Para-Bose coherent states

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(Received 23 January 1978)

In place of the usual commutation relation $[\hat{a},\hat{a}^\dagger]=1$ we consider the generalized commutation relation characteristic of the para-Bose oscillators, viz., $[\hat{a},\hat{H}]=\hat{a}$ where $\hat{H}$ is the Hamiltonian $(1/2)(\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a})$. The number states and the representation of various operators in the basis formed by these states are obtained. We then introduce the para-Bose coherent states defined as the eigenstates of $\hat{a}$ for this generalized case. We consider some of the properties of these coherent states and also show that the uncertainty product $\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle$ in this case could be made arbitrarily small.

1. INTRODUCTION

The classical Hamiltonian of a harmonic oscillator of mass m and angular frequency ω is given by

$$H'=(1/2m)p'^2+\tfrac{1}{2}m\omega^2q'^2, \tag{1.1}$$

with the corresponding equations of motion

$$\dot{q}'=p'/m \quad \text{and} \quad \dot{p}'=-m\omega^2q'. \tag{1.2}$$

The passage to quantum theory is made in two steps as follows[1]:

(1) We replace in H' the c-number variables q' and p' of the classical theory by the operators $\hat{q}'$ and $\hat{p}'$ respectively. It is being assumed that the operators $\hat{q}'$ and $\hat{p}'$ are Hermitian and that they operate on a Hilbert space with positive definite metric.

(2) We postulate the commutation relation

$$[\hat{q}',\hat{p}']=i\hbar. \tag{1.3}$$

For the sake of simplicity, we shall be using in place of the quantities q', p', and H' the dimensionless quantities

$$q=(m\omega/\hbar)^{1/2}q', \tag{1.4a}$$

$$p=(m\omega\hbar)^{-1/2}p', \tag{1.4b}$$

$$H=(\hbar\omega)^{-1}H'. \tag{1.4c}$$

We also introduce the quantities α and α^* defined as

$$\alpha=(q+ip)/\sqrt{2}, \quad \alpha^*=(q-ip)/\sqrt{2}. \tag{1.5}$$

The operators corresponding to H, q, p, α, α^* in quantum theory will be denoted as $\hat{H}$, $\hat{q}$, $\hat{p}$, $\hat{a}$, $\hat{a}^\dagger$, respectively. The operator $\hat{a}^\dagger$ is the Hermitian adjoint of $\hat{a}$. Equations (1.1)–(1.3) then simplify to

$$H=\tfrac{1}{2}(q^2+p^2)=\alpha^*\alpha, \tag{1.6}$$

$$\dot{q}=p, \quad \dot{p}=-q, \tag{1.7a}$$

or equivalently

$$\dot{\alpha}=-i\alpha \tag{1.7b}$$

and

$$[\hat{q},\hat{p}]=i, \tag{1.8a}$$

$$[\hat{a},\hat{a}^\dagger]=1. \tag{1.8b}$$

It may be argued that both of the steps mentioned above for passage to quantum theory are not completely unique. Firstly, since $\hat{q}$ and $\hat{p}$ do not commute with each other, care must be taken in replacing q and p by the corresponding operators $\hat{q}$ and $\hat{p}$, respectively. In fact, depending on different rules of association,[2–4] one may obtain different expressions for $\hat{H}$. Hence for definiteness, we assume Weyl's rule[2] in obtaining the quantum expression for the Hamiltonian, i.e., we write

$$\hat{H}=\tfrac{1}{2}(\hat{q}^2+\hat{p}^2) \tag{1.9}$$

$$=\tfrac{1}{2}(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger). \tag{1.10}$$

Secondly, the commutation relation

$$[\hat{a},\hat{a}^\dagger]=1 \tag{1.11}$$

is *not* a consequence of the equations of motion. In fact, if one is only interested in recovering the equations of motion (1.7) for the operators in quantum theory, one must postulate the more general commutation relation

$$[\hat{a},\hat{H}]=\hat{a}. \tag{1.12}$$

It is readily seen that (1.12) follows from (1.11) but the reverse is in general not true.

