, disjoint G-invariant subsets of R^s_β, say D_1 and D_2. Look at $\pi_\beta(D_1)$ and $\pi_\beta(D_2)$, where $\pi_\beta : \tilde{R}^s_\beta \to \tilde{M}_\beta$. Now $\pi_\beta(D_1)$ and $\pi_\beta(D_2)$ are closed disjoint subsets of $\tilde{M}_\beta$. By G-triviality, the local maps patch together to give a global map $\bar{p} : \tilde{M}_\beta \to \tilde{M}$. Now $\bar{p}$ is a projective morphism of normal varieties. *Assume the existence of a stable bundle in* $\tilde{M}$. Then the generic fibre of $\bar{p}$ is a Flag-variety and hence connected. Thus the special fibre of $\bar{p}$ is also connected. But as $x \in \tilde{\pi}(C_1) \cap \tilde{\pi}(C_2)$, we have $\bar{p}^{-1}(x) \subset \pi_\beta(D_1) \cup \pi_\beta(D_2)$ and $\bar{p}^{-1}(x)$ intersects both $\pi_\beta(D_1)$ and $\pi_\beta(D_2)$, contradicting the connectedness of $\bar{p}^{-1}(x)$. Thus the images of C_1 and C_2 in M are disjoint, completing the proof of Proposition 4.4.

$$\begin{tikzcd}
D_1, D_2 \subset \tilde{R}^s_\beta \arrow[rr, "\text{local maps}"] \arrow[d, "\pi_\beta"'] & & \tilde{R}^{ss} \supset C_1, C_2 \arrow[d, "\pi"'] \arrow[dr] & \\
\tilde{M}_\beta \arrow[rr, "\bar{p}"'] & & \tilde{M} \arrow[r] & M
\end{tikzcd}$$

The existence of a stable bundle in $\tilde{M}$ is proved in Sect. 5.

Remark 4.5. One could ask whether as in [15] the morphism $T : \tilde{R}^{ss} \to Z^{ss}$ could be proved to be *proper* and hence a closed immersion (for a suitable choice of m, N etc.). If this were the case, the proof of Theorem 4.1 would have been a more direct application of Geometric Invariant Theory and would not require Sect. 3 and considerations of Sect. 5. One could extend T to a multivalued mapping $T : \tilde{Q} \to Z$, where $\tilde{Q}$ is a complete variety containing $\tilde{R}$ and what is required for properness is to show that if $q \in \tilde{Q}$ and $T(q) \in Z^{ss}$, then $q \in \tilde{R}^{ss}$. Unfortunately, we are not able to prove this.

Remark 4.6. We now show that the variety $\tilde{M}$ is a *coarse moduli scheme* in the sense of [5, Sect. 1]. Evidently, we only have to show that $\tilde{M}$ is the *categorical quotient* of $\tilde{R}^{ss}$ in the sense of [11, Definition 1.4]. So let $\bar{f} : \tilde{R}^{ss} \to N$ be any G-invariant morphism with $\bar{\Gamma}$ as its graph. Since $\bar{f}$ is constant on the fibres of $\pi : \tilde{R}^{ss} \to \tilde{M}$, $\bar{f}$ induces a set theoretic map $f : \tilde{M} \to N$ with graph Γ. Consider $\pi \times \mathrm{id} : \tilde{R}^{ss} \times N \to \tilde{M}/N$. This is a surjective map which is also *closed* as $\tilde{R}^{ss} \times N \bmod G$ *is complete* in the sense of [11, Definition 4.1]. Thus $\pi \times \mathrm{id}$ maps $\bar{\Gamma}$ onto Γ and hence Γ is a closed subset of $\tilde{M} \times N$. Endow Γ with the reduced structure. Then Γ is birationally isomorphic to $\tilde{M}$ as the map f induces a *morphism* on the stable points of $\tilde{M}$. The normality of $\tilde{M}$ ensures that the canonical projection of Γ onto $\tilde{M}$ is an isomorphism and hence $f : \tilde{M} \to N$ is a morphism.

