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On the complete integrability and linearization of nonlinear ordinary differential equations. IV. Coupled second-order equations

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Coupled second-order nonlinear differential equations are of fundamental importance in dynamics. In this part of our study on the integrability and linearization of nonlinear ordinary differential equations (ODEs), we focus our attention on the method of deriving a general solution for two coupled second-order nonlinear ODEs through the extended Prelle–Singer procedure. We describe a procedure to obtain integrating factors and the required number of integrals of motion so that the general solution follows straightforwardly from these integrals. Our method tackles both isotropic and non-isotropic cases in a systematic way. In addition to the above-mentioned method, we introduce a new method of transforming coupled second-order nonlinear ODEs into uncoupled ones. We illustrate the theory with potentially important examples.

Keywords: nonlinear differential equations; coupled second order; integrability; integrating factors; uncoupling

1. Introduction

In this part of our study on the integrability and linearization of nonlinear ordinary differential equations (ODEs), we focus our attention on the theoretical formulation and applications of the modified Prelle–Singer (PS) procedure (Prelle & Singer 1983; Duarte *et al.* 2001; Chandrasekar *et al.* 2005*a*, 2006) to a set of two coupled second-order ODEs. The need for this demonstration is due to the fact that classifying and studying two-degrees-of-freedom dynamical systems are highly non-trivial problems in the theory of nonlinear dynamical systems. Historically, several techniques have been proposed to identify and obtain general solutions of two coupled second-order ODEs. To cite a few examples, we mention Painlevé analysis, Lie symmetry analysis, generalized Noether symmetries technique, direct methods and so on (Ramani *et al.* 1989; Lakshmanan & Sahadevan 1993; Bluman & Anco 2002; Lakshmanan & Rajasekar 2003). Each of these methods has its own advantages and limitations. For example, among the above-mentioned methods, certain methods fulfil the necessary conditions alone, whereas the others guarantee only sufficient

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conditions for the complete integrability of the system concerned. This factor alone is a motivating factor to search for more and more powerful methods for isolating and classifying integrable and non-integrable dynamical systems. In this way, by extending the PS procedure and its applications to coupled second-order ODEs, we argue that the PS method can be used as a stand-alone technique to solve a wide class of ODEs of any order irrespective of whether it is a single or a coupled equation.

Here, we mention that the present analysis is not a straightforward extension of the scalar case. In fact, by prolonging the theoretical formulation to the coupled second-order ODEs, we deduce the determining equations for the integrating factors and null forms appropriately such that one can obtain the aforementioned functions in a more efficient and straightforward way. Thus, the method of obtaining the integrating factors for the given equation is also augmented in this procedure in an efficient manner. Furthermore, while studying the coupled dynamical systems, one may face both isotropic and non-isotropic cases. Our method covers both of them in a natural way. In addition to the above, in this paper, we also introduce a new method to transform two coupled second-order ODEs to two uncoupled second-order ODEs. Thus, the PS procedure inherits several remarkable features both at the theoretical foundations and in the range of applications, which we have listed already in Chandrasekar *et al.* (2005*a*). Finally, we have carefully fixed the examples so that the basic features associated with this method and the results which it leads to could be explained in an efficient way.

The plan of this paper is as follows. In §2, we describe the PS method applicable for coupled second-order ODEs and indicate the new features in finding the integrating factors and integrals of motion. In §3, we establish a connection between the integrating factors and the form of equations. In §4, the uncoupled equations are briefly considered. In §5, we elaborately discuss the method of constructing integrals and general solutions for the coupled nonlinear ODEs. We support the theory with two non-trivial examples, which are discussed in the contemporary literature. We also briefly discuss the application of our procedure for the case of Liouville integrable systems in §5*d*. We devote §6 to demonstrating yet another method to identify transformation variables from the first integrals, which can be effectively used to rewrite the system of coupled ODEs into uncoupled ones so that one can integrate the resultant equation easily and obtain the general solution. We present our conclusions in §7.

2. The PS method for coupled second-order ODEs

(*a*) *General theory*

Let us consider a system of two coupled second-order ODEs of the form

$$\ddot{x} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{P_1}{Q_1}, \qquad \ddot{y} = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{P_2}{Q_2}, \quad P_i, Q_i \in C[t, x, y, \dot{x}, \dot{y}],\ i = 1, 2, \tag{2.1}$$

where P_i and Q_i are analytic functions of the variables $t, x, y, \dot{x}$ and $\dot{y}$. Let us suppose that the system (2.1) admits a first integral of the form $I(t, x, y, \dot{x}, \dot{y}) = C$, with C constant on the solutions, so that the total differential gives

$$\mathrm{d}I = I_t\mathrm{d}t + I_x\mathrm{d}x + I_y\mathrm{d}y + I_{\dot{x}}\mathrm{d}\dot{x} + I_{\dot{y}}\mathrm{d}\dot{y} = 0, \tag{2.2}$$

where the subscript denotes partial differentiation with respect to that variable. Rewriting (2.1) in the form

$$\frac{P_1}{Q_1}\mathrm{d}t - \mathrm{d}\dot{x} = 0, \qquad \frac{P_2}{Q_2}\mathrm{d}t - \mathrm{d}\dot{y} = 0, \tag{2.3}$$

and adding null terms $s_1(t, x, y, \dot{x}, \dot{y})\dot{x}\,\mathrm{d}t - s_1(t, x, y, \dot{x}, \dot{y})\mathrm{d}x$ and $s_2(t, x, y, \dot{x}, \dot{y})\dot{y}\,\mathrm{d}t - s_2(t, x, y, \dot{x}, \dot{y})\mathrm{d}y$ to the first equation in (2.3), and $u_1(t, x, y, \dot{x}, \dot{y})\dot{x}\,\mathrm{d}t - u_1(t, x, y, \dot{x}, \dot{y})\mathrm{d}x$ and $u_2(t, x, y, \dot{x}, \dot{y})\dot{y}\,\mathrm{d}t - u_2(t, x, y, \dot{x}, \dot{y})\mathrm{d}y$ to the second equation in (2.3), respectively, we obtain that, on the solutions, the 1-forms

$$\left(\frac{P_1}{Q_1} + s_1\dot{x} + s_2\dot{y}\right)\mathrm{d}t - s_1\,\mathrm{d}x - s_2\mathrm{d}y - \mathrm{d}\dot{x} = 0, \tag{2.4a}$$

$$\left(\frac{P_2}{Q_2} + u_1\dot{x} + u_2\dot{y}\right)\mathrm{d}t - u_1\mathrm{d}x - u_2\mathrm{d}y - \mathrm{d}\dot{y} = 0. \tag{2.4b}$$

At this stage, we wish to point out that one can also analyse the coupled second-order ODEs (2.1) by rewriting them as a set of four coupled first-order ODEs of the form $\dot{x} = x_1$, $\dot{x}_1 = P_1/Q_1$, $\dot{y} = y_1$, $\dot{y}_1 = P_2/Q_2$ and its equivalent 1-forms. By introducing four integrating factors, one can deduce the relevant determining equations by following the procedure given by us in the earlier paper, part III (Chandrasekar *et al.* 2009), to the above system of first-order ODEs. However, after examining several examples, we find that it is more advantageous to solve the system (2.1) in the second-order form itself rather than introducing more variables. The procedure is as follows.

