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ON ANALYTIC INDEPENDENCE

BY

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ABSTRACT. This article examines the concept of "analytic independence". Several illustrative examples have been included. The main results are Theorems 1–4 which state the relations between analytic independence and the degree of field extensions, transcendence degree, order of poles and "gap" respectively.

1. **Introduction.** Let k be an algebraically closed field, S an analytic local domain over k with a local subdomain R. Recall that nonunit elements $x_1, \ldots, x_n \in S$ are said to be analytically independent over R iff the mapping φ: $R[[Z_1, \ldots, Z_n]] \longrightarrow S$ defined by $\varphi|_R =$ identity, $\varphi(Z_i) = x_i$, $\forall 1 \leqslant i \leqslant n$, is injective. Otherwise $x_1, \ldots, x_n$ are said to be analytically dependent over R.

The concepts of "algebraic independence" and "analytic independence" are very different even though they are similar in appearance. For instance, Examples 1 and 2 in §4 illustrate that the set of elements in $k[[x, y]]$ which are analytically dependent over $k[[x, xy]]$ is not closed under summation or multiplication.

We shall restrict ourselves to the case that R is a 2-dimensional regular analytic local domain. It has been established (cf. [3]) that if k is of characteristic zero, then one may assume $S = k[[x, y]]$, $R = k[[x, xy]]$ without loss of generality. Throughout we shall assume this even if k is of positive characteristic.

This article establishes some algebraic criteria of analytic independence. The main results are Theorems 1–4 which state the relations between analytic independence and the degree of field extensions, transcendence degree, order of poles or "gap" respectively.

2. **Hasse derivatives, automorphisms.** Let $k[[y]]$ be a power series ring of one variable y.

DEFINITION 1. The ith Hasse derivative $D_i f(y) = f^{(i)}(y)$ of $f(y) \in k[[y]]$ is defined by

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$$f(y+t)=\sum f^{(i)}(y)t^i,$$

where $f(y+t)\in k[[y,t]]$, a power series ring of two variables.

LEMMA 1. *Let $f(y)$ be a power series which is algebraic over $k(y)$. Then all Hasse derivatives of $f(y)\in k(y)(f(y))$.*

PROOF. It is known that $f(y)$ is separable over $k(y)$. Let the minimum equation of $f(y)$ be $a(y)f(y)^n+b(y)f(y)^{n-1}+\cdots+e(y)=0$ with $a(y)$, $b(y),\ldots,e(y)\in k[y]$ and $a(y)\neq 0$. Then

$$a(y+t)f(y+t)^n+b(y+t)f(y+t)^{n-1}+\cdots+e(y+t)=\sum F_i(y)t^i=0.$$

Hence $F_i(y)=0$, $\forall\, i\geqslant 0$, where $F_i(y)$ is of the following form:

$$\begin{aligned}F_i(y)=(&na(y)f(y)^{n-1}+(n-1)b(y)f(y)^{n-2}+\cdots)\\&\cdot f^{(i)}(y)+[\text{polynomial in } y, f(y),\ldots,f^{(i-1)}(y)],\end{aligned}$$

with $f^{(i)}(y)$ the ith Hasse derivative of $f(y)$. Since $a(y)f(y)^n+b(y)f(y)^{n-1}+\cdots+e(y)=0$ is a separable equation, one observes that

$$na(y)f(y)^{n-1}+(n-1)b(y)f(y)^{n-2}+\cdots\neq 0.$$

One concludes that

$$f^{(i)}(y)\in k(y)(f(y),f^{(1)}(y),\ldots,f^{(i-1)}(y)).$$

Our lemma is established by mathematical induction on i. Q.E.D.

Some special automorphisms of $k[[x,y]]$ will be extensively used, therefore we state

DEFINITION 2. A k-automorphism τ of $k[[x,y]]$ is said to be of type A iff $\tau(x)-x\in x^2k[[x,y]]$ and $\tau(y)-y\in xyk[[x,y]]$.

REMARK. All automorphisms of type A form a group.

