Logarithmic correction to the Bekenstein-Hawking entropy

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The exact formula derived by us earlier for the entropy of a four dimensional non-rotating black hole within the quantum geometry formulation of the event horizon in terms of boundary states of a three dimensional Chern-Simons theory, is reexamined for large horizon areas. In addition to the *semiclassical* Bekenstein-Hawking contribution proportional to the area obtained earlier, we find a contribution proportional to the logarithm of the area together with subleading corrections that constitute a series in inverse powers of the area.

The derivation of the Bekenstein-Hawking (BH) area law for black hole entropy from the quantum geometry approach [1] (and also earlier from string theory [2] for some special cases), has lead to a resurgence of interest in the quantum aspects of black hole physics in recent times. However, the major activity has remained focussed on *confirming* the area law for large black holes, which, as is well-known, was obtained originally on the basis of arguments of a semiclassical nature. The question arises as to whether any essential feature of the bona fide quantum aspect of gravity, beyond the domain of the semiclassical approximation, has been captured in these assays. Indeed, as has been most eloquently demonstrated by Carlip [3], a derivation of the area law alone seems to be possible on the basis of some symmetry principle of the (semi)classical theory itself without requiring a detailed knowledge of the actual quantum states associated with a black hole. The result seems to hold for arbitrary number of spatial dimensions, so long as a particular set of isometries of the metric is respected. That quantum gravity has a description in terms of spin networks (or for that matter, in terms of string states in a fixed background) appears to be of little consequence in obtaining the area law, although these proposed underlying structures also lead to the same behaviour via alternative routes, in the semiclassical limit of arbitrarily large horizon area.

Although there is as yet no complete quantum theory of gravitation, one would in general expect key features uncovered so far to lead to modifications of the area law which could not have been anticipated through semiclassical reasoning. Thus, the question as to what is the dominant quantum correction due to these features of quantum gravity becomes one of paramount importance. Already in the string theory literature [4] examples of leading corrections to the area law, obtained by counting D-brane states describing special supersymmetric extremal black holes (interacting with massless vector supermultiplets) have appeared. This has received strong support recently from semiclassical calculations in $N = 2$ supergravity [5] supplemented by ostensible stringy higher derivative corrections which are incorporated using Wald's general formalism describing black hole entropy as Noether charge [6]. However, the geometrical interpretation of these corrections remains unclear. Further, there are subtleties associated with direct application of Wald's formalism which assumes a *non-degenerate* bifurcate Killing horizon, to the case of extremal black holes which have degenerate horizons. Moreover, the string results do not pertain to generic (i.e., non-extremal) black holes of Einstein's general relativity, and are constrained by the unphysical requirement of unbroken spacetime supersymmetry.

In this paper, we consider the corrections to the semiclassical area law of generic four dimensional non-rotating black holes, due to key aspects of *non-perturbative* quantum gravity (or quantum geometry) formulated by Ashtekar and collaborators [7]. In [1], appropriate boundary conditions are imposed on dynamical variables at the event horizon considered as an inner boundary. These boundary conditions require that the Einstein-Hilbert action be supplemented by boundary terms describing a three dimensional $SU(2)$ Chern-Simons theory living on a finite 'patch' of the horizon with a spherical boundary, punctured by links of the spin network bulk states describing the quantum spacetime geometry interpolating between asymptopia and the horizon. On this two dimensional boundary there exists an $SU(2)$ Wess Zumino model whose conformal blocks describe the Hilbert space of the Chern-Simons theory modelling the horizon. An exact formula for the number of these conformal blocks has been obtained by us [8] two years ago, for arbitrary level k and number of punctures p. It has been shown that in the limit of large horizon area given by arbitrarily large k and p, the logarithm of this number duly yields the area law. Here we go one step further, and calculate the dominant sub-leading contribution, as a function of the classical horizon area, or what is equivalent, as a function of the BH entropy itself.

On purely dimensional grounds, one would expect the entropy to have an expansion, for large classical horizon area, in inverse powers of area so that the BH term is the leading one,

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$$S_{bh} \;=\; S_{BH} \;+\; \sum_{n=0}^{\infty} C_n \; A_H^{-n} \tag{1}$$

where, A_H is the classical horizon area and C_n are coefficients which are independent of the horizon area but dependent on the Planck length (Newton constant). Here the Barbero-Immirzi parameter [9] has been 'fitted' to the value which fixes the normalization of the BH term to the standard one. However, in principle, one could expect an additional term proportional to $ln\ A_H$ as the leading quantum correction to the semiclassical S_{BH}. Such a term is expected on general grounds pertaining to breakdown of naïve dimensional analysis due to quantum fluctuations, as is common in quantum field theories in flat spacetime and also in quantum theories of critical phenomena. We show, in what follows, that such a logarithmic correction to the semiclassical area law does indeed arise from the formula derived earlier [8] and derive its coefficient.