Now, on the solutions, the 1-forms (2.2), (2.4a) and (2.4b) must be proportional. Multiplying (2.4a) by the factor $R(t, x, y, \dot{x}, \dot{y})$ and (2.4b) by the factor $K(t, x, y, \dot{x}, \dot{y})$, which act as the integrating factors for (2.4a) and (2.4b), respectively, we have on the solutions that

$$\mathrm{d}I = R(\phi_1 + S\dot{x})\mathrm{d}t + K(\phi_2 + U\dot{y})\mathrm{d}t - RS\mathrm{d}x - KU\mathrm{d}y - R\mathrm{d}\dot{x} - K\mathrm{d}\dot{y} = 0, \tag{2.5}$$

where $\phi_i \equiv P_i/Q_i$, $i = 1,2$; $S = \frac{Rs_1 + Ku_1}{R}$; and $U = \frac{Rs_2 + Ku_2}{K}$. Comparing equations (2.5) and (2.2), we have, on the solutions, the relations

$$\left.\begin{aligned} I_t = R(\phi_1 + S\dot{x}) + K(\phi_2 + Ux), \quad I_x = -RS,\\ I_y = -KU, \quad I\dot{x} = -R, \quad I\dot{y} = -K, \end{aligned}\right\} \tag{2.6}$$

The compatibility conditions between equations (2.6), namely $I_{tx} = I_{xt}$, $I_{t\dot{x}} = I_{\dot{x}t}$, $I_{ty} = I_{yt}$, $I_{t\dot{y}} = I_{\dot{y}t}$, $I_{xy} = I_{yx}$, $I_{x\dot{x}} = I_{\dot{x}x}$, $I_{y\dot{y}} = I_{\dot{y}y}$, $I_{x\dot{y}} = I_{\dot{y}x}$, $I_{y\dot{x}} = I_{\dot{x}y}$ and $I_{\dot{x}\dot{y}} = I_{\dot{y}\dot{x}}$, provide us the following conditions:

$$D[S] = -\phi_{1x} - \frac{K}{R}\phi_{2x} + \frac{K}{R}S\phi_{2\dot{x}} + S\phi_{1\dot{x}} + S^2, \tag{2.7}$$

$$D[U] = -\phi_{2y} - \frac{R}{K}\phi_{1y} + \frac{R}{K}U\phi_{1\dot{y}} + U\phi_{2\dot{y}} + U^2, \tag{2.8}$$

$$D[R] = -(R\phi_{1\dot{x}} + K\phi_{2\dot{x}} + RS), \tag{2.9}$$

$$D[K] = -(K\phi_{2\dot{y}} + R\phi_{1\dot{y}} + KU), \tag{2.10}$$

$$SR_y = -RS_y + UK_x + KU_x, \qquad R_x = SR_{\dot{x}} + RS_{\dot{x}}, \tag{2.11}$$

$$R_y = UK_{\dot{x}} + KU_{\dot{x}}, \qquad K_x = SR_{\dot{y}} + RS_{\dot{y}}, \tag{2.12}$$

$$K_y = UK_{\dot{y}} + KU_{\dot{y}}, \qquad R_{\dot{y}} = K_{\dot{x}}. \tag{2.13}$$

Here, the total differential operator, D, is defined by $D = (\partial/\partial t) + \dot{x}(\partial/\partial x) + \dot{y}(\partial/\partial y) + \phi_1(\partial/\partial \dot{x}) + \phi_2(\partial/\partial \dot{y})$. Integrating equations (2.6), we obtain the integral of motion

$$I = r_1 + r_2 + r_3 + r_4 - \int \left[K + \frac{\mathrm{d}}{\mathrm{d}\dot{y}}(r_1 + r_2 + r_3 + r_4)\right] \mathrm{d}\dot{y}, \tag{2.14}$$

where

$$r_1 = \int \Big(R(\phi_1 + S\dot{x}) + K(\phi_2 + U\dot{y})\Big)\mathrm{d}t, \qquad r_2 = -\int \left(RS + \frac{\mathrm{d}}{\mathrm{d}x}(r_1)\right)\mathrm{d}x,$$

$$r_3 = -\int \left(KU + \frac{\mathrm{d}}{\mathrm{d}y}(r_1 + r_2)\right)\mathrm{d}y, \qquad r_4 = -\int \left(R + \frac{\mathrm{d}}{\mathrm{d}\dot{x}}(r_1 + r_2 + r_3)\right)\mathrm{d}\dot{x}.$$

By solving the determining equations, (2.7)–(2.13), consistently, we can obtain expressions for the functions (S, U, R, K). By substituting them into (2.14) and evaluating the integrals, we can construct the associated integrals of motion. It is also clear that equation (2.1) can be considered as a completely integrable system once we obtain four independent integrals of motion through this procedure.

3. Connection between the integrating factors and the nature of equations

We note that equations (2.7)–(2.13) constitute an overdetermined system for the four unknowns, namely S, U, R and K. Among these equations, the first four equations, (2.7)–(2.10), constitute the evolution equations for the variables S, U, R and K. Here, we mention that, by combining equations (2.7)–(2.10), one can obtain the following two identities:

$$D[RS] = -(R\phi_{1x} + K\phi_{2x}), \qquad D[KU] = -(R\phi_{1y} + K\phi_{2y}). \tag{3.1}$$

Any solution (S, U, R, K) that satisfies equations (2.7)–(2.10) also satisfies equation (3.1). Once the functions S, U, R and K are fixed, then the rest of the problem is to verify whether these functions satisfy the ‘extra determining equations’, i.e. (2.11)–(2.13), or not. If these functions satisfy the extra determining equations, then they form a compatible set of solutions and one can proceed to construct the associated integral of motion from (2.14). On the other hand, if the functions do not satisfy the extra determining equations, then one has to look for alternative ways to obtain compatible solutions. In fact, in practice, one often meets the case in which a certain solution(s) that satisfies (satisfy) the evolutionary determining equations (2.7)–(2.10) does (do) not satisfy the extra determining equations. More specifically, for a class of problems, one often obtains one or two or even three sets of (S, U, R, K) by solving (2.7)–(2.10), which does (do) satisfy the rest and another (other) set(s) does (do) not satisfy the later equations. In this situation, we find an interesting fact that one can use the integral(s) derived from the set(s), which satisfies (satisfy) equations (2.7)–(2.13) and deduce the other compatible solution(s) $(S, U, \hat{R}, \hat{K})$ (definition of $\hat{R}$ and $\hat{K}$ follows). For example, let the set (S_4, U_4, R_4, K_4) be a solution of the evolution equations (2.7)–(2.10) and not of the extra determining equations (2.11)–(2.13). After analysing several examples, we find that one can make the set (S_4, U_4, R_4, K_4) compatible by modifying the form of R_4 and K_4 as

$$\hat{R} = F(t, x, y, \dot{x}, \dot{y})R, \qquad \hat{K} = G(t, x, y, \dot{x}, \dot{y})K, \tag{3.2}$$

where F and G are functions to be determined. However, after making these modifications, S_4 and U_4 remain the same. A motivation to perform this type of modification came from our earlier work on the applicability of the PS procedure to scalar second-order ODEs (Chandrasekar *et al.* 2005*a*). However, unlike the scalar case, here we have to choose two functions, namely F and G, appropriately, such that the compatible forms of $\hat{R}$ and $\hat{K}$ can be fixed. Since we have to choose two functions, one might ask the question of whether these two functions are necessarily different or the same. In the following, we answer this question.

Since $\hat{R}$ and $\hat{K}$ should satisfy equations (2.9) and (2.10), we have

$$D[F]R + FD[R] = -(FR\phi_{1\dot{x}} + GK\phi_{2\dot{x}} + FRS), \tag{3.3}$$

$$D[G]K + GD[K] = -(FR\phi_{1\dot{y}} + GK\phi_{2\dot{y}} + GKU). \tag{3.4}$$

Substituting the expressions for $D[R]$ and $D[K]$ (see (2.9) and (2.10)) in (3.3) and (3.4) and simplifying the resultant equations, we find that

$$D[F]R = (F - G)K\phi_{2\dot{x}}, \qquad D[G]K = (G - F)R\phi_{1\dot{y}}. \tag{3.5}$$

On the other hand, substituting the modified forms (3.2) in the integrability conditions (3.1), we obtain

$$D[F]RS = (F - G)K\phi_{2x}, \qquad D[G]KU = (G - F)R\phi_{1y}. \tag{3.6}$$

Combining equations (3.5) and (3.6), we obtain

$$(F - G)K(S\phi_{2\dot{x}} - \phi_{2x}) = 0, \qquad (G - F)R(U\phi_{1\dot{y}} - \phi_{1y}) = 0. \tag{3.7}$$

From equation (3.7), we can conclude that either $F \neq G$ or $F = G$. In the first case, we have uncoupled equations, i.e.

$$F \neq G \Rightarrow \phi_{2\dot{x}} = \phi_{1\dot{y}} = \phi_{2x} = \phi_{1y} = 0 \quad (\hat{R} = FR, \hat{K} = GK), \tag{3.8}$$

and, in the second case, we have coupled equations, i.e.

$$F = G \Rightarrow \phi_{2\dot{x}}, \phi_{1\dot{y}}, \phi_{2x}, \phi_{1y} \neq 0 \quad (\hat{R} = FR, \hat{K} = FK). \tag{3.9}$$

Note that, in the case of $F \neq G$, the other possible solution $S = \phi_{2x}/\phi_{2\dot{x}}$, $U = \phi_{1y}/\phi_{1\dot{y}}$, when used in (2.7) and (2.8), leads to inconsistencies and so this choice is not considered. Furthermore, the case $R = 0$ and $K = 0$ leads to the trivial solution. The above analysis clearly shows that, for the uncoupled equations, one should choose F and G as different and, for the coupled equations, one should choose them as the same.

