

## Application of Newton's method to a homogenization problem

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**Abstract.** The homogenization of a family  $(P_\epsilon)$  of uniformly elliptic semilinear partial differential equations of second order is studied. The main result is that any non-singular solution  $u$  of the homogenized problem  $(P)$  is the limit of non-singular solutions of  $(P_\epsilon)$ . The method consists of specifying a function  $w_\epsilon$  starting from which the Newton iterates converge to a solution  $u_\epsilon$  of  $P_\epsilon$ . These solutions  $u_\epsilon$  converge to the given solution  $u$  of  $(P)$ .

**Keywords.** Elliptic ; homogenization ; Newton's method ; semilinear.

### 1. Introduction

Let  $\Omega \subset \mathbf{R}^n$  be a bounded open set with boundary  $\Gamma$ . Consider the following family of second-order differential operators defined in  $\Omega$ , depending on a parameter  $\epsilon$  which tends to zero :

$$(1.1) \quad A^\epsilon = - \frac{\partial}{\partial x_i} \left( a_{ij}^\epsilon(x) \frac{\partial}{\partial x_j} \right).$$

(Here, as well as throughout this paper, the summation convention for repeated indices will be assumed.) Various homogenization problems connected with these operators have been considered in the literature. Bensoussan *et al* [1] have studied the linear problem  $A^\epsilon u_\epsilon = f$  with various boundary conditions. Kesavan [4] has analysed the corresponding eigenvalue problems. In this paper the following semilinear problem will be studied :

$$(1.2) \quad A^\epsilon u_\epsilon = f(u_\epsilon) \text{ in } \Omega,$$

$$(1.3) \quad u_\epsilon = 0 \text{ on } \Gamma,$$

where  $f: \mathbf{R} \rightarrow \mathbf{R}$  is a given function.

Indeed the identification of the homogenized problem is not difficult and will not occupy the principal part of this study. In fact Mythily [5] has homogenized (1.2) with a more complicated boundary condition. The more important question, which will be the main preoccupation in this paper, is the following : 'given a solution  $u$  of the homogenized problem, does it approximate a solution  $u_\epsilon$  of the real problem ?' In other words, 'is  $u$  the limit of a sequence  $\{u_\epsilon\}$  of solutions ?'

In the case of the linear problem, the uniqueness of the solutions of the various problems concerned trivially answers the above question in the affirmative. In the case of the eigenvalue problem Kesavan [4] has shown that eigenfunctions  $u$  of a *simple* eigenvalue of the homogenized problem occur as limits of eigenfunctions  $u_\epsilon$  of the given family of problems. The case of multiple eigenvalues is still open. In this paper, it will be proved that, under suitable conditions, a *non-singular* solution  $u$  of the homogenized problem is the limit of a family  $\{u_\epsilon\}$  of solutions of (1.2)–(1.3), which, for sufficiently small  $\epsilon$ , will also be non-singular.

The main idea of the proof is the following: First a function  $w_\epsilon$  is defined as the unique solution of the auxiliary linear problem:

$$(1.4) \quad A^\epsilon w_\epsilon = f(u) \text{ in } \Omega,$$

$$(1.5) \quad w_\epsilon = 0 \text{ on } \Gamma,$$

where  $u$  is the given non-singular solution of the homogenized problem. Then it is shown that for sufficiently small  $\epsilon$ , the hypotheses of the Newton-Kantorovich theorem can be verified if  $w_\epsilon$  is used as the starting vector for the Newton method. Hence the Newton method will converge to a solution  $u_\epsilon$  which will be unique in a neighbourhood of  $w_\epsilon$  and of  $u$ . Thus  $\{u_\epsilon\}$  will converge to  $u$  owing to the property of local uniqueness.

In § 2, the important hypotheses are made and some preliminary results are recalled. In § 3, the existence of solutions to the various problems is proved and the homogenized problem is identified. In § 4, the existence of a sequence  $\{u_\epsilon\}$  converging to a given solution  $u$  is proved. § 5 is reserved for conclusions and various comments.