The case when the particle operators satisfy the more general commutation relation (1.12) has been referred to as the "para-Bose" case.[5–7] Jordan, Mukunda, and Pepper[5] have obtained the "Fock" representation for the para-Bose operators, i.e., they obtained the eigenvalues, and eigenfunctions of the operator $\hat{H}$ and the representation of the other operators in the basis formed by these eigenstates (see also Ref. 8).

In the present paper we are interested in obtaining the "coherent state" representation of these operators, and discuss some of the properties of these states. In analogy with the usual states,[9,10] we define the para-Bose coherent states as the eigenstates of $\hat{a}$,

$$\hat{a}|\alpha\rangle=\alpha|\alpha\rangle, \tag{1.13}$$

where $\hat{a}$ satisfies (1.12) with $\hat{H}$ given by Eq. (1.10).

 0022-2488/78/1910-2089$1.00

In Sec. 2, we give a new derivation for the Fock representation for para-Bose operators. In Sec. 3 we obtain the para-Bose coherent states and discuss some of their properties such as completeness and diagonal coherent state representation in Sec. 4. In Sec. 5 we obtain the uncertainties in the position and momentum variables for the coherent states and observe that in special cases the product of the uncertainties $\langle(\Delta\hat q)^2\rangle \times \langle(\Delta\hat p)^2\rangle$ could be made as small as one likes.

2. PARA-BOSE NUMBER STATES

We start with the basic commutation relation (1.12),

$$[\hat a, \hat H] = \hat a, \tag{2.1}$$

where

$$\hat H = \tfrac{1}{2}(\hat a^\dagger \hat a + \hat a \hat a^\dagger). \tag{2.2}$$

From the fact that $\hat a^\dagger$ is the Hermitian adjoint of $\hat a$ and using the commutation relation (2.1) we readily find that

$$[\hat a^\dagger, \hat H] = -\hat a^\dagger, \tag{2.3}$$

$$[\hat a^2, \hat a^\dagger] = 2\hat a, \tag{2.3a}$$

$$[\hat a^{\dagger 2}, \hat a] = -2\hat a^\dagger. \tag{2.3b}$$

From induction it then follows that

$$[\hat a^{2n}, \hat a^\dagger] = 2n\hat a^{2n-1}, \tag{2.4a}$$

$$[\hat a^{\dagger 2n}, \hat a] = -2n\hat a^{\dagger 2n-1}, \tag{2.4b}$$

whereas

$$[\hat a^{2n+1}, \hat a^\dagger] = \{2n + [\hat a, \hat a^\dagger]\}\hat a^{2n}, \tag{2.5a}$$

$$[\hat a^{\dagger 2n+1}, \hat a] = -\hat a^{\dagger 2n}\{2n + [\hat a, \hat a^\dagger]\}. \tag{2.5b}$$

The commutator $[\hat a, \hat a^\dagger]$ commutes with $\hat a^2$, $\hat a^{\dagger 2}$, and $\hat H$ but not necessarily with $\hat a$ or $\hat a^\dagger$.

Starting in a strictly analogous manner as in the case of an ordinary harmonic oscillator we find that the energy eigenvalues differ by integers:

$$h_0, h_0 + 1, \ldots, h_0 + n, \cdots,$$

where h_0 is the lowest eigenvalue of $\hat H$. In the present case we take h_0 to be completely arbitrary except for the fact that it has to be nonnegative, since $\hat H$ itself is a nonnegative definite operator. We thus label the representation by a parameter h_0, the ground state eigenvalue of $\hat H$. The ordinary harmonic oscillator case is obtained when we take $h_0 = \frac{1}{2}$.

We now introduce the number operator

$$\hat N = \hat H - h_0 = \tfrac{1}{2}(\hat a^\dagger \hat a + \hat a \hat a^\dagger) - h_0, \tag{2.6}$$

and the number states $|n\rangle_{h_0}$ defined by

$$\hat N |n\rangle_{h_0} = n|n\rangle_{h_0}, \quad n = 0, 1, 2, \cdots. \tag{2.7}$$