Finally, we mention the important point that the functions F and G are nothing but functions of integrals of motion. To show this, substituting back the relations (3.8) and (3.9) in (3.5) and (3.6), respectively, we obtain

$$\begin{aligned} &D[F]R=0, \qquad D[G]K=0 \quad \text{and} \quad D[F]RS=0,\\ &D[G]KU=0, \qquad \Rightarrow D[F]=0=D[G]. \end{aligned} \tag{3.10}$$

Thus, irrespective of whether the given equations are coupled ones or uncoupled ones, the functions F and G can always be taken as integrals of motion or functions of them and the problem now is how to determine them. In the following, we discuss this in detail.

4. Case 1: $F \neq G$ (uncoupled equations)

(a) Theory

In this case, we have the following form of equation of motion:

$$\ddot{x}=\phi_1(t,x,\dot{x}), \qquad \ddot{y}=\phi_2(t,y,\dot{y}). \tag{4.1}$$

As a result, the evolution equations for the functions S, U, R and K (see equations (2.7)–(2.10)) are simplified to the forms

$$D[S]=-\phi_{1x}+S\phi_{1\dot{x}}+S^2, \tag{4.2}$$

$$D[U]=-\phi_{2y}+U\phi_{2\dot{y}}+U^2, \tag{4.3}$$

$$D[R]=-RS-R\phi_{1\dot{x}}, \tag{4.4}$$

$$D[K]=-KU-K\phi_{2\dot{y}}. \tag{4.5}$$

By solving equations (4.2) and (4.3), one can obtain explicit forms of S and U. By substituting the known forms of S and U into equations (4.4) and (4.5) and integrating the resultant equations, one can fix the forms of R and K. From the known forms of (S,U,R,K), one can fix the integrals of motion from (2.14). One can also consider equation (4.1) as two independent second-order ODEs and solve the equations independently by adopting the procedure given in our previous paper (Chandrasekar *et al.* 2005*a*). To avoid repetition, we do not discuss the uncoupled case further here.

5. Case 2: $F = G$ (coupled equations)

(a) Theory

Now we focus on the case $F=G$. To obtain the forms (S,U,R,K), one needs to solve the compatibility conditions (2.7)–(2.10). For this purpose, we rewrite equations (2.7)–(2.10) in terms of two variables alone, namely R and K, by eliminating S and U, and analyse the resulting coupled second-order partial differential equations (PDEs) and obtain expressions for R and K. From the latter, we deduce the forms of S and U through the relations (2.9) and (2.10).

To rewrite equations (2.7)–(2.10) in terms of R and K, let us take a total derivative of equations (2.9) and (2.10). In doing so, we obtain

$$D^2[R] = -D[R\phi_{1\dot{x}} + K\phi_{2\dot{x}} + RS], \qquad D^2[K] = -D[K\phi_{2\dot{y}} + R\phi_{1\dot{y}} + KU]. \quad (5.1)$$

Using the identities (3.1), equation (5.1) can be rewritten in a coupled form for R and K as

$$D^2[R] + D[R\phi_{1\dot{x}} + K\phi_{2\dot{x}}] = R\phi_{1x} + K\phi_{2x}, \quad (5.2)$$

$$D^2[K] + D[K\phi_{2\dot{y}} + R\phi_{1\dot{y}}] = R\phi_{1y} + K\phi_{2y}. \quad (5.3)$$

One can note that the above determining equations (5.2) and (5.3) form a system of linear PDEs. To solve equations (5.2) and (5.3), one may assume a specific ansatz either polynomial or rational in $\dot{x}$ and $\dot{y}$ for R and K, and by substituting the known expressions of ϕ_1 and ϕ_2 and their derivatives into (5.2) and (5.3) and solving them, one can obtain expressions for the integrating factors R and K. Once R and K are known, then the functions (S,U) can be fixed through the relations (2.9) and (2.10). Knowing S, U, R and K, one has to make sure that this set (S,U,R,K) also satisfies the remaining compatibility conditions (2.11)–(2.13). The set (S,U,R,K) that satisfies equations (2.7)–(2.13) is then the acceptable solution and one can then determine the associated integral I using the relation (2.14). For complete integrability, we require four independent compatible sets (S_i,U_i,R_i,K_i), $i=1,2,3,4$.

As discussed in §3, suppose the sets (S_i,U_i,R_i,K_i), $i=1,2,3$, are found to satisfy equations (2.7)–(2.13) and the fourth set (S_4,U_4,R_4,K_4) does not satisfy equations (2.11)–(2.13). In this case to identify the correct form of $\hat{R}_4$ and $\hat{K}_4$, one may assume that $\hat{R}_4 = F(I_1,I_2,I_3)R_4$ and $\hat{K}_4 = F(I_1,I_2,I_3)K_4$, where $F(I_1,I_2,I_3)$ is a function of the integrals I_1, I_2 and I_3. To determine the explicit form of $F(I_1,I_2,I_3)$, we proceed as follows. Substituting

$$\hat{R}_4 = F(I_1,I_2,I_3)R_4, \qquad \hat{K}_4 = F(I_1,I_2,I_3)K_4 \quad (5.4)$$

into equations (2.11)–(2.13), we obtain the following set of six relations:

$$\left.\begin{aligned}
\frac{(a_1F_1' + b_1F_2' + c_1F_3')}{d_1} &= F, & \frac{(a_2F_1' + b_2F_2' + c_2F_3')}{d_2} &= F,\\
\frac{(a_3F_1' + b_3F_2' + c_3F_3')}{d_3} &= F, & \frac{(a_4F_1' + b_4F_2' + c_4F_3')}{d_4} &= F,\\
\frac{(a_5F_1' + b_5F_2' + c_5F_3')}{d_5} &= F, & \frac{(a_6F_1' + b_6F_2' + c_6F_3')}{d_6} &= F,
\end{aligned}\right\} \quad (5.5)$$

where

$$a_1=R(I_{1x}-SI_{1\dot{x}}),\quad b_1=R(I_{2x}-SI_{2\dot{x}}),$$
$$c_1=R(I_{3x}-SI_{3\dot{x}}),\quad d_1=(SR_{\dot{x}}+RS_{\dot{x}}-R_x),\quad a_2=K(I_{1y}-UI_{1\dot{y}}),$$
$$b_2=K(I_{2y}-UI_{2\dot{y}}),\quad c_2=K(I_{3y}-UI_{3\dot{y}}),\quad d_2=(UK_{\dot{y}}+KU_{\dot{y}}-K_y),$$
$$a_3=(RI_{1y}-UKI_{1\dot{x}}),\quad b_3=(RI_{2y}-UKI_{2\dot{x}}),\quad c_3=(RI_{3y}-UKI_{3\dot{x}}),$$
$$d_3=(UK_{\dot{x}}+KU_{\dot{x}}-R_y),\quad a_4=(KI_{1x}-SRI_{1\dot{y}}),\quad b_4=(KI_{2x}-SRI_{2\dot{y}}),$$
$$c_4=(KI_{3x}-SRI_{3\dot{y}}),\quad d_4=(SR_{\dot{y}}+RS_{\dot{y}}-K_x),\quad a_5=(RI_{1\dot{y}}-KI_{1\dot{x}}),$$
$$b_5=(RI_{2\dot{y}}-KI_{2\dot{x}}),\quad c_5=(RI_{3\dot{y}}-KI_{3\dot{x}}),\quad d_5=(K_{\dot{x}}-R_{\dot{y}}),$$
$$a_6=(SRI_{1y}-UKI_{1x}),\quad b_6=(SRI_{2y}-UKI_{2x}),$$
$$c_6=(SRI_{3y}-UKI_{3x}),\quad d_6=(UK_x+KU_x-SR_y-RS_y)$$

are all known functions of $t, x, y, \dot{x}$ and $\dot{y}$ and $F'_i=\partial F/\partial I_i$.