## 2. Preliminaries

The following hypotheses are made on the coefficients  $a_{ij}^\epsilon$ :

(H1) There exists a constant  $M > 0$  independent of  $\epsilon$  such that

$$(2.1) \quad |a_{ij}^\epsilon(x)| \leq M \text{ a.e., } 1 \leq i, j \leq n.$$

(H2) There exists a constant  $a_0 > 0$  such that

$$(2.2) \quad \forall \xi = (\xi_i) \in \mathbb{R}^n, a_{ij}^\epsilon(x) \xi_i \xi_j \geq a_0 \xi_i \xi_i \text{ a.e.}$$

**Definition 2.1.** An operator  $A$  of the form,

$$(2.3) \quad A = - \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right),$$

is said to be the *homogenized operator*, w.r.t. the family  $\{A^\epsilon\}$  if the following holds:

Let  $u_\epsilon \rightarrow u$  in  $H_0^1(\Omega)$  weakly and  $A^\epsilon u_\epsilon \rightarrow g$  in  $H^{-1}(\Omega)$  strongly. Then

$$\xi_i^\epsilon = a_{ij}^\epsilon(x) \frac{\partial u_\epsilon}{\partial x_j} \rightarrow a_{ij}(x) \frac{\partial u}{\partial x_j} = \xi_i,$$

in  $L^2(\Omega)$  weakly,  $1 \leq i \leq n$ .

This definition is due to Tartar [8]. As a consequence, given  $\{f_\epsilon\}$  in  $H^{-1}(\Omega)$  converging strongly to  $f$ , and  $\{u_\epsilon\}$  the unique solutions of the problems

$$(2.4) \quad \forall v \in H_0^1(\Omega), \quad a_\epsilon(u_\epsilon, v) = \langle f, v \rangle (H^{-1}, H_0^1)$$

where for  $w, v \in H_0^1(\Omega)$ ,

$$(2.5) \quad a_\epsilon(w, v) = \int_{\Omega} a_{ij}^\epsilon(x) \frac{\partial w}{\partial x_j} \frac{\partial v}{\partial x_i} dx,$$

then,  $u_\epsilon \rightarrow u$  in  $H_0^1(\Omega)$  weakly [and hence in  $L^2(\Omega)$  strongly] to the unique solution,  $u$ , of the homogenized problem,

$$(2.6) \quad \forall v \in H_0^1(\Omega), \quad a(u, v) = \langle f, v \rangle (H^{-1}, H_0^1)$$

where for  $w, v \in H_0^1(\Omega)$ ,

$$(2.7) \quad a(w, v) = \int_{\Omega} a_{ij}(x) \frac{\partial w}{\partial x_j} \frac{\partial v}{\partial x_i} dx.$$

The existence of such an operator is known (cf. Tartar [7]). The operator is also elliptic with the same constant of ellipticity. The coefficients are bounded. (If they are symmetric, the same bound works.) In case of the  $a_{ij}^\epsilon$  having a periodic structure the operator  $A$  can be explicitly written (cf. Bensoussan *et al* [1]).

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function which verifies the following hypothesis :  
(H3)  $f$  is a  $C^1$  function such that both  $f$  and  $f'$  are Lipschitz continuous with constants  $K_0$  and  $K_1$  respectively. Further, it is assumed that  $K_0 < \alpha$  the ellipticity constant of the bilinear forms  $a_\epsilon(\cdot, \cdot)$  and  $a(\cdot, \cdot)$ .

**Proposition 2.1.** Suppose that the hypothesis (H3) is verified. Then if  $u \in L^p(\Omega)$  for some  $p \geq 1$ ,  $f(u)$  and  $f'(u)$  are also in  $L^p(\Omega)$ . If  $u_\epsilon \rightarrow u$  in  $L^p(\Omega)$ , then  $f(u_\epsilon) \rightarrow f(u)$  and  $f'(u_\epsilon) \rightarrow f'(u)$  in  $L^p(\Omega)$ .

*Proof.*

$$\int_{\Omega} |f(u(x)) - f(0)|^p dx \leq K_0^p \int_{\Omega} |u(x)|^p dx < +\infty.$$

Since constant functions are in  $L^p(\Omega)$  for  $\Omega$  bounded it follows that  $f(u) \in L^p(\Omega)$ . Similarly

$$\int_{\Omega} |f(u_\epsilon(x)) - f(u(x))|^p dx \leq K_0^p \int_{\Omega} |u_\epsilon(x) - u(x)|^p dx,$$

which converges to zero. Thus the assertions proved for  $f(u)$ . The proof for  $f'(u)$  is identical.