5. Existence of Stable Parabolic Bundles and Computation of the Dimension of the Moduli Space

We keep the notations of Sect. 4 for the curve X, the point $P \in X$, the category of parabolic vector bundles on X with parabolic structure at P, and $\tilde{M}$ the moduli variety constructed in Sect. 4. In order to prove the existence of a stable parabolic bundle in $\tilde{M}$, we construct the moduli variety over a discrete valuation A, compare the dimensions of the special and general fibres and then use representation methods in characteristic zero to complete the argument.

First, let the curve X be defined over an algebraically closed field k of characteristic $p>0$. Let A be a complete discrete valuation ring of characteristic zero with residue field k and field of fractions K. Now X lifts to a scheme X_A which is smooth and projective over spec A. The point $P\in X$ can be regarded as a section $\sigma:\mathrm{Spec}(k)\to X$, which also extends to a section $\sigma_A:\mathrm{Spec}(A)\to X_A$.

Consider the category $S_A(k,\boldsymbol{\alpha},d,0)$ of all parabolic semi-stable vector bundles on X_A of rank k, weights $\boldsymbol{\alpha}$, degree d and parabolic degree 0. We have the same vanishing theorems as in the geometric case and so we can define the quotient scheme $Q_A(E|P)$, and also the open subset R_A. Similarly we also define $\tilde{R}_A$, which is a fibre bundle over R_A with fibre $\mathscr{F}_A$, and the open subset $\tilde{R}_A^{ss}$ of $\tilde{R}_A$ corresponding to the semi-stable bundles on X_A. We have the map $T_A:\tilde{R}_A^{ss}\to Z_A$, where Z_A is the product of Grassmannians and Flag varieties over $\mathrm{Spec}(A)$. Let M_A be the image of $\tilde{R}_A^{ss}$ in Z_A/G. By the extension theorem of Sect. 3, any semi-stable bundle on the general fibre extends to a semi-stable bundle on the special fibre. So the canonical map $\tilde{R}_A^{ss}$ to $\mathrm{Spec}(A)$ is surjective which implies that the map from M_A to $\mathrm{Spec}(A)$ is surjective. As M_A is an integral sub-scheme of $\tilde{R}_A^{ss}/G$, M_A is hence flat over $\mathrm{Spec}(A)$, with equi-dimensional fibres. So we have that $\dim M_K=\dim M_k$.

Let us assume now that $\dim M_K$ is "correct", i.e. $\dim M_K=k^2(g-1)+1+\dim\mathscr{F}$. Then it follows easily that there must exist a stable bundle in M_k. If not, then every point of M_k would have a non-trivial filtration by stable bundles, which would mean that M_k is a finite union of the images of varieities of strictly smaller dimension, a contradiction. So we have only to prove that $\dim M_K=k^2(g-1)+1+\dim\mathscr{F}$. To that end we may assume that $X=H^+/\Gamma$ as in Sect. 1 with $P\in X$ corresponding to the parabolic cusp of Γ in H^+, which may be assumed to be ∞. Let Γ_∞ be the stabilizer of ∞ and $\sigma:\Gamma\to U(k)$ a unitary representation of Γ, with $B\in U(k)$ the image of the generator of Γ_∞. Define $e=\mathrm{Rank}(\mathrm{Id}\text{-}B)$. Then we have the following (cf. [9])

$$\dim_{\mathbb{R}} H'_{\mathrm{Par}}(\Gamma,\sigma)=2k(g-1)+2\dim_{\mathbb{R}}[H^0(\Gamma,\sigma)]+e.$$