LEMMA 2. *Let τ be an automorphism of type* A *with*

$$\tau(x)-x=x^2\Big(\sum_{i\geqslant 0}f_i(y)x^i\Big),\qquad \tau(y)-y=xy\Big(\sum_{i\geqslant 0}g_i(y)x^i\Big),$$

$$\tau^{-1}(x)-x=x^2\Big(\sum_{i\geqslant 0}F_i(y)x^i\Big),\quad \tau^{-1}(y)-y=xy\Big(\sum_{i\geqslant 0}G_i(y)x^i\Big).$$

Then we have

$$\begin{aligned}(1)_i\qquad &k(y,f_0(y),\ldots,f_i(y),g_0(y),\ldots,g_{i-1}(y))\\&\qquad=k(y,F_0(y),\ldots,F_i(y),G_0(y),\ldots,G_{i-1}(y)),\end{aligned}$$

$$(2)_i \qquad \begin{aligned} &k(y, f_0(y), \ldots, f_{i-1}(y), g_0(y), \ldots, g_i(y)) \\ &\qquad = k(y, F_0(y), \ldots, F_{i-1}(y), G_0(y), \ldots, G_i(y)), \end{aligned}$$

$$(3)_i \qquad \begin{aligned} &k(y, f_0(y), \ldots, f_i(y), g_0(y), \ldots, g_i(y)) \\ &\qquad = k(y, F_0(y), \ldots, F_i(y), G_0(y), \ldots, G_i(y)), \end{aligned}$$

if $f_j(y)$, $g_j(y)$ for $j = 0, 1, \ldots, i-1$ are algebraic power series over $k(y)$.

PROOF. We first observe that $(1)_i$ and $(2)_i$ imply $(3)_i$ by taking compositum of fields. Clearly

$$f_0(y) = -F_0(y), \qquad g_0(y) = -G_0(y).$$

$(3)_0$ is obvious. Hence to establish our lemma it is enough to prove that $(3)_{i-1}$ implies $(1)_i$ and $(2)_i$.

One has

$$\begin{aligned} x &= \tau^{-1}\tau(x) = \tau^{-1}\left(x + x^2\left(\sum_{i\geq 0} f_i(y)x^i\right)\right) \\ &= \tau^{-1}(x) + \tau^{-1}(x)^2\left(\sum_{i\geq 0} f_i(\tau^{-1}(y))(\tau^{-1}(x))^i\right) \\ &= x\left(1 + \sum_{i\geq 0} F_i(y)x^{i+1}\right) \\ &\quad \cdot \left\{1 + \sum_{i\geq 0}\left[\sum_{j\geq 0} f_i^{(j)}(y)\left(y\left(\sum_{i\geq 0} G_i(y)x^{i+1}\right)\right)^j \right.\right. \\ &\qquad\qquad \left.\left. \cdot \left(x\left(1 + \sum_{i\geq 0} F_i(y)x^{i+1}\right)\right)^{i+1}\right]\right\} \\ &= \sum_{i\geq 1} Q_i(y)x^i \end{aligned}$$

where $f_i^{(j)}(y)$ is the jth Hasse derivative of $f_i(y)$. It follows that $Q_1(y) = 1$, $Q_j(y) = 0, \forall j \geq 2$. Clearly

$$(1) \qquad \begin{aligned} &Q_{i+2}(y) = F_i(y) + f_i(y) \\ &\qquad + [\text{polynomial in } F_j(y), G_j(y), y^m f_j^{(m)}(y) \text{ with } j, m < i] \end{aligned}$$

where the said polynomial is a weighted homogeneous polynomial of total weight $i + 1$ when we assign weights $j + 1, j, j + 1 + m$ to $F_j(y), G_j(y), y^m f_j^{(m)}(y)$ respectively; moreover, no term in the said polynomial involves only the G_j's. Hence by Lemma 1 $(3)_{i-1} \Rightarrow (1)_i$.