Equation (5.5) represents an overdetermined system of equations for the unknown F. A simple way to solve this equation is to uncouple it for F'_i, $(=\partial F/\partial I_i)$, $i=1,2,3$, and solve the resultant equations. For example, eliminating F'_2 and F'_3 from the first three relations in equation (5.5), we obtain an equation for F'_1 in the form

$$\frac{F'_1}{F}=\frac{(d_1c_2-c_1d_2)(b_1c_3-b_3c_1)-(d_1c_3-c_1d_3)(b_1c_2-b_2c_1)}{(a_1c_2-c_1a_2)(b_1c_3-b_3c_1)-(a_1c_3-c_1a_3)(b_1c_2-b_2c_1)}.\tag{5.6}$$

On the other hand, eliminating F'_1 and F'_3 from equation (5.5) (again from the first three relations), we arrive at the following equations for F'_2 in the form:

$$\frac{F'_2}{F}=\frac{(d_1c_2-c_1d_2)(a_1c_3-a_3c_1)-(d_1c_3-c_1d_3)(a_1c_2-a_2c_1)}{(b_1c_2-c_1b_2)(a_1c_3-a_3c_1)-(b_1c_3-c_1b_3)(a_1c_2-a_2c_1)}.\tag{5.7}$$

In the similar way, one obtains the following equation for F'_3:

$$\frac{F'_3}{F}=\frac{(d_1b_2-b_1d_2)(a_1b_3-a_3b_1)-(d_1b_3-b_1d_3)(a_1b_2-a_2b_1)}{(c_1b_2-b_1c_2)(a_1b_3-a_3b_1)-(c_1b_3-b_1c_3)(a_1b_2-a_2b_1)}.\tag{5.8}$$

One can easily check that the combination of other relations in equation (5.5) along with the forms (5.6)–(5.8) gives rise to relations that are effectively nothing but the constraint equations (2.11)–(2.13), and hence no new constraint actually arises. Consequently, equations (5.6)–(5.8) can be written as

$$\frac{\partial F}{\partial I_1}=g_1(I_1,I_2,I_3)F,\qquad \frac{\partial F}{\partial I_2}=g_2(I_1,I_2,I_3)F\quad\text{and}\quad \frac{\partial F}{\partial I_3}=g_3(I_1,I_2,I_3)F,\tag{5.9}$$

where g_i's, $i=1,2,3$, are functions of I_1, I_2 and I_3. Now solving equations (5.9), one can obtain the explicit form of $F(I_1,I_2,I_3)$. Once F is known, we can obtain the complete solution $\hat{R}_4$ and $\hat{K}_4$ from which, along with S_4 and U_4, the fourth integral I_4 can be constructed using the expression (2.14).

Finally, we note that, in some cases, one can meet the situation that the sets (S_i,U_i,R_i,K_i), $i=1,2$, alone are found to satisfy equations (2.7)–(2.13) and the third set (S_3,U_3,R_3,K_3) (as well as the fourth set) does not satisfy equations (2.11)–(2.13). In this case, F may be a function of the integrals I_1 and I_2, which can be derived from the sets (S_i, U_i, R_i, K_i), $i=1,2$. We need to find the explicit form of $F(I_1, I_2)$ in order to obtain the compatible solution $(S_3, U_3, \hat{R}_3, \hat{K}_3)$. To recover the complete form of $\hat{R}_3$ and $\hat{K}_3$, one may assume that $\hat{R}_3 = F(I_1, I_2)R_3$ and $\hat{K}_3 = F(I_1, I_2)K_3$ and proceed as before and obtain the determining equations for F. Since F is a function of I_1 and I_2 alone, one essentially obtains the same form (5.5) but without the factor $F_3'(=\partial F/\partial I_3)$. Since c_i's, $i=1, \ldots, 6$, are also exclusive functions of I_3, their derivatives do not appear in the determining equations. By solving the resultant determining equations, one can fix the form of F, which, in turn, provides us $\hat{R}_3$ and $\hat{K}_3$ from which one can construct the third integral I_3 for the given problem. Now, from the known forms I_1, I_2 and I_3, one can proceed as before to obtain the fourth compatible set $(S_4, U_4, \hat{R}_4, \hat{K}_4)$ and thereby obtain the fourth integral I_4 also.

Similarly, if the set (S_1, U_1, R_1, K_1) alone is found to satisfy equations (2.7)–(2.13) and the second set (S_2, U_2, R_2, K_2) does not satisfy equations (2.11)–(2.13), then, in this case, the determining equations for F take the form (5.5), with $F_2 = F_3 = b_i = c_i = 0$, $i=1, \ldots, 6$. Following the procedure mentioned above, one can first derive the second integral. From a knowledge of (I_1, I_2), one can again follow the above procedures and construct I_3 and I_4. In the following, we illustrate the theory with specific examples.

(*b*) *Example 2: coupled modified Emden equation*

The modified Emden equation (MEE) and its variants arise in different branches of physics (Erwin *et al.* 1984; Dixon & Tuszynski 1990), and considerable attention has recently been paid to explore the mathematical and geometrical properties of the MEE and its variant equations (e.g. Mahomed & Leach 1985; Steeb 1993; Chandrasekar *et al.* 2005*b*; Euler *et al.* 2007). The equation can be written as

$$\ddot{x} + 3kx\dot{x} + kx^3 + \lambda x = 0, \tag{5.10}$$

where k and λ are arbitrary parameters, which can be completely integrated in terms of elementary functions (Chandrasekar *et al.* 2005*b*). It exhibits some unusual features such as amplitude-independent frequency, non-standard Lagrangian and time-independent Hamiltonian forms and transition from periodic to front-like solutions as the sign of the parameter changes (Chandrasekar *et al.* 2005*b*).

We now consider a coupled two-dimensional version of (5.10) and demonstrate how it can be integrated by following the theory given in §5*a*. The coupled system of equations can be written as

$$\left.\begin{aligned} &\ddot{x} + 2(k_1x + k_2y)\dot{x} + (k_1\dot{x} + k_2\dot{y})x + (k_1x + k_2y)^2x + \lambda_1 x = 0,\\ &\ddot{y} + 2(k_1x + k_2y)\dot{y} + (k_1\dot{x} + k_2\dot{y})y + (k_1x + k_2y)^2y + \lambda_2 y = 0, \end{aligned}\right\} \tag{5.11}$$

where k_i and λ_i, $i=1, 2$, are arbitrary parameters.

To determine the integrating factors, we seek a rational form of the ansatz for R and K in the form (suggested by equation (5.10))

$$R = \frac{a_1 + a_2\dot{x} + a_3\dot{y}}{(a_4 + a_5\dot{x} + a_6\dot{y})^q}, \qquad K = \frac{b_1 + b_2\dot{x} + b_3\dot{y}}{(a_4 + a_5\dot{x} + a_6\dot{y})^r}, \tag{5.12}$$

where q and r are arbitrary numbers (to be fixed) and a_i's, $i=1, 2, \ldots, 6$, and b_i's, $i=1, 2, 3$, are arbitrary functions of t, x and y. Substituting (5.12) into (5.2) and (5.3) and equating the coefficients of different powers of $\dot{x}$ and $\dot{y}$ to zero, we obtain a set of linear PDEs for the variables a_i's, $i=1, 2, \ldots, 6$, and b_j's, $j=1, 2, 3$. By solving the resultant equations, we find the following sets of particular solutions for R and K:

$$\left.\begin{aligned} R_1 &= -\frac{e^{-\sqrt{-\lambda_1}t}(\lambda_1(k_2\dot{y}+k_2^2y^2+\lambda_2)+k_1x(k_2\lambda_1y-\lambda_2\sqrt{-\lambda_1}))}{g(x,y,\dot{x},\dot{y})^2},\\ K_1 &= \frac{e^{-\sqrt{-\lambda_1}t}\lambda_1k_2(\dot{x}+k_1x^2+x(\sqrt{-\lambda_1}+k_2y))}{g(x,y,\dot{x},\dot{y})^2}, \end{aligned}\right\} \tag{5.13}$$

$$\left.\begin{aligned} R_2 &= \frac{e^{-\sqrt{-\lambda_2}t}\lambda_2k_1(\dot{y}+k_2y^2+y(\sqrt{-\lambda_2}+k_1x))}{g(x,y,\dot{x},\dot{y})^2}\\ K_2 &= -\frac{e^{-\sqrt{-\lambda_2}t}(\lambda_2(k_1\dot{x}+k_1^2x^2+\lambda_1)+k_2y(\lambda_2k_1x-\lambda_1\sqrt{-\lambda_2}))}{g(x,y,\dot{x},\dot{y})^2} \end{aligned}\right\} \tag{5.14}$$

$$R_3 = \frac{x}{g(x,\ y,\dot{x},\dot{y})^2}, \qquad K_3 = 0, \tag{5.15}$$

$$R_4 = 0, \qquad K_4 = \frac{y}{g(x,\ y,\dot{x},\dot{y})^2}, \tag{5.16}$$

where $g(x,\ y,\dot{x},\dot{y}) = (k_1\lambda_2\dot{x}+k_2\lambda_1\dot{y}+k_1^2\lambda_2x^2+k_2^2\lambda_1y^2+k_1k_2(\lambda_1+\lambda_2)xy+\lambda_1\lambda_2)$. Now substituting the above forms of R_i's and K_i's, $i=1, 2, 3, 4$, into (2.9) and (2.10), we can obtain the corresponding S_i's and U_i's, $i=1, 2, 3, 4$. As a result, now we have four sets of independent solutions (S_i, U_i, R_i, K_i), $i=1, 2, 3, 4$, for equations (2.7)–(2.10). Now we check the compatibility of these solutions with the remaining equations (2.11)–(2.13). We find that only the first two sets (S_i, U_i, R_i, K_i), $i=1, 2$, satisfy the extra constraints (2.11)–(2.13) and become compatible solutions. Substituting their forms separately into equation (2.14) and evaluating the integrals, we obtain

$$\left.\begin{aligned} I_1 &= \frac{e^{-\sqrt{-\lambda_1}t}(\dot{x}+(k_1x+k_2y)x+\sqrt{-\lambda_1}x)}{g(x,y,\dot{x},\dot{y})},\\ I_2 &= \frac{e^{-\sqrt{-\lambda_2}t}(\dot{y}+(k_1x+k_2y)y+\sqrt{-\lambda_2}y)}{g(x,y,\dot{x},\dot{y})}. \end{aligned}\right\} \tag{5.17}$$