The problem (1.2)–(1.3) can now be written as follows in its weak form :

( $P_\epsilon$ ) To find  $u_\epsilon \in H_0^1(\Omega)$  such that

$$(2.8) \quad \forall v \in H_0^1(\Omega), \quad a_\epsilon(u_\epsilon, v) = \int_{\Omega} f(u_\epsilon) \cdot v dx.$$

One can define the nonlinear operator

$T_\epsilon: H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  as follows;

$$(2.9) \quad \forall v \in H_0^1(\Omega), \quad a_\epsilon(T_\epsilon(u), v) = \int_{\Omega} f(u) \cdot v \, dx.$$

Then the problem  $(P_\epsilon)$  reduces to finding fixed points of  $T_\epsilon$  or the zeros of  $F_\epsilon$  where

$$(2.10) \quad F_\epsilon = I - T_\epsilon.$$

*Remark 2.1.* Since the injection  $H_0^1(\Omega) \rightarrow L^2(\Omega)$  is compact, it follows that  $T_\epsilon$  is compact.

Analogously, the operators  $T$  and  $F(=I - T)$ , associated to the homogenized operator  $A$ , can be defined.

Finally, an abstract theorem relating to solution of nonlinear equations in Banach spaces will now be recalled.

*Theorem 2.1.* Let  $X$  and  $Y$  be Banach spaces and  $F: D \subset X \rightarrow Y$ . Suppose that on an open convex set  $D_0 \subset D$ ,  $F$  is Fréchet differentiable and

$$(2.11) \quad \forall x, y \in D_0, \quad \|F'(x) - F'(y)\| \leq K \|x - y\|.$$

For some  $x_0 \in D_0$ , assume that  $\Gamma_0 = (F'(x_0))^{-1}$  is defined on all of  $Y$  and that  $h \equiv \beta K \eta \leq \frac{1}{2}$  where  $\|\Gamma_0\| \leq \beta$ , and  $\|\Gamma_0 F x_0\| \leq \eta$ . Set

$$(2.12) \quad t^* = \frac{1}{\beta K} (1 - \sqrt{1 - 2h}), \quad t^{**} = \frac{1}{\beta K} (1 + \sqrt{1 - 2h}).$$

Suppose that  $S \equiv \{x \mid \|x - x_0\| \leq t^*\} \subset D_0$ .

Then the Newton iterates  $\{x_k\}$  given by

$$(2.13) \quad x_{k+1} = x_k - (F'(x_k))^{-1} F(x_k), \quad k = 0, 1, \dots$$

are all well defined, lie in  $S$ , and converge to a solution  $x^*$  of  $F(x) = 0$ , which is unique in  $D_0 \cap \{x \mid \|x - x_0\| < t^{**}\}$ . Moreover, if  $h < \frac{1}{2}$  the order of convergence is at least quadratic.

Known as the Newton-Kantorovich Theorem, the above was proved by Kantorovich (cf. [2], [3]). Ortega [6] has given a simpler proof to the same. This theorem will be used to prove the main result in § 4.

### 3. Existence and homogenization results

Throughout this section we assume that the hypotheses (H1)–(H3) are verified.

*Proposition 3.1.* There exists a constant  $C > 0$ , independent of  $\epsilon$ , such that

$$(3.1) \quad \|u_\epsilon\|_{1, \Omega} \leq C.$$

*Proof.* Set  $v = u_\epsilon$  in (2.8). Then

$$\begin{aligned} a \|u_\epsilon\|_{1, \Omega}^2 &\leq a_\epsilon(u_\epsilon, u_\epsilon) = \int_{\Omega} f(u_\epsilon) u_\epsilon \, dx \\ &= \int_{\Omega} (f(u_\epsilon) - f(a)) u_\epsilon \, dx + \int_{\Omega} f(a) u_\epsilon \, dx, \quad f(a) = 0 \\ &\leq K_0 \int_{\Omega} |u_\epsilon - a| |u_\epsilon| \, dx + C_1 |u_\epsilon|_{0, \Omega} \\ &\leq K_0 |u_\epsilon|_{0, \Omega}^2 + C_2 |u_\epsilon|_{0, \Omega}^2. \end{aligned}$$

or

$$\|u_\epsilon\|_{1,\Omega}^2 \leq \frac{C_2}{\alpha - K_0} \|u_\epsilon\|_{0,\Omega} \leq \frac{C_2}{\alpha - K_0} \|u_\epsilon\|_{1,\Omega},$$

which proves (3.1).