Now consider

$$y=\tau^{-1}\tau(y)=\tau^{-1}\Big(y+xy\Big(\sum_{i\geqslant 0} g_i(y)x^i\Big)\Big)$$

$$=\Big(y+xy\Big(\sum_{i\geqslant 0} G_i(y)x^i\Big)\Big)\Big\{1+\sum_{i\geqslant 0}\Big[\Big(\sum_{j\geqslant 0} g_i^{(j)}(y)\Big(y\Big(\sum_{i\geqslant 0} G_i(y)x^{i+1}\Big)\Big)^j\Big)$$

$$\cdot\Big(x\Big(1+\sum_{i\geqslant 0} F_i(y)x^{i+1}\Big)\Big)^{i+1}\Big]\Big\}$$

$$=\Big(\sum_{i\geqslant 0} P_i(y)x^i\Big)y,$$

where $g_i^{(j)}(y)$ is the jth Hasse derivative of $g_i(y)$. It follows that $P_0(y)=1$, $P_j(y)=0$, $\forall j\geqslant 1$. Clearly

$$P_{i+1}(y)=G_i(y)+g_i(y) \tag{2}$$
$$+\ [\text{polynomial in } F_j(y), G_j(y), y^m g_j^{(m)}(y) \text{ with } j, m<i].$$

Again by Lemma 1 we conclude $(3)_{i-1}\Rightarrow(2)_i$. The process of mathematical induction is thus finished. Q.E.D.

In Lemma 3, we shall use the notation and assumptions of Lemma 2.

LEMMA 3. *Let* $z=\sum_{i=0}^{\infty} h_i(y)x^i$ *and* $\tau(z)=\sum_{i=0}^{\infty} H_i^*(y)x^i$. *Then*

(1*) $h_i(y)\in k(y, f_0(y),\dots,f_{i-1}(y), g_0(y),\dots,g_{i-1}(y), H_0^*(y),\dots,H_i^*(y))$,

(2*) $H_i^*(y)\in k(y, f_0(y),\dots,f_{i-1}(y), g_0(y),\dots,g_{i-1}(y), h_0(y),\dots,h_i(y))$,

if $h_j(y), f_j(y), g_j(y)$ *for* $j=0,1,\dots,i-1$ *are algebraic power series over* $k(y)$.

PROOF. In view of Lemma 2, it is enough to prove (2*). We have

$$\tau(z)=\sum_{i\geqslant 0} h_i(\tau(y))(\tau(x))^i$$

$$=\sum_{i\geqslant 0}\Big[h_i(y)+\sum_{j\geqslant 1} h_i^{(j)}(y)\Big(xy\Big(\sum_{i\geqslant 0} g_i(y)x^i\Big)\Big)^j\Big]$$

$$\cdot\Big[x+x^2\Big(\sum_{i\geqslant 0} f_i(y)x^i\Big)\Big]^i$$

$$=\sum H_i^*(y)x^i$$

with $H_0^*(y)=h_0(y)$ and for $i>0$:

$$H_i^*(y)=h_i(y)+y^i h_0^{(i)}(y)g_0(y)^i \tag{3}$$
$$+\ [\text{polynomial in } f_j(y), g_j(y), y^j h_m^{(j)}(y) \text{ with } m, j<i]$$

where the said polynomial is a weighted homogeneous polynomial of total weight i when we assign weights $j+1, j, j+m$ to $f_j(y), g_j(y), y^j h_m^{(j)}(y)$ respectively. Hence (2*) follows from Lemma 1 by induction.

3. **Algebraic criteria of analytic independence.** Let R be a field of characteristic zero, k an algebraic closure of R. We shall establish

PROPOSITION 1. *Let* $f(x) = \Sigma_{i \geq 0} a_i x^i \in k[[x]]$ *be integral over* $R[[x]]$. *Then* $[R(\{a_i\}_{i=0}^{\infty}):R] < \infty$.