Obviously $|n\rangle_{h_0}$ is also the eigenstate of $\hat H$ with the eigenvalue $n + h_0$. Because of the relations

$$[\hat N, \hat a] = -\hat a, \tag{2.8a}$$

$$[\hat N, \hat a^\dagger] = \hat a^\dagger, \tag{2.8b}$$

which follow from (2.1) and (2.6), we may interpret $\hat a$ and $\hat a^\dagger$ as annihilation and creation operators, respectively. In order to obtain the representation for $\hat a$ and $\hat a^\dagger$, we write

$$\hat a|n\rangle_{h_0} = \lambda_n |n-1\rangle_{h_0}, \tag{2.9}$$

$$\hat a^\dagger |n\rangle_{h_0} = \mu_n |n+1\rangle_{h_0}, \tag{2.10}$$

where λ_n and μ_n are some constants to be determined. From the Hermiticity requirement ${}_{h_0}\langle n-1|a|n\rangle_{h_0} = {}_{h_0}\langle n|a^\dagger|n-1\rangle^*_{h_0}$ it follows that

$$\lambda_n = \mu^*_{n-1}, \tag{2.11}$$

Further on taking the norms of Eqs. (2.9), (2.10) and adding we obtain

$$|\lambda_n|^2 + |\mu_n|^2 = 2(n + h_0). \tag{2.12}$$

Equations (2.11) and (2.12) then determine λ_n and μ_n apart from the phase factors. For definiteness, we take these constants to be real. We then find that (cf. Refs. 5,8),

$$\lambda_{2n} = (2n)^{1/2}, \tag{2.13a}$$

$$\lambda_{2n+1} = \{2(n + h_0)\}^{1/2}, \tag{2.13b}$$

$$\mu_n = \lambda_{n+1}, \tag{2.14}$$

$n = 0, 1, 2, \cdots$.

The Para-Bose number states thus satisfy the following properties:

$$\hat N|n\rangle_{h_0} = n|n\rangle_{h_0}, \tag{2.15}$$

$$\hat a|2n\rangle_{h_0} = (2n)^{1/2}|2n-1\rangle_{h_0}, \tag{2.16a}$$

$$\hat a|2n+1\rangle_{h_0} = \{2(n+h_0)\}^{1/2}|2n\rangle_{h_0}, \tag{2.16b}$$

$$\hat a^\dagger|2n\rangle_{h_0} = \{2(n+h_0)\}^{1/2}|2n+1\rangle_{h_0}, \tag{2.17a}$$

$$a^\dagger|2n+1\rangle_{h_0} = \{2n+2\}^{1/2}|2n+2\rangle_{h_0}, \tag{2.17b}$$

$$[\hat a, \hat a^\dagger]|2n\rangle_{h_0} = 2h_0|2n\rangle_{h_0}, \tag{2.18a}$$

$$[\hat a, \hat a^\dagger]|2n+1\rangle_{h_0} = 2(1-h_0)|2n+1\rangle_{h_0}. \tag{2.18b}$$

Further, we have the completeness relation

$$\sum_{n=0}^{\infty} |n\rangle_{h_0}\, {}_{h_0}\langle n| = 1 \tag{2.19a}$$

and the orthogonality relation

$${}_{h_0}\langle n|m\rangle_{h_0} = \delta_{nm}. \tag{2.19b}$$

Relations (2.17) and (2.18) also give

$$|n\rangle_{h_0} = \left\{\frac{\Gamma(h_0)}{2^n\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}\right\}^{1/2} \hat a^{\dagger n}|0\rangle_{h_0},$$
$$h_0 \neq 0, \tag{2.20}$$

where $[K]$ stands for the largest integer smaller than or equal to K.

It has been mentioned earlier that the constant h_0 is an arbitrary nonnegative number. When $h_0 = \frac{1}{2}$, we recover the familiar case of the ordinary oscillator in which case [cf. Eqs. (2.18)] the commutator $[\hat a, \hat a^\dagger]$ becomes unity. Relation (2.20) is not valid for the case when $h_0 = 0$. In this case

$$\hat a^\dagger|0\rangle_0 = 0, \tag{2.21}$$

(and also $\hat a|1\rangle_0 = 0$). We therefore find that $|0\rangle_0$ is an isolated state with no possible interaction with any of the other states. For all practical purposes $|1\rangle_0$ is then the ground state. In fact this situation is identical to the case when $h_0=1$, so that we may associate the state $|n\rangle_0$ with the state $|n-1\rangle_1$. Hence without any loss of generality the case $h_0 \equiv 0$ may be ignored.