*Remark 3.1.* In the above proof, the hypothesis  $K_0 < \alpha$  has been used.

*Theorem 3.1.* There exists at least one solution, to the problem  $(P_\epsilon)$ .

*Proof.* First of all it was remarked in the previous section that  $T_\epsilon$  is compact. Further if  $0 < \lambda < 1$  it can be proved that any solution of the problem

$$(3.2) \quad z_\epsilon = \lambda T_\epsilon z_\epsilon,$$

is such that

$$(3.3) \quad \|z_\epsilon\|_{1,\Omega} \leq C,$$

where  $C > 0$  is as in Proposition 3.1. (The proof of this fact is identical to that of the above-mentioned proposition.) Thus by Schaeffer's Theorem  $T_\epsilon$  admits at least one fixed point in  $B(0; C)$ , the ball centre 0 and radius  $C$  in  $H_0^1(\Omega)$ .

*Theorem 3.2.* There is at least one solution to the homogenized problem  $(P) : u = Tu$

*Proof.* The proof is identical to that of the preceding theorem.

*Theorem 3.3.* Let  $u_\epsilon$  be a solution to  $(P_\epsilon)$ . Then there exists a subsequence (again indexed by  $\epsilon$ ) such that  $u_\epsilon \rightarrow u$  in  $H_0^1(\Omega)$  weakly,  $u$  being a solution to the homogenized problem  $(P)$ .

*Proof.* By Proposition 3.1, the  $u_\epsilon$  are all bounded in  $H_0^1(\Omega)$ . Thus we can extract a weakly convergent subsequence with limit, say,  $u$ . Then  $u_\epsilon \rightarrow u$  in  $L^2(\Omega)$  strongly. Hence by proposition 2.1,  $f(u_\epsilon) \rightarrow f(u)$  in  $L^2(\Omega)$  strongly and thus in  $H^{-1}(\Omega)$  strongly. Now by the definition of the homogenized operator (cf. definition 2.1) we have

$$(3.4) \quad \forall v \in H_0^1(\Omega), a(u, v) = \int_{\Omega} f(u) v \, dx,$$

which proves the theorem.

The idea of using Schaeffer's Theorem is quite classical in the existence theory of nonlinear equations (cf. Rabinowitz [7]).

#### 4. The main theorem

Under the hypotheses (H1)–(H3), we already know that the problems  $(P)$  and  $(P_\epsilon)$  admit at least one solution each. We will prove now that if  $n \leq 5$ , and  $u$  a non-singular solution of  $(P)$ , among the solution of  $(P_\epsilon)$  there exists a solution  $u_\epsilon$  which is non singular, tending to  $u$ . This is an important step toward the study of the stability of homogenization for the associated evolution equation.

*Proposition 4.1.* Let  $n \leq 6$ . Then  $T_\epsilon$  (resp.  $T$ ) is Fréchet differentiable on all of  $V$ . The family  $\{T_\epsilon, T\}$  is uniformly Lipschitz continuous.

*Proof.* Since  $n \leq 6$ , the following inclusion holds :

$$H_0^1(\Omega) \rightarrow L^3(\Omega).$$

Then for  $u, v, w \in H_0^1(\Omega)$ , the integral

$$\int f'(u) v w \, dx,$$

is well defined and is continuous w.r.t.  $v$  and  $w$ . It is easy to check that the Fréchet derivative of  $T_\epsilon$  at  $u$  is defined by

$$(4.1) \quad \forall w \in H_0^1(\Omega), a_\epsilon(T'_\epsilon(u) v, w) = \int_\Omega f'(u) v w \, dx,$$