Let $\mathbf{U(k)}$ be the Lie algebra of $U(k)$, the space of all $k\times k$ skew-hermitian matrices. Let $\mathrm{Ad}\,B$ be the map $\mathbf{U(k)}\to\mathbf{U(k)}$ given by

$$M\to BMB^{-1},\qquad M\in\mathbf{U(k)}.$$

Assume that

$$B=A\begin{pmatrix}\exp(2\pi i\alpha_1) & & 0\\ & \ddots & \\ 0 & & \exp(2\pi i\alpha_r)\end{pmatrix}A^{-1},$$

where $A\in U(k)$ and let $k_1,\dots,k_r$ be the multiplicities of $\alpha_1,\dots,\alpha_r$. Then we have

$$\mathrm{Rank}(\mathrm{Id}-\mathrm{Ad}\,B)=k^2-\sum_{i=1}^{r}k_i^2=2\sum_{i\neq j}k_ik_j.$$

Let $U(k)$ act on itself by inner conjugation. The isotropy group at B has real dimension $\sum_{i=1}^{r}k_i^2$. Hence the dimension of the orbit through B has real dimension $k^2-\sum_{i=1}^{r}k_i^2$. So we get that $\mathrm{Rank}(\mathrm{Id}-\mathrm{Ad}\,B)=$ dimension of the space of all matrices conjugate to B. It is easy to check that the above number is equal to twice the complex dimension of the flag-variety $\mathscr{F}$, which consists of all flags $(V=V_1\supset V_2\ldots\supset V_r)$ in a k-dimensional complex vector space V with $\dim V_i-\dim V_{i+1}=k_i$, $1\leqq i\leqq r-1$ and $\dim V_r=k_r$.

Now let R be the set of all representations $\tau:\Gamma\to U(k)$ such that if C is the generator of Γ_∞, then $\tau(C)$ is conjugate to B. We know that a presentation of Γ is given by:

$2g+1$ generators $(X_1, Y_1, X_2, Y_2, \ldots, X_g, Y_g, Z)$ with one relation:

$$\prod_{i=1}^{g}(X_iY_iX_i^{-1}Y_i^{-1})Z=\mathrm{Id}.$$

Denoting by W the conjugacy class of B, we define a map

$$\chi: U(k)^{2g}\times W\to U(k)$$

by:

$$M_1, N_1, \ldots, M_g, N_g, P\to\prod_{i=1}^{g}(M_iN_iM_i^{-1}N_i^{-1})P,$$

where M_i and $N_i\in U(k)$ and $P\in W$.

Then R is precisely $\chi^{-1}(\mathrm{Id})$ and thus acquires the structure of a real analytic space. We want the smooth locus of R and we first quote.

Lemma 5.1 (cf. [12]). *If $\tau\in R$, then the kernel of the differential map $d\chi$ at τ can be identified with $Z^1_{\mathrm{Par}}(\Gamma, \mathrm{Ad}, \tau)$.*

Thus we find

$$\begin{aligned}\dim Z^1_{\mathrm{Par}}&(\Gamma, \mathrm{Ad}\,\tau)\\ &=\dim H'_{\mathrm{Par}}(\Gamma, \mathrm{Ad}\,\tau)+\dim\\ &\quad\text{of space of coboundaries}\\ &=\dim H'_{\mathrm{Par}}(\Gamma, \mathrm{ad}\,\tau)+\dim[\mathbf{U(k)}-\mathit{dim}\,H^0(\Gamma, \mathrm{Ad}\,\tau)].\end{aligned}$$

In particular,

$$\dim Z^1_{\mathrm{Par}}(\Gamma, \mathrm{Ad}\,\tau)=2k^2(g-1)+k^2+\dim H^0(\Gamma, \mathrm{Ad}\,\tau)+\dim W.$$

Now $H^0(\Gamma, \mathrm{Ad}\,\tau)=\Gamma$-invariants of $\mathbf{U(k)}$. But $\dim H^0(\Gamma, \mathrm{Ad}\,\tau)$ is always bigger than or equal to one as the scalar matrices of $\mathbf{U(k)}$ always belong to $\mathbf{U(k)}^\Gamma$. Hence by semi-continuity and implicit function theorems, it follows that the set of all $\tau\in R$ with $H^0(\Gamma, \mathrm{Ad}\,\tau)=1$ is open and smooth. Now $H^0(\Gamma, \mathrm{Ad}\,\tau)=1$ if and only if τ is an irreducible representation. So we get:

