Imaginary potential as a counter of delay time for wave reflection from a one-dimensional random potential

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We show that the delay time distribution for wave reflection from a one-dimensional (one-channel) random potential is related directly to that of the reflection coefficient, derived with an arbitrarily small but uniform imaginary part added to the random potential. Physically, the reflection coefficient, being exponential in the time dwelt in the presence of the imaginary part, provides a natural counter for it. The delay time distribution then follows straightforwardly from our earlier results for the reflection coefficient, and coincides with the distribution obtained recently by Texier and Comtet [C. Texier and A. Comtet, Phys. Rev. Lett. **82**, 4220 (1999)], with all moments infinite. The delay time distribution for a random amplifying medium is then derived. In this case, however, all moments work out to be finite.

When a wave packet centered at an energy E is scattered elastically from a scattering potential, it suffers a time delay before spreading out dispersively. This delay is related to the time for which the wave dwells in the interaction region. For the general case of a scatterer coupled to N open channels leading to the continuum, one defines the phase-shift time delays through the Hermitian energy derivative of the S matrix, $-i\hbar S^{-1}\partial S/\partial E$, whose eigenvalues give the proper delay times. These delay times then averaged over the N channels give the Wigner-Smith delay times introduced by Wigner[1] for the one-channel case, and generalized later by Smith[2] to the case of N open channels. Thus the scattering delay time is the single most important quantity describing the time-dependent aspect, i.e., physically, the reactive aspect of the scattering in open quantum systems, e.g., the chaotic microwave cavity and the quantum billiard (whose classical motion is chaotic) and the solid-state mesoscopic dots coupled capacitively to open leads terminated in the reservoir. The delay time is, however, not self-averaging and one must have its full probability distribution over a statistical ensemble of random samples. The latter may be related ergodically to the ensembles generated parametrically, e.g., by energy E variation over a sufficient interval. Thus we have the random matrix theory (RMT) for circular ensembles of the S matrix giving delay times for all three Dyson universality classes for the case of a chaotic cavity connected to a single open channel.[3] Generalization to the case of N channels corresponded to the Laguarre ensemble[4] of RMT. The RMT approach has been treated earlier through the supersymmetric technique for the case of a quantum chaotic cavity having a few equivalent open channels.[5] However, it has been suspected for quite sometime that the RMT-based results and the universality claimed thereby may not extend to a strictly one-dimensional (1D) random system where Anderson localization dominates, and that the 1D random system may constitute after all a different universality class.[6] This important problem has been reexamined recently by Texier and Comtet[7] who have derived the delay time distribution for a 1D conductor with the Frish-Lloyd model randomness in the limit of high energy and weak disorder and the sample length $\gg$ the localization length. The universality of the distribution is amply supported by numerical simulations for different models of disorder.[7,8]

In this work we reexamine this question of the universality of the delay time distribution for a 1D random system and relate it to the universality of the distribution of the reflection coefficient, a quantity that we have direct access to from our earlier work.[9] To this end we introduce a counter that literally clocks the time dwelt by the wave in the scattering region, obviating the need for calculating the energy derivative of the phase shift.[10] This involves adding formally an arbitrarily small but uniform imaginary part iV_i to the 1D random potential V_r. Now, the reflection coefficient, being exponential in the time dwelt in the scattering region in the presence of iV_i, provides a literal ''counter'' for this time. The distribution derived by us agrees exactly with the universal time-delay distribution of Texier and Comtet.[7] Besides, our technique allows us to treat the time-delay distribution for the important case of light reflected from a random amplifying medium equally well. In this case, however, unlike the case for the passive random medium, all moments of the delay time are finite for long samples.

Consider first the electronic case for a 1D disordered sample of length L having a random potential V_r, $0\leqslant x\leqslant L$, and connected to infinitely long perfect leads at the two ends. Let the electron wave of energy $E=\hbar^2k^2/2m$ be incident from the right at $x=L$, and be partially reflected with a complex amplitude reflection coefficient $R(L)=|R(L)|\exp[i\theta(L)]$ and $|R(L)|^2=r(L)$, the real reflection coefficient. Inside the sample we have the Schrödinger equation

$$\frac{d^2\psi(x)}{dx^2}+k^2[1+\eta_r(x)]\psi(x)=0, \qquad (1)$$

with $\eta_r(x)=-V_r(x)/E$.