However, the sets (S_i, U_i, R_i, K_i)'s, $i=3, 4$, do not satisfy the extra constraints (2.11)–(2.13), which means that the form of R_3 in the third set and K_4 in the fourth set may not be in the 'complete form' (since $K_4=R_3=0$) but might only be a factor of the complete form.

To deduce the compatible set $(S_3, U_3, \hat{R}_3, K_3)$, let us substitute $\hat{R}_3 = F(I_1, I_2) \times R_3$ into equations (5.6)–(5.8). As a result, we obtain

$$\frac{1}{2} I_1 F_1' + F = 0, \qquad F_2' = 0, \qquad F_3' = 0, \qquad \left(F_i = \frac{\partial F}{\partial I_i},\ i = 1, 2, 3 \right). \tag{5.18}$$

Upon integrating (5.18), we obtain $F = 1/I_1^2$ (the integration constants are set to zero for simplicity), which fixes the form of R_3 as

$$\hat{R}_3 = \frac{e^{2\sqrt{-\lambda_1}t} x}{(\dot{x} + (k_1 x + k_2 y)x + \sqrt{-\lambda_1} x)^2}. \tag{5.19}$$

Now one can easily check that the set $(S_3, U_3, \hat{R}_3, K_3)$ is a compatible solution for the full set of determining equations (2.7)–(2.13), which, in turn, provides I_3 through the relation (2.14) as

$$I_3 = \frac{e^{2\sqrt{-\lambda_1}t}(\dot{x} + (k_1 x + k_2 y)x - \sqrt{-\lambda_1} x)}{(\dot{x} + (k_1 x + k_2 y)x + \sqrt{-\lambda_1} x)}. \tag{5.20}$$

Finally, in the fourth set, we take $\hat{K}_4 = F(I_1, I_2, I_3) K_4$ and substitute it into equations (5.6)–(5.8) to obtain

$$F_1' = 0, \qquad \frac{1}{2} I_2 F_2' + F = 0, \qquad F_3' = 0. \tag{5.21}$$

Upon integrating (5.21), we obtain $F = 1/I_2^2$, which fixes the form of $\hat{K}_4$ as

$$\hat{K}_4 = \frac{e^{2\sqrt{-\lambda_2}t} y}{(\dot{y} + (k_1 x + k_2 y)y + \sqrt{-\lambda_2} y)^2}. \tag{5.22}$$

Now, the set $(S_4, U_4, R_4, \hat{K}_4)$ is a compatible solution for equations (2.7)–(2.13), which, in turn, provides I_4 through the relation (2.14) as

$$I_4 = \frac{e^{2\sqrt{-\lambda_2}t}(\dot{y} + (k_1 x + k_2 y)y - \sqrt{-\lambda_2} y)}{(\dot{y} + (k_1 x + k_2 y)y + \sqrt{-\lambda_2} y)}. \tag{5.23}$$

Using the explicit forms of the integrals, I_1, I_2, I_3 and I_4, the general solution to equation (5.11) can be deduced directly as

$$\left.\begin{aligned} x(t) &= \frac{\sqrt{-\lambda_1}\lambda_2 I_1 (e^{2\sqrt{-\lambda_1}t} - I_3)}{(\hat{I}_1 + I_2 k_2 \lambda_1 e^{\hat{\lambda}_1 t} + I_2 I_4 k_2 \lambda_1 e^{\hat{\lambda}_2 t} + I_1 k_1 \lambda_2 e^{2\sqrt{-\lambda_1}t} - 2e^{\sqrt{-\lambda_1}t})}, \\ y(t) &= \frac{\sqrt{-\lambda_2}\lambda_1 I_2 (e^{2\sqrt{-\lambda_2}t} - I_4)}{(\hat{I}_2 + I_1 k_1 \lambda_2 e^{\hat{\lambda}_1 t} + I_1 I_3 k_1 \lambda_2 e^{-\hat{\lambda}_2 t} + I_2 k_2 \lambda_1 e^{2\sqrt{-\lambda_2}t} - 2e^{\sqrt{-\lambda_2}t})}, \end{aligned}\right\} \tag{5.24}$$

where $\hat{\lambda}_1 = \sqrt{-\lambda_1} + \sqrt{-\lambda_2}$, $\hat{\lambda}_2 = \sqrt{-\lambda_1} - \sqrt{-\lambda_2}$, $\hat{I}_1 = I_1 I_3 k_1 \lambda_2$ and $\hat{I}_2 = I_2 I_4 k_2 \lambda_1$.

Finally, we note that, for the case $\lambda_i > 0$, $i=1, 2$, the above general solution becomes a periodic one, i.e.

$$x(t) = \frac{A\sin(\omega_1 t + \delta_1)}{1-(Ak_1/\omega_1)\cos(\omega_1 t + \delta_1) - (Bk_2/\omega_2)\cos(\omega_2 t + \delta_2)}, \tag{5.25}$$

$$y(t) = \frac{B\sin(\omega_2 t + \delta_2)}{1-(Ak_1/\omega_1)\cos(\omega_1 t + \delta_1) - (Bk_2/\omega_2)\cos(\omega_2 t + \delta_2)}, \tag{5.26}$$

where $\omega_j = \sqrt{\lambda_j}$, $j=1,2$, $A=\omega_1\omega_2^2 I_1 e^{-i\delta_1}$, $B=\omega_2\omega_1^2 I_2 e^{-i\delta_2}$, $I_3 = e^{-2i\delta_1}$ and $I_4 = e^{-2i\delta_2}$ are arbitrary constants and $(Ak_1/\omega_1 + Bk_2/\omega_2) < 1$. More details on the classical dynamics and characteristic features of this system will be published elsewhere.

(c) *Example 3: the coupled Mathews–Lakshmanan oscillator*

Let us consider the coupled Mathews–Lakshmanan oscillator discussed by Cariñena *et al.* (2004*a*,*b*, 2007), namely

$$\left.\begin{aligned} \ddot{x} &= \frac{\lambda(\dot{x}^2 + \dot{y}^2 + \lambda(x\dot{y} - y\dot{x})^2)x - \alpha^2 x}{(1+\lambda r^2)} = \phi_1, \\ \ddot{y} &= \frac{\lambda(\dot{x}^2 + \dot{y}^2 + \lambda(x\dot{y} - y\dot{x})^2)y - \alpha^2 y}{(1+\lambda r^2)} = \phi_2, \end{aligned}\right\} \tag{5.27}$$

where $r = \sqrt{x^2 + y^2}$ and λ and α are arbitrary parameters. Equation (5.27) is a two-dimensional generalization of the one-dimensional non-polynomial oscillator introduced by Mathews & Lakshmanan (1974),

$$(1+\lambda x^2)\ddot{x} - (\lambda\dot{x}^2 - \alpha^2)x = 0. \tag{5.28}$$

Now we use the following ansatz for R and K to explore the integrating factors for equation (5.27) in the form (suggested by the one-dimensional oscillator (5.28)):