PROOF. If $a_i \in R$, $\forall i \geq 0$, then we have nothing to prove. Otherwise assume that $a_i \in R$, $\forall 0 \leq i < m$, and $a_m \notin R$. Replacing $f(x)$ by $f(x) - \Sigma_{i=0}^{m-1} a_i x$, we shall assume $a_i = 0$, $\forall\, 0 \leq i < m$. Let the minimal equation of $f(x)$ over $R[[x]]$ be

$$Y^n + b_1(x)Y^{n-1} + \cdots + b_n(x) = 0. \tag{4}$$

Let $s = \min_i \{(\text{ord } b_i(x))/i : i = 1, 2, \ldots, n\} = a/b$. Let $t = x^{1/b}$. Consider equation (4) as one over $R[[t]]$ and replace Y by $Y/t^a = Z$. Then we have

$$Z^n + b_1(t^b)/t^a Z^{n-1} + \cdots + b_n(t^b)/t^{na} \tag{5}$$

$$= Z^n + C_1(t)Z^{n-1} + \cdots + C_n(t) = 0$$

and

$$C_1(t)Z^{n-1} + \cdots + C_n(t) \in R[[t]][Z] \text{ but } \notin t \cdot R[[t]][Z].$$

There are two possibilities: either $s < m$, or $s = m$. In the first case $Z^n + C_1(0)Z^{n-1} + \cdots + C_n(0) \neq Z^n$ and will have 0 as a root (cf. [7]). In the second case a_m will be a root of $Z^n + C_1(0)Z^{n-1} + \cdots + C_n(0)$, hence $Z^n + C_1(0)Z^{n-1} + \cdots + C_n(0)$ is not of the form $(Z+C)^n$ (cf. [7]). In either case, $Z^n + C_1(t)Z^{n-1} + \cdots + C_n(t)$ will be reducible in $R(a_m)[[t]][Z]$ because $R(a_m)[[t]]$ is Henselian. It is clear that $f(x) \in R(a_m, \ldots, a_{m_q})[[\eta]]$ after repeating the above process where η is a suitable root of x. It is trivial that

$$R(a_{m_1}, \ldots, a_{m_q})[[\eta]] \cap k[[x]] = R(a_{m_1}, \ldots, a_{m_q})[[x]].$$

Hence $f(x) \in R(a_{m_1}, \ldots, a_{m_q})[[x]]$. Q.E.D.

The following theorem establishes an algebraic criterion of analytic independence by the degree of field extension.

THEOREM 1. *Let* $z = \Sigma_{i \geq 1}^{\infty} h_i(y)x^i \in k[[x, y]]$ *with* $h_i(y)$ *algebraic power series over* $k(y)$ *where* k *is of characteristic zero. If* $[k(y)\{h_i(y)\}_{i \geq 1} : k(y)] = \infty$, *then* z *is analytically independent over* $k[[x, xy]]$.

PROOF. Suppose z is analytically dependent over $k[[x, xy]]$. Let $F(x, xy, Z) \in k[[x, xy]][[Z]]$ be one of the nontrivial irreducible analytic relations satisfied by x, xy, z. Now $F(x, 0, Z) \neq 0$. Choose a in k such that $F(x - aZ^3, 0, Z)$ is regular with respect to Z. Then $F(x - aZ^3, xy, Z) = G(x, xy, Z)$ will be regular with respect to Z and $G(x + az^3, xy, z) = 0$. Define a k-automorphism τ of $k[[x, y]]$ as follows:

$$(6)\quad \begin{cases} \tau(x) = x + az^3 = x + ax^3\left(\sum_{i\geq 1}^{\infty} h_i x^{i-1}\right)^3 = x + x^2\left(\sum_{i\geq 0}^{\infty} f_i(y)x^i\right), \\ \tau(y) = y\left[1 + ax^2\left(\sum_{i\geq 1}^{\infty} h_i(y)x^{i-1}\right)^3\right]^{-1} = y\left[1 + x\left(\sum_{i\geq 0}^{\infty} g_i(y)x^i\right)\right]. \end{cases}$$

Then τ is of type A and $\tau(xy) = xy$. Clearly z is integral over $k[[\tau(x), \tau(xy)]]$ and hence $\tau^{-1}(z)$ is integral over $k[[x, xy]]$.

It follows trivially from binomial expansion and inverse formula that $f_j \in k(y, h_1(y), \ldots, h_i(y)), \forall j \leq i + 1$, and $g_j \in k(y, h_1(y), \ldots, h_i(y)), \forall j \leq i + 1$.