3. PARA-BOSE COHERENT STATES

The para-Bose coherent states are defined as the eigenstates of the annihilation operator $\hat{a}$,

$$\hat{a}\,|\alpha\rangle_{h_0}=\alpha\,|\alpha\rangle_{h_0}. \qquad (3.1)$$

The matrix representation of $\hat{a}$ in the para-Bose number states has been obtained in the previous section. We now expand $|\alpha\rangle_{h_0}$ in terms of these number states

$$|\alpha\rangle_{h_0}=\sum_{n=0}^{\infty} C_n\,|n\rangle_{h_0} \qquad (3.2)$$

and use (2.16). From (3.1) we then find the recurrence relations

$$C_{2n}=(\alpha^2/2)\{n(h_0+n-1)\}^{1/2}C_{2n-2}, \qquad (3.3)$$

$$C_{2n+1}=\alpha\{2(n+h_0)\}^{-1/2}C_{2n}. \qquad (3.4)$$

From these relations, we readily obtain

$$C_{2n}=\left\{\frac{\Gamma(h_0)}{n!\,\Gamma(n+h_0)}\right\}^{1/2}\left(\frac{\alpha}{\sqrt{2}}\right)^{2n}C_0 \qquad (3.5)$$

and

$$C_{2n+1}=\left\{\frac{\Gamma(h_0)}{n!\,\Gamma(h_0+n+1)}\right\}^{1/2}\left(\frac{\alpha}{\sqrt{2}}\right)^{2n+1}C_0. \qquad (3.6)$$

Equations (3.5) and (3.6) may equivalently be written in the form

$$C_n=\left\{\frac{\Gamma(h_0)}{2^n\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}\right\}^{1/2}\alpha^n C_0, \qquad (3.7)$$

where $[K]$ as before stands for the largest integer smaller than or equal to K. We now require the coherent state $|\alpha\rangle_{h_0}$ to be normalized,

$$_{h_0}\langle\alpha\,|\,\alpha\rangle_{h_0}=1. \qquad (3.8)$$

Equation (3.8) then determines C_0 except for a phase factor

$$C_0=\left\{\sum_{n=0}^{\infty}\frac{\Gamma(h_0)}{\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}\left(\frac{1}{2}\,|\alpha|^2\right)^n\right\}^{-1/2}. \qquad (3.9)$$

Let us define

$$f(x)=\sum_{n=0}^{\infty}\frac{\Gamma(h_0)}{\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}\left(\frac{1}{2}\,x\right)^n, \qquad (3.10)$$

or equivalently

$$f(x)=\Gamma(h_0)\left(\tfrac{1}{2}x\right)^{1-h_0}\{I_{h_0-1}(x)+I_{h_0}(x)\}, \qquad (3.11)$$

where I_k is the modified Bessel function of the kth order.[11]

From Eqs. (3.2), (3.7), (3.9), and (3.11) we obtain the following expression for the para-bose coherent state

$$|\alpha\rangle_{h_0}=\{f(|\alpha|^2)\}^{-1/2}\times\sum_{n=0}^{\infty}\left\{\frac{\Gamma(h_0)}{2^n\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}\right\}^{1/2}\alpha^n|n\rangle_{h_0}. \qquad (3.12)$$

Using Eq. (2.20) we may also write

$$|\alpha\rangle_{h_0}=[f(\alpha\hat{a}^\dagger)/\{f(|\alpha|^2)\}^{1/2}]\,|0\rangle_{h_0}. \qquad (3.13)$$

The familiar case of the ordinary oscillator, is recovered when we set $h_0=\frac{1}{2}$. In this case we find from Eq. (3.10) that

$$f(x)=\exp(x). \qquad (3.14)$$

Equations (3.13) and (3.14) then give

$$|\alpha\rangle_{1/2}=\exp[-(1/2)|\alpha|^2]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle_{1/2}, \qquad (3.15a)$$

$$=\exp[-(1/2)|\alpha|^2]\exp(\alpha a^\dagger)\,|0\rangle_{1/2}, \qquad (3.15b)$$

which are the well-known expressions for the ordinary coherent states.[9,10]

The Hermitian adjoint of Eq. (3.1) reads

$$_{h_0}\langle\alpha\,|\hat{a}^\dagger=\alpha^*\,{}_{h_0}\langle\alpha\,|. \qquad (3.16)$$