for any  $v \in H_0^1(\Omega)$ . Now, let  $u_1, u_2 \in H_0^1(\Omega)$ . Set

$$(4.2) \quad z_i^\epsilon = T'_\epsilon(u_i) v, \quad i = 1, 2,$$

where  $v \in H_0^1(\Omega)$ . Then

$$\begin{aligned} \alpha \|z_1^\epsilon - z_2^\epsilon\|_{1,\Omega}^2 &\leq a_\epsilon(z_1^\epsilon - z_2^\epsilon, z_1^\epsilon - z_2^\epsilon) \\ &= \int_\Omega (f'(u_1) - f'(u_2))(z_1^\epsilon - z_2^\epsilon) v \, dx \\ &\leq K_1 \int |u_1 - u_2| |z_1^\epsilon - z_2^\epsilon| |v| \, dx \\ &\leq K_1 \|u_1 - u_2\|_{0,3,\Omega} \|z_1^\epsilon - z_2^\epsilon\|_{0,3,\Omega} \|v\|_{0,3,\Omega} \\ &\leq C K_1 \|u_1 - u_2\|_{1,\Omega} \|z_1^\epsilon - z_2^\epsilon\|_{1,\Omega} \|v\|_{1,\Omega}, \end{aligned}$$

$$\text{or} \quad \|(T'_\epsilon(u_1) - T'_\epsilon(u_2)) v\|_{1,\Omega} \leq K \|u_1 - u_2\|_{1,\Omega} \|v\|_{1,\Omega},$$

which gives

$$(4.3) \quad \|T'_\epsilon(u_1) - T'_\epsilon(u_2)\|_{1,\Omega} \leq K \|u_1 - u_2\|_{1,\Omega}$$

$K$  being independent of  $\epsilon$ . The proof for  $T$  is identical.

**Proposition 4.2.** Let  $n \leq 5$ . Then for any  $u \in H_0^1(\Omega)$ ,  $T'_\epsilon(u)$  (resp.  $T'(u)$ ) is compact.

*Proof.* For  $n \leq 5$ , the injection

$$(4.4) \quad H_0^1(\Omega) \rightarrow L^3(\Omega)$$

is compact. The result is a direct consequence of this fact.

**Remark 4.1.** If  $n = 6$ , then the inclusion (4.4) is not compact and the above result does not hold.

Henceforth, it will be assumed that  $u$  is a given solution of the homogenized problem (P). Let, for each  $\epsilon$ ,  $w_\epsilon \in H_0^1(\Omega)$  be defined as the unique solution of the following linear problem.

$$(4.5) \quad \forall v \in H_0^1(\Omega), a_\epsilon(w_\epsilon, v) = \int_\Omega f(u) v \, dx.$$

From the theory of homogenization of linear problems, it follows that  $w_\epsilon \rightarrow u$  in  $H_0^1(\Omega)$  weakly [and in  $L^2(\Omega)$  strongly].

**Proposition 4.3.** Let  $n \leq 5$ . Let  $u$  be a non-singular solution of problem (P), i.e.  $F'(u) = I - T'(u)$  is invertible. Then for sufficiently small  $\epsilon$ ,  $F'_\epsilon(w_\epsilon) =$

$I - T'_\epsilon(w_\epsilon)$  is also invertible. Further, there exists a  $\beta > 0$ , independent of  $\epsilon$ , such that

$$(4.6) \quad \|(I - T'_\epsilon(w_\epsilon))^{-1}\| \leq \beta.$$

*Proof.* Assume the first assertion false. Then there exists a sequence  $\epsilon_n \rightarrow 0$  such that

$$I - T'_{\epsilon_n}(w_{\epsilon_n}),$$

is singular. (Henceforth, purely as a matter of convenience, the index  $n$  will be suppressed.) Since  $T'_\epsilon(w_\epsilon)$  is compact, there exists a non-zero function  $z_\epsilon \in H_0^1(\Omega)$  such that

$$(4.7) \quad z_\epsilon = T'_\epsilon(w_\epsilon) z_\epsilon.$$

It can be further assumed that

$$(4.8) \quad \|z_\epsilon\|_{1,\Omega} = 1.$$

Now (4.7) can be rewritten as

$$(4.9) \quad \forall w \in H_0^1(\Omega), a_\epsilon(z_\epsilon, w) = \int_\Omega f'(w_\epsilon) z_\epsilon w \, dx.$$