As we will be interested in the reflection coefficient, it is apt to follow the invariant imbedding technique[9–12] and reduce the Schrödinger equation (1) to an equation for the emergent quantity $R(L)$:

$$\frac{dR(L)}{dL}=2ikR(L)+\frac{ik}{2}\eta_r(L)[1+R(L)]^2. \tag{2}$$

We now introduce a uniform imaginary part iV_i, with $V_i>0$, and accordingly define $\eta(L)=\eta_r+i\eta_i$, with $\eta_i=-V_i/E$. For an analytical treatment, we take for $V_r(x)$ a Gaussian δ-correlated random potential (the Halperin model) with $\langle\eta_r(L)\rangle=0$ and $\langle\eta_r(L)\eta_r(L')\rangle=\Delta^2\delta(L-L')$. The Fokker-Planck equation corresponding to the stochastic equation (2) can be solved analytically in the limit $L\to\infty$, giving[9]

$$P_\infty(r)=\frac{D\exp\left(-\frac{D}{r-1}\right)}{(r-1)^2},\quad r\geqslant 1,$$
$$=0,\qquad r<1, \tag{3}$$

with $D=(4V_i)/(E\Delta^2k)$. This result is obtained in the high-energy and weak-disorder limit. Now, clearly for a passive medium, i.e., with $V_i=0$, the distribution $P_\infty(r)$ must collapse to a δ function $\delta(r-1)$ as $L\to\infty$. However, with $V_i\neq 0$, for a short dwell time T in the sample, the reflection coefficient $r=|R|^2=\exp(2V_iT/\hbar)$, giving $r-1=2V_iT/\hbar$ to first order in V_i as V_i is taken to be arbitrarily small. Thus, $P_\infty(r)$ can at once be translated into the dwell time distribution $P^0_\infty(\tau)$:

$$P^0_\infty(\tau)=\frac{\alpha}{\tau^2}\exp\left(-\frac{\alpha}{\tau}\right), \tag{4}$$

where $\alpha=2(\Delta^2k)^{-1}$ and the dimensionless time $\tau=ET/\hbar$. This is precisely the result of Texier and Comtet.[7] Note that V_i, the counter, drops out in the limit $V_i\to 0$, as it should. It should also be noted that the invariant imbedding equation for the energy derivative of the phase shift[10] also yields the same result for the delay time distibution when the high-energy limit ($k\to\infty$ while keeping V_r/k constant) is explicitly taken. This again reconfirms our delay time distribution given above.

At this point, it is perhaps apt to demystify our time delay counter, viz., the introduction of an imaginary potential (V_i) in the limit $V_i\to 0$ as a mathematical artifice for the electronic case, in terms of the well-known analytic property of the *S* matrix, corresponding to wave reflection from the 1D infinitely long disordered system. The *S* matrix in this case is simply the complex *amplitude* reflection coefficient $R(E)=\exp[i\theta(E)]$ with $|R|^2=1$ for real E. Now from the analyticity of the *S* matrix in the complex energy plane, we have $\partial(\mathrm{Re}\,\theta)/\partial(\mathrm{Re}\,E)=\partial(\mathrm{Im}\,\theta)/\partial(\mathrm{Im}\,E)$, where ''Re'' and ''Im'' denote the real and imaginary parts, respectively. As we approach the real axis, i.e., in the limit $\mathrm{Im}\,E\to 0$, we have $\partial(\mathrm{Re}\,\theta)/\partial(\mathrm{Re}\,E)=T/\hbar$ (Wigner time delay), while $\partial(\mathrm{Im}\,\theta)/\partial(\mathrm{Im}\,E)\to\mathrm{Im}\,\theta/V_i$ as $V_i\to 0$ (along with $\mathrm{Im}\,\theta$). Thus we have $|R|^2=\exp[2V_iT/\hbar]$, giving $|R|^2-1=2V_iT/\hbar$ in the limit $V_i\to 0$ (the latter corresponds to treating our electronic problem as a limit of vanishing imaginary part of the scattering potential). This is what has been used above to obtain the delay time distribution from the reflection coefficient distribution given by Eq. (3) in the limit $V_i\to 0$.