Let $\tau^{-1}(z) = \sum_{i\geq 1}^{\infty} H_i(y)x^i$. Then it follows from Lemmas 2 and 3 that $H_i(y) \in k(y, h_1(y), \ldots, h_i(y))$ and $h_i(y) \in k(y, H_1(y), \ldots, H_i(y))$. In other words

$$[k(y, \{h_i(y)\}_{i\geq 1}): k(y)] = [k(y, \{H_i(y)\}_{i\geq 1}): k(y)].$$

Finally, $\tau^{-1}(z)$ is integral over $k[[x, xy]]$ implies that $\tau^{-1}(z)$ is integral over $k(y)[[x]] \supset k[[x, xy]]$. Our theorem follows from Proposition 1. Q.E.D.

Let $z_i = g_i(y)x$ for $i = 1, \ldots, n$ with $y, g_1(y), \ldots, g_n(y) \in k[[y]]$ and algebraically independent over k. Then it has been established in [1] that $z_1, \ldots, z_n$ are analytically independent over $k[[x, xy]]$. The following theorem is a generalization of the result stated above.

THEOREM 2. *For $i = 1, \ldots, n$, given $z_i = \sum_{j\geq 1}^{\infty} f_{ij}(y)x^j \in k[[x, y]]$ such that $f_{ij}(y)$ is transcendental over $k(y)$ for some j, let $f_{im_i}(y)$ be the first $f_{ij}(y)$ which is transcendental over $k(y)$. If $f_{1m_1}(y), \ldots, f_{nm_n}(y)$ are algebraically independent over $k(y)$, then $z_1, \ldots, z_n$ are analytically independent over $k[[x, xy]]$.*

PROOF. Upon reordering $z_1, \ldots, z_n$ we may assume $1 \leq m_1 \leq m_2 \leq \cdots \leq m_n$ from the very beginning. Let us order all n-tuples $(m_1, \ldots, m_n)$ by lexicographic ordering where $1 \leq m_1 \leq m_2 \leq \cdots \leq m_n$. We shall induct on $(m_1, \ldots, m_n)$.

The case that $m_1 = m_2 = \cdots = m_n = 1$ can proceed essentially by the same method used in [1]. We shall base our induction thereon.

Suppose $z_1, \ldots, z_n$ satisfy our assumption with $1 \leq m_1 \leq \cdots \leq m_n$ and are analytically dependent over $k[[x, xy]]$. Note that $m_n > 1$. Let

$F(x, xy, Z_1, \ldots, Z_n) \in k[[x, xy, Z_1, \ldots, Z_n]]$ be one of the nontrivial analytic relations satisfied by $x, xy, z_1, \ldots, z_n$. Then $F(x, 0, Z_1, \ldots, Z_n) \neq 0$. Choose $a_0, a_1, \ldots, a_{n-1} \in k$ and integer $m > 3$ such that

$$F(x - a_0 Z_n^m, 0, Z_1 - a_1 Z_n^m, \ldots, Z_{n-1} - a_{n-1} Z_n^m, Z_n)$$

is regular with respect to Z_n. Let

$$F(x - a_0 Z_n^m, xy, Z_1 - a_1 Z_n^m, \ldots, Z_n) = G(x, xy, Z_1, \ldots, Z_n).$$

Then $G(x + a_0 z_n^m, xy, z_1 + a_1 z_n^m, \ldots, z_n) = 0$. In other words $x + a_0 z_n^m, xy, z_1 + a_1 z_n^m, \ldots, z_n$ are analytically dependent and G is a nontrivial relation among them. Moreover z_n is integral over

$$k[[x + a_0 z_n^m, xy, z_1 + a_1 z_n^m, \ldots, z_{n-1} + a_{n-1} z_n^m]].$$

Let τ be the k-automorphism of $k[[x, y]]$ defined by

$$\tau(x) = x + a_0 z_n^m = x + x^2 \left(\sum_{i \geqslant 0}^{\infty} g_i(y) x^i \right),$$

$$\tau(y) = y \left[1 + x \left(\sum_{i \geqslant 0} g_i(y) x^i \right) \right]^{-1} = y \left[1 + x \left(\sum_{i \geqslant 0} h_i(y) x^i \right) \right].$$