Now for a subsequence (again indexed by  $\epsilon$ )  $z_\epsilon \rightarrow z$  and  $w_\epsilon \rightarrow u$  in  $H_0^1(\Omega)$  weakly. Consider, for any  $w \in H_0^1(\Omega)$ ,

$$\begin{aligned} & \left| \int_\Omega (f'(w_\epsilon) z_\epsilon - f'(u) z) w \, dx \right| \\ & \leq \int_\Omega |f'(w_\epsilon)| |z_\epsilon - z| |w| \, dx + \int_\Omega |f'(w_\epsilon) - f'(u)| |z| |w| \, dx \\ & \leq C(|z_\epsilon - z|_{0,3,\Omega} + |f'(w_\epsilon) - f'(u)|_{0,3,\Omega}) \|w\|_{1,\Omega}. \end{aligned}$$

Since  $z_\epsilon \rightarrow z$  in  $L^3(\Omega)$  strongly ( $n \leq 5$ ) and  $f'(w_\epsilon) \rightarrow f'(u)$  in  $L^3(\Omega)$  strongly [cf. Proposition (2.11)] it follows that

$$f'(w_\epsilon) z_\epsilon \rightarrow f'(u) z,$$

in  $H^{-1}(\Omega)$  strongly. Then by the definition of the homogenized operator,

$$(4.10) \quad \forall w \in H_0^1(\Omega), a(z, w) = \int_\Omega f'(u) z w \, dx,$$

or, equivalently,

$$(4.11) \quad z = T'(u) z.$$

By assumption, this is possible only if  $z = 0$ . Hence  $z_\epsilon \rightarrow 0$  in  $H_0^1(\Omega)$  weakly ( $L^3(\Omega)$  strongly).

Now, choosing  $w = z_\epsilon$  in (4.9) and taking into account (4.8), it follows that

$$0 < \alpha \leq \int_\Omega f'(w_\epsilon) z_\epsilon z_\epsilon \, dx \leq |f'(w_\epsilon)|_{0,3,\Omega} |z_\epsilon|_{0,3,\Omega}$$

which converges to zero, thus giving a contradiction. Hence the first assertion is true.

Assume now that (4.6) is false. Then there exists a sequence (denoted, as usual, by  $\epsilon$  itself)  $y_\epsilon$  and a sequence  $v_\epsilon$  in  $H_0^1(\Omega)$  such that

$$(4.12) \quad \begin{cases} v_\epsilon = (I - T'_\epsilon(w_\epsilon))^{-1} y_\epsilon, \\ \|v_\epsilon\|_{1, \Omega} = 1, \|y_\epsilon\|_{1, \Omega} \rightarrow 0. \end{cases}$$

Set,

$$(4.13) \quad z_\epsilon = T'_\epsilon(w_\epsilon) v_\epsilon.$$

Then

$$(4.14) \quad y_\epsilon = v_\epsilon - z_\epsilon.$$

Clearly,  $\|z_\epsilon\|_{1, \Omega} \leq C$  and hence (for a subsequence)  $v_\epsilon \rightarrow v$ ,  $z_\epsilon \rightarrow z$ , say, in  $H_0^1(\Omega)$  weakly. Since  $y_\epsilon \rightarrow 0$  in  $H_0^1(\Omega)$  strongly it follows that  $z = v$ . As before it can be shown that

$$f'(w_\epsilon) v_\epsilon \rightarrow f'(u) v,$$

in  $H^{-1}(\Omega)$  strongly. Passing to the limit,

$$(4.15) \quad \forall w \in H_0^1(\Omega), a(z, w) = \int_{\Omega} f'(u) z w \, dx,$$

which again implies  $z = 0 = v$ . Once again,

$$(4.16) \quad \alpha \|z_\epsilon\|_{1, \Omega}^2 \leq a_\epsilon(z_\epsilon, z_\epsilon) = \int_{\Omega} f'(w_\epsilon) v_\epsilon z_\epsilon \, dx.$$

The integral converges to zero since  $v_\epsilon \rightarrow 0$ ,  $z_\epsilon \rightarrow 0$  in  $L^3(\Omega)$  strongly. Hence it follows from (4.16) that  $z_\epsilon \rightarrow 0$  in  $H_0^1(\Omega)$  strongly and hence from (4.14) that  $v_\epsilon \rightarrow 0$  in  $H_0^1(\Omega)$  strongly. But this contradicts the fact that  $\|v_\epsilon\|_{1, \Omega} = 1$ . Thus, (4.6) is established.