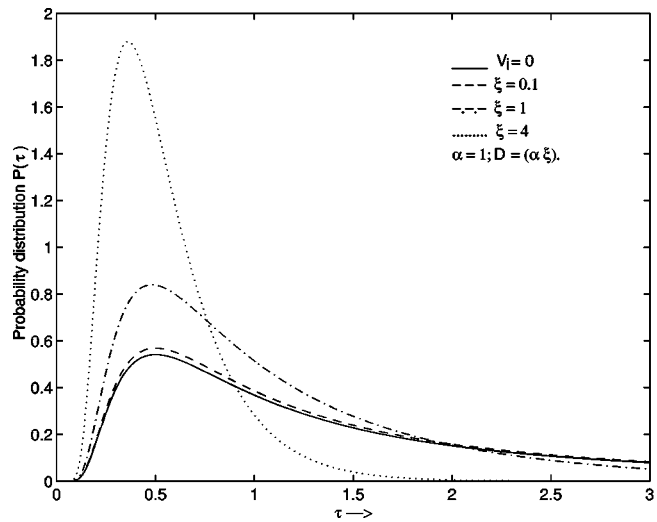

FIG. 1. The delay time distribution from an amplifying medium.

Encouraged by this result for for the electronic case, we now turn to the case of a light wave reflected from a random amplifying medium. The latter has received much attention in recent years in the context of random lasers.[13–15] To fix ideas, consider the case of a single-mode optical fiber doped with Er^{3+}, say, optically pumped and intentionally disordered refractively. All we have to do now is to keep V_i finite, a measure of medium gain, and use $T=(\hbar/2)(\partial\ln r/\partial V_i)$ for the dwell time, and translate $P_\infty(r)$ into $P_\infty(\tau)$:

$$P_\infty(\tau)=(D\xi)\frac{\exp\left(-\frac{D}{e^{\xi\tau}-1}\right)}{(e^{\xi\tau}-1)^2}e^{\xi\tau}, \tag{5}$$

where $\xi=2V_i/E$. Again, P_∞ vanishes in the limit $\tau\to\infty$ as also for $\tau\to 0$. Also, $P_\infty(\tau)\to P^0_\infty$ as $V_i\to 0$. All moments $\langle\tau^n\rangle$ are, however, finite in this case. An explicit expression can be obtained for the first moment as

$$\langle\tau\rangle=\frac{1}{\xi}[\ln D+C-e^D\,\mathrm{Ei}(-D)], \tag{6}$$

where C is the Euler's constant[16] and ''Ei'' is the exponential integral.[16] This expression diverges as $V_i\to 0$. In Fig. 1, we show the delay time distributions given by Eq. (5) for different values of the parameter ξ, keeping α fixed corresponding to different values of the imaginary potential V_i while keeping the disorder fixed.

Several interesting points are to be noted here. The counter introduced by us literally counts the dwell time in the interaction region for total reflection in the 1D, i.e., the one-channel case. A large delay time is dominated by the dwell time when the wave penetrates deeper into the sample, which is true at high energy and low disorder. It is this ''equilibrated'' part of the reflected wave, and not the prompt part, that is expected to give universality. Hence the universal $1/\tau^2$ tail in Eq. (4). Indeed, the universality of the delay time distribution directly reflects that of the reflection coefficient given by Eq. (3).[17–19] Indeed, we have verified that Eq. (3) is obtained for telegraph disorder also. It is to be remarked here that this universal delay time distribution as in Eq. (4) is not obtained for a chaotic cavity connected to a reservoir by a single open channel.[3] Here the localization picture may not hold. As for the finiteness of all the moments $\langle \tau^n \rangle$ for the case of the random amplifying medium, it is quite consistent with the known fact that amplification enhances localization and thus prevents deep penetration in the random sample. Of course, there is also an enhanced prompt part of the reflection resulting from the increased refractive index mismatch with respect to its imaginary part at the sample-lead interface.

In conclusion, we have introduced a ''counter'' that measures the dwell time in the scattering medium. We have used it successfully to derive the delay time distribution in terms of that of the reflection time. Both passive and amplifying media have been treated. Our counter can be used equally well in principle to calculate the traversal time for the problem of tunneling across a potential barrier.[20,21]

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