However, one may readily show that there are no right eigenstates of $\hat{a}$, i.e., there is no state $|\lambda\rangle$ which satisfies a relation of the type

$$\hat{a}^\dagger|\lambda\rangle_{h_0}=\lambda\,|\lambda\rangle_{h_0}.$$

We close this section by giving the average values of the various operators in the para-Bose coherent states:

$$_{h_0}\langle\alpha\,|\,\hat{a}\,|\alpha\rangle_{h_0}={}_{h_0}\langle\alpha\,|\,\hat{a}^\dagger\,|\alpha\rangle^*_{h_0}=\alpha, \qquad (3.17)$$

$$_{h_0}\langle\alpha\,|\,\hat{q}\,|\alpha\rangle_{h_0}=\frac{1}{\sqrt{2}}\,(\alpha+\alpha^*), \qquad (3.18)$$

$$_{h_0}\langle\alpha\,|\,[\hat{a},\hat{a}^\dagger]\,|\alpha\rangle_{h_0}=2\,\frac{h_0 I_{h_0-1}(|\alpha|^2)+(1-h_0)I_{h_0}(|\alpha|^2)}{I_{h_0-1}(|\alpha|^2)+I_{h_0}(|\alpha|^2)}, \qquad (3.19)$$

$$_{h_0}\langle\alpha|\,\hat{H}\,|\alpha\rangle_{h_0}=\frac{(h_0+|\alpha|^2)I_{h_0-1}(|\alpha|^2)+(1-h_0+|\alpha|^2)I_{h_0}(|\alpha|^2)}{I_{h_0-1}(|\alpha|^2)+I_{h_0}(|\alpha|^2)}, \qquad (3.20)$$

$$_{h_0}\langle\alpha\,|\,\hat{N}\,|\alpha\rangle=|\alpha|^2+(1-2h_0)\,\frac{I_{h_0}(|\alpha|^2)}{I_{h_0-1}(|\alpha|^2)+I_{h_0}(|\alpha|^2)}. \qquad (3.21)$$

These relations are readily derived from Eqs. (3.1), (1.5), (2.6), (2.18), and (3.12).

4. COMPLETENESS AND THE DIAGONAL COHERENT STATE REPRESENTATION

In this section we show that the para-Bose coherent states introduced in the last section form a complete set, in fact an over-complete set. Analogous to ordinary coherent states, we find the possibility of the diagonal[10] para-Bose coherent state representation.

Since $\hat{a}$ is not Hermitian, the coherent states are not expected to be orthogonal. From Eq. (3.12) we obtain the following expression for the scalar product of two coherent states:

$$_{h_0}\langle\beta\,|\,\alpha\rangle_{h_0}=f(\beta^*\alpha)\{f(|\alpha|^2)f(|\beta|^2)\}^{-1/2}. \qquad (4.1)$$

We show below that these coherent states form an over-complete set. For this purpose we assume the existence of some function $\mu(\alpha)$ such that

$$\int |\alpha\rangle_{h_0 h_0}\langle\alpha| \mu(\alpha) d^2\alpha = 1, \tag{4.2}$$

where $d^2\alpha = dxdy$, x and y being the real and imaginary parts of α and the integration is performed over the whole complex α plane. We substitute for $|\alpha\rangle_{h_0}$ from Eq. (3.12) and find that

$$\sum_{n,m} \left\{\frac{\Gamma(h_0)}{2^n\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}\right\}^{1/2} \times\left\{\frac{\Gamma(h_0)}{2^m\Gamma([m/2]+1)\Gamma([(m+1)/2]+h_0)}\right\}^{1/2} \times \int \{f(|\alpha|^2)\}^{-1}\alpha^{*m}\alpha^n\mu(\alpha)d^2\alpha\, |n\rangle_{h_0 h_0}\langle m| = 1. \tag{4.3}$$

From the orthogonality of the number states [Eq. (2.19b)], we may write

$$\int \{f(|\alpha|^2)\}^{-1}\alpha^{*m}\alpha^n\mu(\alpha)d^2\alpha = \frac{2^n\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}{\Gamma(h_0)}\ \delta_{nm}. \tag{4.4}$$