We first briefly recapitulate the derivation [8] of the general formula for the number of conformal blocks of the $SU(2)_k$ Wess Zumino model on a punctured 2-sphere appropriate to the black hole situation. This number can be computed in terms of the so-called fusion matrices ${N_{ij}}^r$ [10]

$$N_{\mathcal{P}} \;=\; \sum_{\{r_i\}} {N_{j_1 j_2}}^{r_1} \; {N_{r_1 j_3}}^{r_2} \; {N_{r_2 j_4}}^{r_3} \; \ldots\ldots \; {N_{r_{p-2} j_{p-1}}}^{j_p} \tag{2}$$

Diagrammatically, this can be represented as shown in fig. 1 below.

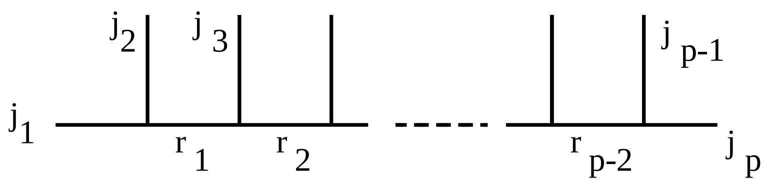

FIG. 1. Diagrammatic representation of composition of spins j_i for $SU(2)_k$

Here, each matrix element ${N_{ij}}^r$ is 1 *or* 0, depending on whether the primary field $[\phi_r]$ is allowed or not in the conformal field theory fusion algebra for the primary fields $[\phi_i]$ and $[\phi_j]$ $\quad (i, j, r \;=\; 0, 1/2, 1, \ldots.k/2)$:

$$[\phi_i] \;\otimes\; [\phi_j] \;=\; \sum_r {N_{ij}}^r [\phi_r] \; . \tag{3}$$

Eq. (2) gives the number of conformal blocks with spins $j_1, j_2, \ldots, j_p$ on p external lines and spins $r_1, r_2, \ldots, r_{p-2}$ on the internal lines.

We then use the Verlinde formula [10] to obtain

$${N_{ij}}^r \;=\; \sum_s \frac{S_{is} S_{js} S_s^{\dagger r}}{S_{0s}} \; , \tag{4}$$

where, the unitary matrix S_{ij} diagonalizes the fusion matrix. Upon using the unitarity of the S-matrix, the algebra (2) reduces to

$$N_{\mathcal{P}} \;=\; \sum_{r=0}^{k/2} \frac{S_{j_1 \; r} S_{j_2 \; r} \ldots S_{j_p \; r}}{(S_{0r})^{p-2}} \; . \tag{5}$$

Now, the matrix elements of S_{ij} are known for the case under consideration ($SU(2)_k$ Wess-Zumino model); they are given by

$$S_{ij} \;=\; \sqrt{\frac{2}{k+2}} \; sin \left(\frac{(2i+1)(2j+1)\pi}{k+2} \right) \; , \tag{6}$$

where, $i,\ j$ are the spin labels, $i,\ j \;=\; 0, 1/2, 1, \ldots.k/2$. Using this S-matrix, the number of conformal blocks for the set of punctures $\mathcal{P}$ is given by

$$N_{\mathcal{P}} \;=\; \frac{2}{k+2} \sum_{r=0}^{k/2} \frac{\prod_{l=1}^{p} sin \left(\frac{(2j_l+1)(2r+1)\pi}{k+2} \right)}{\left[sin \left(\frac{(2r+1)\pi}{k+2} \right) \right]^{p-2}} \; . \tag{7}$$

Eq. (7) thus gives the dimensionality of the $SU(2)$ Chern-Simons states corresponding to a three-fold bounded by a two-sphere punctured at p points. The black hole microstates are counted by summing $N_{\mathcal{P}}$ over all sets of punctures $\mathcal{P}$, $N_{bh} = \sum_{\{\mathcal{P}\}} N_{\mathcal{P}}$. Then, the entropy of the black hole is given by $S_{bh} = \log N_{bh}$.

We are however interested only in the leading correction to the semiclassical entropy which ensues in the limit of arbitrarily large A_H. To this end, recall that the eigenvalues of the area operator [7] are given by

$$A_H = 8\pi\beta \, l_{Pl}^2 \sum_{l=1}^{p} [j_l(j_l+1)]^{\frac{1}{2}} , \tag{8}$$

where, l_{Pl} is the Planck length, j_l is the spin on the lth puncture on the 2-sphere and β is the Barbero-Immirzi parameter [9]. Clearly, the large area limit corresponds to the limits $k \rightarrow \infty$, $p \rightarrow \infty$. Now, from eq. (8), it follows that the number of punctures p is largest for a given A_H provided *all* spins $j_l = \frac{1}{2}$. Thus, for a fixed classical horizon area, we obtain the largest number of punctures p_0 as

$$p_0 = \frac{A_H}{4l_{Pl}^2} \frac{\beta_0}{\beta} , \tag{9}$$

where, $\beta_0 = 1/\pi\sqrt{3}$. In this approximation, the set of punctures $\mathcal{P}_0$ with all spins equal to one-half dominates over all other sets, so that the black hole entropy is simply given by

$$S_{bh} = ln \; N_{\mathcal{P}_0} , \tag{10}$$

with $N_{\mathcal{P}_0}$ being given by eq. (7) with $j_l = 1/2$.