Then τ is of type A and $\tau(xy) = xy$. Note that $g_i(y), h_i(y)$ are algebraic power series $\forall i \leqslant m_n$. Let $\eta_i = z_i + a_i z_n^m$ for $i = 1, \ldots, n-1$. Then z_n is integral over $k[[\tau(x), \tau(xy), \eta_1, \ldots, \eta_{n-1}]]$ and hence $\tau^{-1}(z_n)$ is integral over $k[[x, xy, \tau^{-1}(\eta_1), \ldots, \tau^{-1}(\eta_{n-1})]]$. Let

$$\tau^{-1}(\eta_i) = \sum_{j \geqslant 1}^{\infty} p_{ij}(y) x^j \quad \text{for } i = 1, \ldots, n-1,$$

$$\tau^{-1}(z_n) = \sum_{j \geqslant 1}^{\infty} p_{nj}(y) x^j.$$

Then it follows from Lemmas 1 and 2 that $p_{im_i}(y)$ is the first $p_{ij}(y)$ which is transcendental over $k(y)$ and $p_{1m_1}(y), \ldots, p_{nm_n}(y)$ are algebraically independent over $k(y)$.

Note that $\tau^{-1}(z_n)$ is integral over $k[[x, xy, \tau^{-1}(\eta_1), \ldots, \tau^{-1}(\eta_{n-1})]]$. Hence $\tau^{-1}(z_n)$ is algebraic over $k[[x, xy, \tau^{-1}(\eta_1), \ldots, \tau^{-1}(\eta_{n-1})]]$. It follows trivially that

$$\eta_n = \frac{\tau^{-1}(z_n) - \sum_{j=1}^{m_n - 1} p_{nj}(y) x^j}{x^{m_n - 1}} = p_{nm_n}(y) x + \cdots$$

is algebraically, hence analytically, dependent over

$$k[[x, xy, \tau^{-1}(\eta_1), \ldots, \tau^{-1}(\eta_{n-1})]].$$

Note that the n-tuple for $\eta_n, \tau^{-1}(\eta_1), \ldots, \tau^{-1}(\eta_{n-1})$ is $(1, m_1, \ldots, m_{n-1}) < (m_1, \ldots, m_{n-1}, m_n)$. By the hypothesis of mathematical induction, this is impossible. Q.E.D.

4. **Poles, gaps and analytic independence.** The converses of Theorems 1 and 2 are false. In fact, as indicated by Theorems 3 and 4, there are many elements in $k[y][[x]]$ which are analytically independent over $k[[x, xy]]$. For the convenience of the reader, we shall state the following proposition (cf. [4]).

PROPOSITION 2. *Let R be a field with a valuation V. Let $R[\langle x\rangle]$ be a convergent power series ring of one variable with respect to V. Then $R[\langle x\rangle]$ is algebraically closed in $R[[x]]$.*

REMARK. Let V be a k-valuation of $k(y)$. Then $z = \Sigma h_i(y)x^i \in k(y)[[x]]$ is a convergent power series with respect to V iff $V(h_i(y)) \geqslant -mi$ for some positive integer m.

THEOREM 3. *Let $z \in k[[x, y]] \cap k(y)[[x]]$ with k an algebraically closed field of characteristic zero. Moreover assume that $z = \Sigma_{i\geqslant 1}^{\infty} h_i(y)x^i$ is not convergent with respect to the k-valuation V of $k(y)$ with $V(y) = 0$. Then z is analytically independent over $k[[x, xy]]$.*

PROOF. We shall use the terminology of the proof of Theorem 1. There are two things to be verified. We have to establish that $\tau^{-1}(z) \in k(y)[[x]]$ and $\tau^{-1}(z)$ is not convergent. Our theorem will follow from the preceding proposition thereafter.

The fact that $\tau^{-1}(z) \in k(y)[[x]]$ follows trivially from Lemmas 1–3.