$$\left.\begin{aligned} R &= a_1 + a_2\dot{x} + a_3\dot{y} + a_4\dot{x}^2 + a_5\dot{x}\dot{y} + a_6\dot{y}^2 + a_7\dot{x}^3 + a_8\dot{x}^2\dot{y} + a_9\dot{y}^2\dot{x} + a_{10}\dot{y}^3, \\ K &= b_1 + b_2\dot{x} + b_3\dot{y} + b_4\dot{x}^2 + b_5\dot{x}\dot{y} + b_6\dot{y}^2 + b_7\dot{x}^3 + b_8\dot{x}^2\dot{y} + b_9\dot{y}^2\dot{x} + b_{10}\dot{y}^3, \end{aligned}\right\} \tag{5.29}$$

where a_i's and b_i's, $i=1, 2, \ldots, 10$, are arbitrary functions of t, x and y. Substituting (5.29) into (5.2) and (5.27) and equating the coefficients of different powers of $\dot{x}$ and $\dot{y}$ to zero and solving the resultant PDEs as before, one obtains the following non-trivial solutions for R and K:

$$R_1 = y, \qquad K_1 = -x, \tag{5.30}$$

$$R_2=\frac{2\lambda((1+\lambda y^2)\dot{x}-\lambda xy\dot{y})}{1+\lambda r^2},\qquad K_2=\frac{2\lambda((1+\lambda x^2)\dot{y}-\lambda xy\dot{x})}{1+\lambda r^2},\tag{5.31}$$

$$\left.\begin{aligned}R_3&=\dot{x}+\dot{y}+\frac{\lambda(x+y)^2(\lambda xy\dot{y}-(1+\lambda y^2)\dot{x})}{1+\lambda r^2},\\K_3&=\dot{x}+\dot{y}-\frac{\lambda(x+y)^2((1+\lambda x^2)\dot{y}-\lambda xy\dot{x})}{1+\lambda r^2},\end{aligned}\right\}\tag{5.32}$$

$$\begin{aligned}R_4=\frac{-1}{1+\lambda r^2}\Bigg[&t\lambda\left((\dot{x}+\dot{y})^2-\frac{(x+y)^2\phi_1}{x}\right)\Big((1+\lambda y^2)\dot{x}-\lambda xy\dot{y}\Big)\\&-(x+y)(\alpha^2+\lambda(1+\lambda y(x+y))\dot{x}\dot{y}-\lambda(1+\lambda x(x+y))\dot{y}^2)\Bigg],\end{aligned}\tag{5.33}$$

$$\begin{aligned}K_4=\frac{-1}{1+\lambda r^2}\Bigg[&t\lambda\left((\dot{x}+\dot{y})^2-\frac{(x+y)^2\phi_2}{y}\right)\Big((1+\lambda x^2)\dot{y}-\lambda xy\dot{x}\Big)\\&-(x+y)(\alpha^2-\lambda y(1+\lambda(x+y))\dot{x}^2+\lambda(1+\lambda x(x+y))\dot{x}\dot{y})\Bigg].\end{aligned}\tag{5.34}$$

Now substituting the above forms of R_i's and K_i's, i=1, 2, 3, 4, into equations (2.9) and (2.10), we obtain the corresponding forms of S_i's and U_i's, i=1, 2, 3, 4. As a result, now we have four sets of independent solutions for equations (2.7)–(2.10). Out of these, we find that only the first three sets (i=1, 2, 3) satisfy the extra constraints (2.11)–(2.13) and become compatible solutions. Substituting these forms separately into equation (2.14) and evaluating the integrals, we arrive at

$$I_1=(y\dot{x}-x\dot{y}),\qquad I_2=\frac{(\alpha^2-\lambda(\dot{x}^2+\dot{y}^2+\lambda(y\dot{x}-x\dot{y})^2))}{1+\lambda r^2},\tag{5.35}$$

$$I_3=(\dot{x}+\dot{y})^2+\frac{(x+y)^2(\alpha^2-\lambda(\dot{x}^2+\dot{y}^2+\lambda(y\dot{x}-x\dot{y})^2))}{1+\lambda r^2}.\tag{5.36}$$

However, the set (S_4, U_4, R_4, K_4) does not satisfy the extra constraints (2.11)–(2.13), which means that the forms of R_4 and K_4 are incomplete. So we assume that $\hat{R}_4$ and $\hat{K}_4$ are in the forms (5.4). As a result, equations (5.6)–(5.8) lead us to the following determining equations for the unknown F:

$$F_1'=0,\qquad 2I_2F_2'+F=0,\qquad I_3F_3'+F=0,\qquad \left(F_i=\frac{\partial F}{\partial I_i},\ i=1,2,3\right).\tag{5.37}$$

Upon integrating (5.37), we obtain $F=1/(\sqrt{I_2}I_3)$ (the integration constants are set to zero for simplicity), which fixes the form of $\hat{R}_4$ and $\hat{K}_4$ as

$$\hat{R}_4=\frac{R_4}{\sqrt{I_2}I_3}\qquad \hat{K}_4=\frac{K_4}{\sqrt{I_2}I_3},\tag{5.38}$$

where I_2 and I_3 are given in equations (5.35) and (5.36). Now one can easily check that the set $(S_4, U_4, \hat{R}_4, \hat{K}_4)$ is a compatible solution for equations (2.7)–(2.13), which, in turn, provides I_4 through the relation (2.14) in the form

$$I_4 = -\sqrt{\frac{t^2(\alpha^2 - \lambda(\dot{x}^2 + \dot{y}^2 + \lambda(y\dot{x} - x\dot{y})^2))}{1 + \lambda r^2}} + \tan^{-1}\left[\sqrt{\frac{(x+y)^2(\alpha^2 - \lambda(\dot{x}^2 + \dot{y}^2 + \lambda(y\dot{x} - x\dot{y})^2))}{(\dot{x} + \dot{y})^2(1 + \lambda r^2)}}\right]. \tag{5.39}$$

Note that the above integral is given for the first time in the literature.

Using the explicit forms of the integrals I_1, I_2, I_3 and I_4, the solution to equation (5.27) can be deduced directly as

$$\left.\begin{aligned} x(t) &= \sqrt{\frac{I_3}{4I_2}}\sin(\sqrt{I_2}t + I_4) + \sqrt{\frac{\alpha^2 - I_2 - 2\lambda I_3 - \lambda^2 I_1^2}{2\lambda I_2}}\sin(\sqrt{I_2}t + \delta),\\ y(t) &= \sqrt{\frac{I_3}{4I_2}}\sin(\sqrt{I_2}t + I_4) - \sqrt{\frac{\alpha^2 - I_2 - 2\lambda I_3 - \lambda^2 I_1^2}{2\lambda I_2}}\sin(\sqrt{I_2}t + \delta), \end{aligned}\right\} \tag{5.40}$$

where $\delta = I_4 - \frac{1}{2}\cos^{-1}\left[\frac{(\lambda I_1^2(4I_2 + \lambda I_3) + I_3(7I_2 - \alpha^2 + 2\lambda I_3))}{I_3(I_2 - \alpha^2 + 2\lambda I_3 + \lambda^2 I_1^2)}\right]$.

(d) *Example 4: known two-dimensional integrable systems*

We have also studied the integrability properties of certain well-known two-dimensional nonlinear Hamiltonian systems, namely the Hénon–Heiles system (Ramani *et al.* 1989; Lakshmanan & Sahadevan 1993; Lakshmanan & Rajasekar 2003) and the generalized van der Waals potential equation (Ganesan & Lakshmanan 1990; Lakshmanan & Rajasekar 2003) through the modified PS method. Our analysis shows that these systems do admit only the known integrable cases. In the following, we present the salient features of our analysis.

(i) *The Hénon–Heiles system*

Let us consider the generalized Hénon–Heiles system (Hénon & Heiles 1964)

$$\ddot{x} = -(Ax + 2\alpha xy) \qquad \ddot{y} = -(By + \alpha x^2 - \beta y^2), \tag{5.41}$$

where A, B, α and β are arbitrary parameters. We obtain the integrating factors $R_1 = \dot{x}$ and $K_1 = \dot{y}$ for arbitrary values of the parameters using the ansatz (5.29). Substituting the forms R_1 and K_1 into equations (2.9) and (2.10), we obtain the corresponding forms of S_1 and U_1. Furthermore, we find that this set also satisfies the extra constraints (2.11)–(2.13) and becomes a compatible solution. Substituting these forms into equation (2.14) and evaluating the integrals, we obtain

$$I_1 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + Ax^2 + By^2) + \alpha x^2 y - \frac{\beta}{3}y^3, \tag{5.42}$$

which is nothing but the total energy. Furthermore, we find that only for the specific parametric choices (i) $A=B$, $\alpha=-\beta$, (ii) A, B arbitrary, $\beta=-6\alpha$, and (iii) $B=-16A$, $\beta=-6\alpha$, one can obtain an additional set of integrating factors, i.e.

$$\text{(i)}\quad R_2=-\dot{y},\qquad K_2=-\dot{x},$$

$$\text{(ii)}\quad R_2=2\dot{x}y-x\dot{y}-\left(\frac{4A-B}{2\alpha}\right)\dot{x},\qquad K_2=-x\dot{x},$$

$$\text{(iii)}\quad R_2=\alpha(x\dot{y}-6y\dot{x})x^2-3(\dot{x}^2+Ax^2)\dot{x},\qquad K_2=\alpha\dot{x}x^3,$$

respectively.