*Proposition 4.4.*

$$\lim_{\epsilon \rightarrow 0} \|F_\epsilon(w_\epsilon)\|_{1, \Omega} = 0.$$

*Proof.* Set

$$(4.17) \quad y_\epsilon = T_\epsilon w_\epsilon.$$

$$\text{Then } \|F_\epsilon(w_\epsilon)\|_{1, \Omega} = \|w_\epsilon - y_\epsilon\|_{1, \Omega}.$$

Now by definition,

$$(4.18) \quad \forall w \in H_0^1(\Omega), a_\epsilon(w_\epsilon - y_\epsilon, w) = \int_{\Omega} (f(u) - f(w_\epsilon)) w \, dx.$$

Thus, it follows that

$$\alpha \|w_\epsilon - y_\epsilon\|_{1, \Omega}^2 \leq K_0 \|u - w_\epsilon\|_{0, \Omega} \|w_\epsilon - y_\epsilon\|_{0, \Omega},$$

or, for some  $C > 0$ , independent of  $\epsilon$ ,

$$(4.19) \quad \|w_\epsilon - y_\epsilon\|_{1, \Omega} \leq C \|u - w_\epsilon\|_{0, \Omega},$$

which converges to zero, thus proving the result.

The main theorem can now be stated and proved.



**Theorem 4.1.** Let the hypotheses (H1-H3) be true. Let  $n \leq 5$ . If  $u$  is a given non-singular solution of the homogenized problem (P), then for sufficiently small  $\epsilon$ , the problem  $(P_\epsilon)$  admits a solution  $u_\epsilon$  such that  $u_\epsilon \rightarrow u$  in  $H_0^1(\Omega)$  weakly and  $L^2(\Omega)$  strongly. Further, the solution  $u_\epsilon$  is also non-singular.

*Proof.* For the first assertion, it will be shown that all the hypotheses of the Newton-Kantorovich theorem are verified.

**Step 1.** First of all for every  $\epsilon$ ,  $F'_\epsilon$  exists on all of  $H_0^1(\Omega)$  and is Lipschitz continuous with a constant  $K$ , independent of  $\epsilon$  (Proposition 4.1).

**Step 2.** Consider  $w_\epsilon$  defined by (4.5). Then for sufficiently small  $\epsilon$ ,  $\Gamma_{0,\epsilon} = (F'_\epsilon(w_\epsilon))^{-1}$  exists and  $\|\Gamma_{0,\epsilon}\| \leq \beta$ , independent of  $\epsilon$  (Proposition 4.3).

**Step 3.** Finally set

$$(4.20) \quad \eta_\epsilon = C\beta \|u - w_\epsilon\|_{0,\Omega},$$

where  $C$  is as in (4.19). Then

$$\|\Gamma_{0,\epsilon} F_\epsilon(w_\epsilon)\| \leq \eta_\epsilon,$$

and  $\eta_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Hence for sufficiently small  $\epsilon$ ,

$$(4.21) \quad h_\epsilon = K\beta\eta_\epsilon < 1/2.$$

In fact,  $h_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Thus the Newton iterates

$$(4.22) \quad \begin{cases} u_\epsilon^{k+1} = u_\epsilon^k - (F'_\epsilon(u_\epsilon^k))^{-1} F_\epsilon(u_\epsilon^k) \\ u_\epsilon^0 = w_\epsilon \end{cases}$$

are all well-defined, lie in the ball

$$(4.23) \quad S_\epsilon = \{v \mid \|v - w_\epsilon\|_{1,\Omega} \leq t_\epsilon^*\},$$

with

$$(4.24) \quad t_\epsilon^* = 1/\beta K(1 - \sqrt{1 - 2h_\epsilon}),$$

and converge to a solution  $u_\epsilon$  of  $(P_\epsilon)$  which also lies in  $S_\epsilon$ , by the Newton-Kantorovich theorem (cf. Theorem 2.1).