It suffices to prove that $\tau^{-1}(z)$ is not convergent. Since z is not convergent, for any given positive integer $S > 1$ there is an index i such that $V(h_i(y)) < -S(i-1)$ and $V(h_j(y)) \geqslant -S(j-1), \forall j < i$. Since every k-evaluation is nonarchimedean, it is obvious that $V(f_j(y)) \geqslant -S(j-1)$ and $V(g_j(y)) \geqslant -S(j-1), \forall j < i$.

Let us point out that the ith Hasse derivative $= i!$ (the ith ordinary derivative under our assumption that k is of characteristic zero). Hence $V(f^{(i)}(y)) \geqslant V(f(y)) - i$ for all $f(y) \in k(y)$. Consequently, in view of the description of the polynomial occurring in equation (1) of Lemma 1 and the fact that $\tau(xy) = xy$, by induction on j we can deduce that $V(F_j(y)) \geqslant -S(j-1)$ and $V(G_j(y)) \geqslant -S(j-1), \forall j < i$. By equation (3) of Lemma 3, as applied to $\tau^{-1}(z) = \Sigma_{u\geqslant 1}^{\infty} H_u(y)x^u$, we also have

$$H_i(y) = h_i(y) + [\text{polynomial in } F_j(y), G_j(y), y^j h_m^{(j)}(y) \text{ with } m, j < i]$$

with the polynomial in the parentheses a weighted homogeneous polynomial of total weight i with $F_j(y)$, $G_j(y)$, $y^j h_m^{(j)}(y)$ of weight $j+1$, j and $j+m$ respectively. It follows that V (the above polynomial in the parentheses) $\geqslant -S$ (its total weight -1) $= -S(i-1)$. Since $V(h_i(y)) < -S(i-1)$, we conclude that $V(H_i(y)) = V(h_i(y)) < -S(i-1)$. Thus $\tau^{-1}(z)$ is not convergent. Q.E.D.

We shall use a very arithmetic method to prove the following "gap theorem". For this purpose we need the following:

NOTATION. Let $f(x_1, \ldots, x_n) \in k[[x_1, \ldots, x_n]]$. By $x_1^{a_1} \cdots x_n^{a_n} \in f$ we mean that $f_{a_1,\ldots,a_n} \neq 0$ in the following expansion:

$$f(x_1, \ldots, x_n) = \sum f_{i_1,\ldots,i_n} x_1^{i_1}, \ldots, x_n^{i_n}.$$

THEOREM 4. *Let k be a field of characteristic zero and $z = \Sigma_{i\geqslant 1}^{\infty} h_i(y)x^i \in k[y][[x]] \subset k[[x, y]]$. Let $\alpha_i = \deg h_i(y)$ and $\beta_i = \operatorname{ord} h_i(y)$. Assume*:

(1′) $\beta_i \geqslant i$, $\forall i$; *and*

(2′) *given any integers $m > 0$ and $s \geqslant 0$, there exists an integer $n > 0$ with $h_n(y) \neq 0$ such that $\beta_{n+i} > m\alpha_n$ $\forall i > 0$ and $(n+s)/\alpha_n < \min\{1, i/r\}$ where r and i run through all $i \leqslant n$ and $y^r \in h_i(y)$ with either $i \neq n$ or $r \neq \alpha_n$. Then z is analytically independent over $k[[x, xy]]$.*

PROOF. Suppose the converse. Let $F(x, xy, Z)$ be one of the irreducible analytic relations among x, xy and z.

We shall write $F(x, xy, Z) = \Sigma_{i\geqslant 0}^{\infty} F_i(x, xy)Z^i$. Since $\beta_i \geqslant i$, $z \in yk[[x, y]]$. Thus $F_0(x, xy) \in yk[[x, y]]$ and hence $F_0(x, xy) \in xyk[[x, y]]$. It follows from the irreducibility of $F(x, xy, Z)$ that $F_i(x, xy) \notin xyk[[x, xy]]$ for some $i > 0$. Let m be the smallest i with this property.

Since $F_m(x, xy) \notin xyk[[x, y]]$, it follows that $F_m(x, 0) \neq 0$. Let $s = \operatorname{ord} F(x, 0)$.

Let n be an integer such that condition (2′) is satisfied with respect to the integers m and s just determined.