Theorem 5.2. *The set of all irreducible unitary representation of Γ of* rank *k and fixed conjugacy class of the image of Γ_∞ is a complex manifold of dimension $2k^2(g-1)+(k^2-1)+2+\dim\mathscr{F}$.*

Now $U(k)$ acts on R and the irreducible subset R_0 of R is $U(k)$ stable. The scalars in $U(k)$ operate trivially and hence $\mathrm{PU}(k)$ acts on R_0. This action is free and hence $R_0/\mathrm{PU}(k)$ is a complex manifold of dimension $2k^2(g-1)+(k^2-1)+2+\dim\mathscr{F}-\dim\mathrm{PU}(k)$. Consequently, we have

Theorem 5.3. *The equivalence classes of irreducible unitary representations of Γ with fixed conjugacy class of the image of Γ_∞ is a complex manifold of dimension $k^2(g-1)+1+\dim\mathscr{F}$.*

So to complete the proof of Proposition 4.4, we have only to show that a parabolic bundle is semi-stable (stable) of degree zero if and only if it is a unitary (irreducible unitary) Γ-bundle. Note that the "if" part has been proved in Proposition 1.12. Let T^{iu}, T^u and P be the set of irreducible unitary, unitary and all representations of Γ respectively with fixed conjugacy class of Γ_∞. Recall the varieties R, $\tilde{R}$, $\tilde{R}^{ss}$ and R^s constructed in Sect. 4. Now the analytic space P parametrizes a family of parabolic vector bundles on X and so by the local universality of $\tilde{R}$ we get a family of local maps $P\to\tilde{R}$ with T^u mapping into $\tilde{R}^{ss}$ and T^{iu} mapping into $\tilde{R}^s$. Let M^{ss} and M^s be the (invariant-theoretic) quotients of R^{ss} and R^s by G. We get well defined maps

$$\pi_0:\frac{T^{iu}}{\mathrm{PU}(k)}\to M^s$$

$$\pi:\frac{T^u}{\mathrm{PU}(k)}\to M^{ss}.$$

π_0 is an injective map between complex manifolds of the same dimension and this π_0 is an open map. On the other hand, Image of $\pi_0=(\text{Image of }\pi)\cap M^s$ and hence Image of π_0 is also closed. So π_0 maps onto M^s and consequently π also maps onto M^{ss}. Now for the group Γ, with generators and relations as given before, it is easily seen that the space T^{iu} is non-empty. Thus M^s is non-empty, which proves the existence of stable parabolic bundles on X in characteristic zero and hence in any characteristic. This completes the proof of Proposition 4.4 and also the proof of Theorem 4.1.

Remark 5.4. Let V be a vector bundle on X, a smooth projective curve, of rank two and degree zero. Suppose we are given a parabolic (or just a quasi-parabolic) structure at a point $P\in X$ defined by a 1-dimensional subspace F_2V_P of V_P. Let T be the torsion $\mathcal{O}_X$-module defined by

$$T_P=V_P|F_2V_P|;T_Q=0 \quad \text{if} \quad Q\sim P.$$

We have a homomorphism of V onto T (or $\mathcal{O}_X$-modules); let W be the kernel of this map. Then the subsheaf W of V is locally free of rank 2 and degree -1. Conversely such a subsheaf W of V determines a *quasi-parabolic structure* on V at P. If V and W vary say over schemes M and N respectively, then pairs (W,V) as above determine a *correspondence* between M and N. This is essentially the definition of a *Hecke correspondence* in the sense of [6c].

Let $\tilde{M}$ denote the variety of parabolic stable bundles of rank 2, degree zero and sufficiently small weights. Then it can be shown (cf. [6a]) that $\tilde{M}$ is in fact a *correspondence variety* between the moduli space of semi-stable vector bundles of rank 2 and degree 0 and the moduli space of stable vector bundles of rank 2 and degree -1.

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