Finding the corresponding forms of S and U from (2.9) and (2.10) and substituting the above forms of R and K along with the corresponding S and U into (2.14) and evaluating the integrals, we obtain the following second integrals for the above three parametric choices:

$$\text{(i)}\quad I_2=\dot{x}\dot{y}+\left(Ay+\alpha\left(y^2+\frac{1}{3}x^2\right)\right)x,$$

$$\text{(ii)}\quad I_2=4(x\dot{y}-\dot{x}y)\dot{x}+(4Ay+\alpha x^2+4\alpha y^2)x^2+\frac{(4A-B)}{\alpha}(\dot{x}^2+Ax^2),$$

$$\text{(iii)}\quad I_2=9(\dot{x}^2+Ax^2)^2+12\alpha\dot{x}x^2(3y\dot{x}-x\dot{y})-2\alpha^2x^4(6y^2+x^2)-12\alpha Ax^4y,$$

respectively.

(ii) *Generalized van der Waals potential*

The generalized van der Waals potential equation in two dimensions is given by

$$\ddot{x}=-\left(2\gamma x+\frac{x}{r^3}\right),\qquad \ddot{y}=-\left(2\gamma\beta^2y+\frac{y}{r^3}\right),\qquad r=(x^2+y^2)^{1/2},\tag{5.43}$$

where γ and β are arbitrary parameters, which can be derived from the Hamiltonian

$$H=\frac{1}{2}(\dot{x}^2+\dot{y}^2)+\gamma(x^2+\beta^2y^2)-\frac{1}{r}.\tag{5.44}$$

Repeating the above procedure, we find that the system (5.43) admits the integrating factors

$$\left.\begin{array}{lll}\text{(i)}\quad R_2=-y, & K_2=x, & \beta^2=1,\\[2mm] \text{(ii)}\quad R_2=x\dot{y}-2y\dot{x}, & K_2=x\dot{x}, & \beta^2=4,\\[2mm] \text{(iii)}\quad R_2=y\dot{y}, & K_2=y\dot{x}-2x\dot{y}, & \beta^2=\dfrac{1}{4},\end{array}\right\}\tag{5.45}$$

and the corresponding second integrals of motion are (with γ arbitrary)

$$\left.\begin{aligned} &\text{(i)} \quad I_2 = (y\dot{x} - x\dot{y}), \quad \beta^2 = 1,\\ &\text{(ii)} \quad I_2 = (\dot{x}y - \dot{y}x)\dot{x} - \frac{y}{r} - 2\gamma x^2 y, \qquad \beta^2 = 4,\\ &\text{(iii)} \quad I_2 = (\dot{y}x - \dot{x}y)\dot{y} - \frac{x}{r} - \frac{\gamma y^2 x}{2}, \qquad \beta^2 = \frac{1}{4}. \end{aligned}\right\} \tag{5.46}$$

The integrals I_1 and I_2 ensure the integrability of the above systems for the respective parametric choices through Liouville's theorem.

6. Method of deriving general solution

In the previous sections, we derived the integrals of motion by deducing the functions S, U, R, from which we are able to derive only two sets of solutions (S_i, U_i, R_i, K_i), $i=1$, 2, and their associated integrals of motion, and the remaining two sets of functions are difficult to derive from the determining equations. Under this situation, one may be able to deduce, in many cases, the remaining two integrals from the known integrals themselves without worrying further about the functions (S_i, U_i, R_i, K_i), $i=3$, 4. The underlying idea is the following.

From the known integrals, we deduce a set of coordinate transformations and, using these transformations, we rewrite the original ODEs as a set of two first-order ODEs. By integrating the latter, we obtain the remaining two integration constants. The solution to the original problem can then be obtained just by inverting the variables. In the following, we first describe the theory and then illustrate the ideas with an example.

We start the procedure with two known integrals of motion, namely

$$I_1 = \mathcal{F}(t, x, y, \dot{x}, \dot{y}) \quad \text{and} \quad I_2 = \mathcal{G}(t, x, y, \dot{x}, \dot{y}). \tag{6.1}$$

Now we split the functional form of the integrals I_1 and I_2 into two terms such that one involves all the variables $(t, x, y, \dot{x}, \dot{y})$, while the other excludes $\dot{x}$ and $\dot{y}$, i.e.

$$I_1 = F_1(t, x, y, \dot{x}, \dot{y}) + F_2(t, x, y), \qquad I_2 = F_3(t, x, y, \dot{x}, \dot{y}) + F_4(t, x, y). \tag{6.2}$$

Let us split the functions F_1 and F_3 further in terms of two functions, i.e.

$$\left.\begin{aligned} I_1 &= F_1\left(\frac{1}{G_2(t, x, y, \dot{x}, \dot{y})}\frac{\mathrm{d}}{\mathrm{d}t}G_1(t, x, y)\right) + F_2(G_1(t, x, y)),\\ I_2 &= F_3\left(\frac{1}{G_4(t, x, y, \dot{x}, \dot{y})}\frac{\mathrm{d}}{\mathrm{d}t}G_3(t, x, y)\right) + F_4(G_3(t, x, y)). \end{aligned}\right\} \tag{6.3}$$

Now identifying a set of new variables in terms of functions G_1, G_2, G_3 and G_4 as

$$\left.\begin{aligned} w_1 &= G_1(t, x, y), \qquad z_1 = \textstyle\int_0^t G_2(t', x, y, \dot{x}, \dot{y})\mathrm{d}t',\\ w_2 &= G_3(t, x, y), \qquad z_2 = \textstyle\int_0^t G_4(t', x, y, \dot{x}, \dot{y})\mathrm{d}t', \end{aligned}\right\} \tag{6.4}$$

equation (6.3) can be rewritten in the form

$$I_1 = F_1\left(\frac{\mathrm{d}w_1}{\mathrm{d}z_1}\right) + F_2(w_1), \qquad I_2 = F_3\left(\frac{\mathrm{d}w_2}{\mathrm{d}z_2}\right) + F_4(w_2). \tag{6.5}$$

In other words,

$$F_1\left(\frac{\mathrm{d}w_1}{\mathrm{d}z_1}\right) = I_1 - F_2(w_1), \qquad F_1\left(\frac{\mathrm{d}w_2}{\mathrm{d}z_2}\right) = I_2 - F_2(w_2). \tag{6.6}$$

By rewriting equation (6.6), one arrives at the decoupled equations

$$\frac{\mathrm{d}w_1}{\mathrm{d}z_1} = f_1(w_1), \qquad \frac{\mathrm{d}w_2}{\mathrm{d}z_2} = f_2(w_2), \tag{6.7}$$

which can in principle be integrated and two additional integration constants identified. Now rewriting the solution in terms of the original variables, one obtains the general solution for the given coupled second-order ODE. Thus, the new variables w_1, w_2, z_1 and z_2 correspond to transformations that effectively decouple the original coupled nonlinear second-order ODEs (2.1), provided such transformation variables can be identified. In the following, we illustrate the procedure with an example.

(*a*) *Example 5: force-free coupled Duffing–van der Pol oscillators*

Let us consider the force-free coupled Duffing–van der Pol oscillators of the form

$$\left.\begin{aligned} \ddot{x} + (\alpha + \beta(x+y)^2)\dot{x} - \omega_1 x + \delta_1 x^3 + \gamma_1 xy^2 + \lambda_1 x^2 y = 0,\\ \ddot{y} + (\alpha + \beta(x+y)^2)\dot{y} - \omega_2 y + \delta_2 y^3 + \gamma_2 x^2 y + \lambda_2 xy^2 = 0, \end{aligned}\right\} \tag{6.8}$$

where α, β, ω_i, δ_i, γ_i and λ_i, $i=1, 2$, are arbitrary parameters. Although the general equation (6.8) has not been discussed in detail in the literature, special cases of the above coupled Duffing–van der Pol equation have been used to represent physical and biological systems (Linkens 1974; Datardina & Linkens 1978; Kawahara 1980; Rajasekar & Murali 2004).