If we now use polar coordinates

$$\alpha = r\exp(i\theta), \quad d^2\alpha = r\,dr\,d\theta, \tag{4.5}$$

we may readily show that μ cannot depend on θ, and is thus a function of $|\alpha|^2$ only. We then write

$$\mu(\alpha) = \mu(|\alpha|^2). \tag{4.6}$$

Substituting $r^2 = x$, and integrating over θ, we then find that

$$\int_0^\infty \{f(x)\}^{-1}x^n\mu(x)dx = \frac{2^n\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}{\pi\Gamma(h_0)}. \tag{4.7}$$

Thus our problem of showing the completeness of the para-Bose coherent states reduces to determining $\mu(x)$ whose moments are given by (4.7). If we define a function

$$M(y) = \sum_{n=0}^\infty \frac{(2iy)^n}{n!}\ \frac{\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}{\pi\Gamma(h_0)}, \tag{4.8}$$

we may write

$$\int_0^\infty \{f(x)\}^{-1}\mu(x)\exp(ixy)\,dx = M(y). \tag{4.9}$$

The series on the right-hand side of (4.8) is an absolutely convergent series for $|y| < 1$ (and is divergent for $|y| > 1$). For $|y| > 1$, we define $M(y)$ by analytic countinuation. Assuming that $M(y)$ thus defined is well behaved such that we may take the inverse Fourier transform of (4.9), we obtain

$$\mu(x) = \frac{1}{2\pi}\ f(x)\int_{-\infty}^\infty M(y)\exp(-ixy)\,dy. \tag{4.10}$$

We thus obtain the resolution of the identity operator

$$\int |\alpha\rangle_{h_0 h_0}\langle\alpha| \mu(|\alpha|^2)d^2\alpha = 1, \tag{4.11}$$

where $\mu(x)$ is given by (4.10), thereby showing that the para-Bose coherent states are complete. Using Eqs. (4.1) and (4.11) we may write

$$|\beta\rangle_{h_0} = \int \frac{f(\alpha^*\beta)\mu(|\alpha|^2)}{\{f(|\alpha|^2)f(|\beta|^2)\}^{1/2}}\ |\alpha\rangle_{h_0} d^2\alpha. \tag{4.12}$$

Equation (4.12) show that these coherent states in fact form an over complete set. On multiplying (4.12) by $\langle\gamma|$ on the left, we obtain the "self-reproducing" property[12] of $f(x)$:

$$\int \mu(|\alpha|^2)\frac{f(\gamma^*\alpha)f(\alpha^*\beta)}{f(|\alpha|^2)}\ d^2\alpha = f(\gamma^*\beta). \tag{4.13}$$

The possibility of the existence of the diagonal coherent state representation of an arbitrary operator may also be considered. If we write

$$\hat{G} = \int \phi(\alpha)|\alpha\rangle_{h_0 h_0}\langle\alpha|\mu(|\alpha|^2)d^2\alpha, \tag{4.14}$$

we obtain using (3.12),

$$\int \frac{\alpha^m\alpha^{*n}\phi(\alpha)\mu(|\alpha|^2)}{f(|\alpha|^2)}\ d^2\alpha = {}_{h_0}\langle m|\hat{G}|n\rangle_{h_0}\{\Gamma(h_0)\}^{-1} \times\{2^n\Gamma([n/2]+1)\,\Gamma([(n+1)/2]+h_0) \times 2^m\Gamma([m/2]+1)\,\Gamma([(m+1)/2]+h_0)\}^{1/2}. \tag{4.15}$$

Equation (4.15) gives all the moments of $\phi(\alpha)\mu(|\alpha|^2)/f(|\alpha|^2)$.