Observe that $N_{\mathcal{P}_0}$ can now be written as

$$N_{\mathcal{P}_0} = \frac{2^{p_0+2}}{k+2} \; [F(k,p_o) - F(k,p_0+2)] \tag{11}$$

where,

$$F(k,p) = \sum_{\nu=1}^{[\frac{1}{2}(k+1)]} cos^p \left(\frac{\nu\pi}{k+2}\right) . \tag{12}$$

The sum over ν in eq. (12) can be approximated by an integral in the limit $k \rightarrow \infty$, $p_0 \rightarrow \infty$, with appropriate care being taken to restrict the domain of integration; one obtains

$$F(k,p_0) \approx \left(\frac{k+2}{\pi}\right) \int_0^{\pi/2} dx \; cos^{p_0} x , \tag{13}$$

so that,

$$N_{\mathcal{P}_0} \approx \frac{2^{p_0+2}}{\pi(p_0+2)} \; B \; (\frac{p_0+1}{2} , \frac{1}{2}) , \tag{14}$$

where, $B(x,y)$ is the standard B-function [11]. Using well-known properties of this function, it is straightforward to show that

$$\begin{aligned} ln \; N_{\mathcal{P}_0} &= p_0 \; ln2 - \frac{3}{2} \; ln \; p_0 - ln \; (2\pi) \\ &- \frac{5}{2} \; p_0^{-1} + O(p_0^{-2}) . \end{aligned} \tag{15}$$

Substituting for p_0 as a function of A_H from eq. (9) and setting the Barbero-Immirzi parameter β to the 'universal' value $\beta_0 \; ln2$ [1], one obtains our main result

$$S_{bh} = S_{BH} - \frac{3}{2} \; ln \left(\frac{S_{BH}}{ln2}\right) + const. + \cdots , \tag{16}$$

where, $S_{BH} = A_H/4l_{Pl}^2$, and the ellipses denote corrections in inverse powers of A_H or S_{BH}.

Admittedly, the above calculation is restricted to the leading correction to the semiclassical approximation. It has been done for a fixed large A_H by taking the spins on all the punctures to be 1/2 so that we have the largest number of punctures. But it is not difficult to argue that the coefficient of the lnA_H term is robust in that inclusion of spin values higher than 1/2 do not affect it, although the constant term and the coefficients of sub-leading corrections with powers of $O(A_H^{-1})$ might get affected. The same appears to be true for values of the level k away from the asymptotic value which we have assigned it above: the coefficient of the lnA_H is once again unaffected. Thus, the leading logarithmic correction with coefficient -3/2 that we have discerned for the black hole entropy is in this sense *universal*. Moreover, although we have set $\beta = \beta_0\ ln2$ in the above formulae, the coefficient of the lnA_H term is independent of β, a feature not shared by the semiclassical area law.

It is therefore clear that the leading correction (and maybe also the subleading ones) to the BH entropy is negative. One way to understand this could be the information-theoretic approach of Bekenstein [12]: black hole entropy represents lack of information about quantum states which arise in the various ways of gravitational collapse that lead to formation of black holes with the same mass, charge and angular momentum. Thus, the BH entropy is the 'maximal' entropy that a black hole can have; incorporation of leading quantum effects reduces the entropy. The logarithmic nature of the leading correction points to a possible existence of what might be called a 'non-perturbative fixed point'. That this happens in the physical world of four dimensions is perhaps not without interest.

Recently, the zeroth and first law of black hole mechanics have been derived for situations with radiation present in the vicinity of the horizon, using the notion of the isolated horizon [13]. Our conclusions above for the case of non-rotating black holes hold for such generalizations [15] as well. Note however that while, the foregoing analysis involves $SU(2)_k$ Chern Simons theory, for large k this reduces to a specific $U(1)$ theory presumably related to the 'gauge fixed' classical theory discussed in [13]. The charge spectrum of this $U(1)$ theory is discrete and bounded from above by k. The $SU(2)$ origin of the theory thus provides a natural 'regularization' for calculation of the number of conformal blocks.

Note Added: After the first version of this paper appeared in the Archives, it has been brought to our attention that corrections to the area law in the form of logarithm of horizon area have been obtained earlier [14] for extremal Reissner-Nordstrom and dilatonic black holes. These corrections are due to quantum scalar fields propagating in fixed classical backgrounds appropriate to these black holes. The coefficient of the lnA_H term that appears in ref. [14] is different from ours. This is only expected, since in contrast to ref. [14], our corrections originate from non-perturbative quantum fluctuations of spacetime geometry (for generic non-rotating black holes), *in the absence of matter fields.* Thus, this correction is *finite* and independent of any arbitrary 'renormalization scale' associated with divergences due to quantum matter fluctuations in a fixed classical background.

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