**Step 4.** Now by Theorem 3.3, given a subsequence of  $\{u_\epsilon\}$ , a further subsequence can be extracted converging to a solution, say,  $\hat{u}$  of (P). But since  $h_\epsilon \rightarrow 0$ , it follows that  $t_\epsilon^* \rightarrow 0$  and hence  $\|u_\epsilon - w_\epsilon\|_{1,\Omega} \rightarrow 0$ . Since both  $u_\epsilon$  and  $w_\epsilon$  converge weakly, the limits must coincide and hence  $\hat{u} = u$ . Thus irrespective of the subsequence, the limit is always  $u$  and so it can be deduced the entire family  $u_\epsilon$  converges to  $u$  weakly in  $H_0^1(\Omega)$ .

**Step 5.** To prove the non-singularity of  $u_\epsilon$ , the invertibility of  $I - T'_\epsilon(u_\epsilon)$  must be proved. This is done exactly as in Proposition (4.3).

Thus the theorem is completely proved.

**Remark 4.2.** It may be wondered that  $u_0$  could be used as a starting point for the Newton iterates for each  $\epsilon$  instead of  $w_\epsilon$ , thus avoiding several estimates. But then  $F_\epsilon(u_0) = u_0 - w_\epsilon$  which does not converge to zero in  $H_0^1(\Omega)$ . Thus  $h_\epsilon < 1/2$  cannot be guaranteed. Thus  $u_\epsilon$  converges to  $u$  only weakly while  $w_\epsilon$  is always sufficiently close to  $u_\epsilon$  to guarantee the convergence of the Newton method

## 5. Conclusion

It has been proved that non-singular solutions of homogenized problem are limits of non-singular solutions of the given family of problems,  $(P_\epsilon)$ . The hypotheses on the coefficients of the differential operator and on the given function  $f$  are fairly minimal.

The hypotheses (H1) and (H2) are standard ones used in the linear theory of homogenization. The Lipschitz continuity of  $f$  and  $f'$  laid down in hypothesis (H3) was used to prove the Fréchet derivability of  $T_\epsilon$  and the Lipschitz continuity of  $T'_\epsilon$ , an essential condition in the Newton-Kantorovich Theorem. It seems possible that, working with classical solutions (cf. Mythily [5]), the local Lipschitz continuity of  $f$  and  $f'$  may suffice. The only simplifying assumption is  $K_0 < \alpha$ , which has been used only to prove the existence of a solution *via a priori* estimates. Indeed this condition can be removed and replaced by any other which guarantees the existence of a non-singular solution of the homogenized problem alone. Then the method of this paper can be used to prove (without new hypotheses) the existence of a solution of  $(P_\epsilon)$ . This solution will be non-singular and will converge to  $u$ .

The principal drawback seems to be the restriction on the dimension, viz.,  $n \leq 5$ . In the case  $n = 6$ , the theorem can be established with the additional hypotheses that  $f'$  is bounded. Then  $f'(u)v \in L^2(\Omega)$  for  $u, v \in H_0^1(\Omega)$  and the compactness of the operators  $T'_\epsilon(u)$  and  $T''(u)$  will follow.

The case  $n > 6$  seems to be open. The main difficulty in this case is the differentiability of  $T_\epsilon$ . It is not obvious [even for  $f'(x) = x$ ] that  $f'(u)v \in H^{-1}(\Omega)$ . Of course, it will be so if  $f'$  is assumed bounded. Even then while  $T'(u)$  can be *formally* defined by (4.1), it cannot be *proved* that it is indeed the Fréchet derivative of  $T_\epsilon$  at  $u$ . This is because in assuming  $f'$  to be bounded, while  $f'(u)v \in L^2(\Omega) \rightarrow H^{-1}(\Omega)$ ,  $f'(u)$  does not play any role in the estimates. Again for the same reasons  $T'_\epsilon$  cannot be proved to be Lipschitz continuous.

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