The scheme is to prove that $x^{s+nm}y^{\alpha_n m} \in G(x, y) = F(x, xy, z)$. Clearly this will imply $F(x, xy, z) \neq 0$.

Note that any term in $F_j(x, xy)z^j = F_j(x, xy)(\Sigma h_i(y)x^i)^j$ can be written as

$$cx^u(xy)^v \prod (x^i y^{\gamma_i}) \tag{7}$$

where $0 \neq c \in k$, $x^u T^v \in F_j(x, T)$, $y^{\gamma_i} \in h_i(y)$ with the last product one of j

terms. To prove that $x^{s+nm}y^{\alpha_n m} \in G(x, y) = F(x, xy, z)$, it is enough to show that this particular term happens once and only once in all possible expressions of (7).

In the product of j terms, $\Pi(x^i y^{\gamma_i})$, if $i > n$ for one term, then $\gamma_i \geqslant \beta_i > m\alpha_n$. Hence the y-degree is obviously too big. So to produce $x^{s+nm}y^{\alpha_n m}$, one requires that $i \leqslant n$ for all j terms in the product $\Pi(x^i y^{\gamma_i})$. Moreover

$$x^u (xy)^v \prod (x^i y^{\gamma_i}) = x^{u+v+\Sigma i} y^{v+\Sigma\gamma_i} = x^{s+nm} y^{\alpha_n m}$$

implies

$$u + v + \sum i = s + nm, \qquad v + \sum \gamma_i = \alpha_n m.$$

Let the number of terms of the form $x^n y^{\alpha_n}$ in $\Pi(x^i y^{\gamma_i})$ be p. Then we have

$$u + v + \sum{}' i = s + n(m - p), \qquad v + \sum{}' \gamma_i = \alpha_n (m - p)$$

with both summations running through terms in $\Pi(x^i y^{\gamma_i})$ which are not of the form $x^n y^{\alpha_n}$. Suppose $m - p \neq 0$. Then by conditions (1′) and (2′) we get

$$\frac{u + v + \Sigma' i}{v + \Sigma' \gamma_i} \geqslant \frac{\Sigma' i}{\Sigma' \gamma_i} > \frac{n + s}{\alpha_n} \geqslant \frac{s + n(m - p)}{\alpha_n (m - p)}$$

with all denominators $\geqslant 1$. That is clearly a contradiction. Hence we conclude that $m - p = 0$. Therefore $m = p = j$ for all terms in $\Pi(x^i y^{\gamma_i})$ we have $i = n$ and $\gamma_i = \alpha_n$. Moreover, it clearly follows that $v = 0, u = s$. Thus the uniqueness of the appearance of such terms is established. The existence part is trivial. Q.E.D.

We shall conclude with a few examples.

EXAMPLE 1. Let $f(y) \in k[[y]]$ be a transcendental power series over $k(y)$ with ord $f(y) = 1$. Then $k[[x, y]] = k[[x, f(y)]]$ and $xy \in k[[x, f(y)]]$. Thus there exists a nontrivial analytic relation among xy, x and $f(y)$. Hence $f(y)$ is analytically dependent over $k[[x, xy]]$. Clearly x is analytically dependent over $k[[x, xy]]$. As indicated by Theorem 2, $f(y)x$ is analytically independent over $k[[x, xy]]$. This example shows that all elements in $k[[x, y]]$ which are analytically dependent over $k[[x, xy]]$ do not form a set which is closed under usual multiplication.

EXAMPLE 2. Let $f(y)$ be as in Example 1. Let $g(x, y) = f(y)x - y$. Then $k[[x, g(x, y)]] = k[[x, y]]$, which shows that $g(x, y)$ is analytically dependent over $k[[x, xy]]$. Clearly $g(x, y) + y = f(y)x$ is not. This example shows that the set of analytic dependent elements is not closed under summation.

EXAMPLE 3. Let $z = \Sigma_{n \geqslant 3} c_n x^n y^{n!}$ with $c_n \in k$ and $c_n \neq 0$ for infinitely many n. Then it follows from Theorem 4 that z is analytically independent over $k[[x, xy]]$.

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