We seek a rational form of the ansatz for R and K in the form given by (5.12). Upon solving equations (5.2) and (5.3) with this ansatz, we find that three non-trivial solutions exist for the choice

$$\omega_i = -\frac{3\alpha^2}{16}, \qquad \delta_i = \gamma_i = \frac{\alpha\beta}{4} \quad \text{and} \quad \lambda_i = \frac{\alpha\beta}{2}, \quad i = 1, 2. \tag{6.9}$$

The corresponding forms of R and K turn out to be

$$\left.\begin{aligned} (R_1, K_1) = (\mathrm{e}^{(3/4)\alpha t}, \mathrm{e}^{(3/4)\alpha t}), \quad (R_2, K_2) = \left(-\frac{1}{(\dot{y} + \frac{\alpha}{4}y)}, \frac{(\dot{x} + \frac{\alpha}{4}x)}{(\dot{y} + \frac{\alpha}{4}y)^2}\right),\\ (R_3, K_3) = \left(\frac{y\mathrm{e}^{(1/4)\alpha t}}{(\dot{y} + \frac{\alpha}{4}y)}, -\frac{y(\dot{x} + \frac{\alpha}{4}x)}{(\dot{y} + \frac{\alpha}{4}y)^2}\mathrm{e}^{(1/4)\alpha t}\right). \end{aligned}\right\} \tag{6.10}$$

Now substituting the form of R_i's and K_i's, $i=1$, 2, 3, into equations (2.9) and (2.10), we obtain the corresponding S_i's and U_i's, $i=1$, 2, 3. As a result, now we have three sets of independent solutions for equations (2.7)–(2.10), which are all found to be compatible with (2.11)–(2.13). Substituting the forms (S_i, U_i, R_i, K_i)'s, $i=1$, 2, 3, separately into equation (2.14) and evaluating the integrals, we obtain

$$\left.\begin{aligned} I_1 &= \left(\dot{x}+\dot{y}+\frac{\alpha}{4}(x+y)+\frac{\beta}{3}(x+y)^3\right)\mathrm{e}^{(3/4)\alpha t}, \\ I_2 &= \frac{(\dot{x}+\frac{\alpha}{4}x)}{(\dot{y}+\frac{\alpha}{4}y)}, \qquad I_3 = \left(\frac{(x\dot{y}-y\dot{x})}{(\dot{y}+\frac{\alpha}{4}y)}\right)\mathrm{e}^{(1/4)\alpha t}. \end{aligned}\right\} \tag{6.11}$$

To illustrate the theory given in the first part of this section, we consider the first two integrals I_1 and I_2 given by equation (6.11), and rewrite them in the form (6.2) as

$$\left.\begin{aligned} I_1 &= \left(\dot{x}+\dot{y}+\frac{\alpha}{4}(x+y)\right)\mathrm{e}^{(3/4)\alpha t}+\frac{\beta}{3}((x+y)\mathrm{e}^{1/4\alpha t})^3, \\ I_2 &= \frac{(\dot{x}+\frac{\alpha}{4}x)\mathrm{e}^{(1/4)\alpha t}}{(\dot{y}+\frac{\alpha}{4}y)\mathrm{e}^{(1/4)\alpha t}}. \end{aligned}\right\} \tag{6.12}$$

Now splitting the first term in I_1 and I_2 further in the form (6.3) as

$$\left.\begin{aligned} I_1 &= \mathrm{e}^{(1/2)\alpha t}\left(\frac{\mathrm{d}}{\mathrm{d}t}[(x+y)\mathrm{e}^{(1/4)\alpha t}]\right)+\frac{\beta}{3}(x+y)^3\mathrm{e}^{(3/4)\alpha t}, \\ I_2 &= \frac{\left(\dot{x}+\dfrac{\alpha}{4}x\right)\mathrm{e}^{(1/4)\alpha t}}{\left(\dot{y}+\dfrac{\alpha}{4}y\right)\mathrm{e}^{(1/4)\alpha t}} = \frac{\mathrm{e}^{-(1/4)\alpha t}}{(\dot{y}+(\alpha/4)y)}\left(\frac{\mathrm{d}}{\mathrm{d}t}(x\mathrm{e}^{(1/4)\alpha t})\right), \end{aligned}\right\} \tag{6.13}$$

we identify the new dependent and independent variables from (6.13) using the relations (6.4),

$$w_1 = (x+y)\mathrm{e}^{(1/4)\alpha t}, \quad z_1 = -\frac{2}{\alpha}\mathrm{e}^{-(1/2)\alpha t}, \quad w_2 = x\mathrm{e}^{(1/4)\alpha t}, \quad z_2 = y\mathrm{e}^{(1/4)\alpha t}. \tag{6.14}$$

One can easily check that, in the new variables, equations (6.11) become uncoupled as

$$\frac{\mathrm{d}^2 w_1}{\mathrm{d}^2 z_1}+\beta w_1^2\frac{\mathrm{d}w_1}{\mathrm{d}z_1}=0, \qquad \frac{\mathrm{d}^2 w_2}{\mathrm{d}^2 z_2}=0. \tag{6.15}$$

The integrals of (6.15) can be easily obtained. These are nothing but the integrals I_1 and I_2 given in (6.12), but in terms of the new variables (6.14), they can be written as

$$I_1 = \frac{\mathrm{d}w_1}{\mathrm{d}z_1}+\frac{\beta}{3}w_1^3, \qquad I_2 = \frac{\mathrm{d}w_2}{\mathrm{d}z_2}. \tag{6.16}$$

Solving equations (6.16), we obtain (Gradshteyn & Ryzhik 1980)

$$\left.\begin{aligned} z_1 - z_0 &= \frac{a}{3I_1}\left[\frac{1}{2}\log\left(\frac{(w_1+a)^2}{w_1^2 - aw_1 + a^2}\right) + \sqrt{3}\tan^{-1}\left(\frac{w_1\sqrt{3}}{2a - w_1}\right)\right], \\ w_2 &= I_2 z_2 + I_3, \end{aligned}\right\} \tag{6.17}$$

where $a = \sqrt[3]{-\frac{3I_1}{\beta}}$, I_3 and z_0 are the third and fourth integration constants. Rewriting w_i and z_i, $i=1,2$, in terms of the old variables, one can obtain the explicit solution for equation (6.8) for the parametric choice (6.9) in the form

$$\left.\begin{aligned} -\frac{2}{\alpha}\mathrm{e}^{-(1/2)\alpha t} - z_0 &= \frac{a}{3I_1}\left[\frac{1}{2}\log\left(\frac{((x+y)\mathrm{e}^{(1/4)\alpha t} + a)^2}{(x+y)^2\mathrm{e}^{(1/2)\alpha t} - a(x+y)\mathrm{e}^{(1/4)\alpha t} + a^2}\right)\right. \\ &\qquad \left. + \sqrt{3}\tan^{-1}\left(\frac{(x+y)\mathrm{e}^{(1/4)\alpha t}\sqrt{3}}{2a - (x+y)\mathrm{e}^{(1/4)\alpha t}}\right)\right], \\ x &= I_2 y + I_3 \mathrm{e}^{-(1/4)\alpha t}, \end{aligned}\right\} \tag{6.18}$$

from which one can obtain implicit relations between x and t and y and t, corresponding to the general solution.

7. Conclusion

In this paper, as a first part of our investigations on the complete integrability and linearization of two coupled second-order ODEs, we have focused our attention only on the integrability aspects. In particular, we have introduced a general method of finding integrable parameters, integrating factors, integrals of motion and general solution associated with a set of two coupled second-order ODEs through the extended PS procedure. The procedure can also be extended straightforwardly to analyse any number of coupled second-order ODEs. The proposed method is simple, straightforward and very useful to solve a class of coupled second-order ODEs. We have illustrated the theory with potentially important examples. We have also introduced a novel idea of transforming coupled second-order nonlinear ODEs into uncoupled second-order ODEs. We have deduced the transformations from the first two integrals themselves.

It is also of interest to study the problem of linearization of coupled nonlinear ODEs by transforming them into linear ODEs as in the case of single second-order nonlinear ODEs. However, it is a more difficult and challenging problem than quadratures. The primary reasons are as follows. (i) It is not known, in general, whether the given equation is linearizable or not since, in the case of two-degrees-of-freedom systems, there can exist many types of linearizing transformations and it is not obvious which one will be successful. (ii) The possible transformations that could exist in the case of two coupled second-order ODEs are also not known in the literature. (iii) Finally, there is no simple procedure that gives us the required transformation in a straightforward way. We would like to address all these questions in the follow-up paper (part V).

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