We close this section by observing that the properties of the usual coherent states are reproduced if we set $h_0 = \frac{1}{2}$. In this case we find from (4.8) that

$$M(y) = \frac{1}{\pi}\ \frac{1}{1-iy}, \tag{4.16}$$

and if we substitute this expression in (4.10) we obtain $\mu(x) = 1/\pi$. This gives

$$\frac{1}{\pi}\int |\alpha\rangle_{1/2\ \ 1/2}\langle\alpha|\,d^2\alpha = 1. \tag{4.17}$$

5. UNCERTAINTY RELATIONS

It is well known that for two Hermitian operators $\hat{A}$ and $\hat{B}$ which satisfy the commutation relation

$$[\hat{A},\hat{B}] = i\hat{C}, \tag{5.1}$$

the uncertainty relation

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \tfrac{1}{4}|\langle\hat{C}\rangle|^2 \tag{5.2}$$

holds. Relation (5.2) is an equality if the state under consideration is an eigenstate of $\hat{A} + i\lambda\hat{B}$, where λ is some real constant. If we identify $\hat{A}$ and $\hat{B}$ with the position and momentum variables $\hat{q}$ and $\hat{p}$ of the para-Bose operator, we find that

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle \geq \tfrac{1}{4}|\langle[\hat{q},\hat{p}]\rangle|^2, \tag{5.3}$$

We may readily verify that relation (5.3) is an equality for the para-Bose coherent states (being eigenstates of the operator $(\hat{q}+i\hat{p})$. From Eq. (1.5) we may write

$$\hat{q} = (1/\sqrt{2})(\hat{a}+\hat{a}^\dagger), \tag{5.4}$$

$$\hat{p} = -(i/\sqrt{2})(\hat{a}-\hat{a}^\dagger). \tag{5.5}$$

The commutator of $\hat{q},\hat{p}$ is therefore given by

$$[\hat{q},\hat{p}] = i[\hat{a},\hat{a}^\dagger]. \tag{5.6}$$

We also have

$$\hat{q}^2 = \hat{H} + \tfrac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2}), \tag{5.7}$$

$$\hat{p}^2 = \hat{H} - \tfrac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2}), \tag{5.8}$$

so that

$$\langle(\Delta\hat{q})^2\rangle = \langle\hat{H}\rangle - \langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle + \tfrac{1}{2}\{\langle(\Delta\hat{a})^2\rangle + \langle(\Delta\hat{a}^\dagger)^2\rangle\} \tag{5.9}$$

and

$$\langle(\Delta\hat{p})^2\rangle = \langle\hat{H}\rangle - \langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle - \tfrac{1}{2}\{\langle(\Delta\hat{a})^2\rangle + \langle(\Delta\hat{a}^\dagger)^2\rangle\}. \tag{5.10}$$

For the coherent state $|\alpha\rangle_{h_0}$, $\langle(\Delta\hat{a})^2\rangle = 0$ and we then find from Eqs. (3.20) that

$$\langle(\Delta\hat{q})^2\rangle = \langle(\Delta\hat{p})^2\rangle = \frac{h_0 I_{h_0-1}(|\alpha|^2) + (1-h_0) I_{h_0}(|\alpha|^2)}{I_{h_0-1}(|\alpha|^2) + I_{h_0}(|\alpha|^2)}. \tag{5.11}$$

Comparing (5.11) with (3.19) and using (5.6) we obtain

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle = \tfrac{1}{4}\,|\langle[\hat{q},\hat{p}]\rangle|^2. \tag{5.12}$$

We thus find that for para-Bose coherent states, the uncertainty relation (5.3) reduces to an equality. However, since $[\hat{q},\hat{p}]$ is in general not a c number the right-hand side of (5.12) itself depends on the given state. Hence the para-Bose coherent states are not the minimum uncertainty states in the absolute sense (except for the case of ordinary oscillator, $h_0 = \frac{1}{2}$ when $[\hat{q},\hat{p}]$ becomes a c number). It is obvious that one may find states for which the uncertainty product $\langle(\Delta\hat{q})^2\rangle \times \langle(\Delta\hat{p})^2\rangle$ is in fact less than $\frac{1}{4}$, which is the minimum value for the ordinary oscillator. For small values of $|\alpha|$, we know that[11]

$$I_k(|\alpha|^2) \sim (\tfrac{1}{2}|\alpha|^2)^k/\Gamma(k+1), \quad (k \neq -1, -2, \ldots)$$

and hence from (5.11) we obtain

$$\langle(\Delta\hat{q})^2\rangle = \langle(\Delta\hat{p})^2\rangle \sim h_0$$

and

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle \sim h_0^2.$$

Thus for para-Bose operators with $h_0 < \frac{1}{2}$, we find that the ground state (or the coherent state with $\alpha = 0$) gives the uncertainty product which is less than $\frac{1}{4}$.

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