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A Simple Finite Element Formulation of a Higher-order Theory for Unsymmetrically Laminated Composite Plates

T. Kant and B. N. Pandya

Department of Civil Engineering, Indian Institute of Technology, IIT Powai, Bombay 400 076, India

ABSTRACT

A higher-order theory which satisfies zero transverse shear stress conditions on the bounding planes of a generally laminated fibre-reinforced composite plate subjected to transverse loads is developed. The displacement model accounts for non-linear distribution of inplane displacement components through the plate thickness and the theory requires no shear correction coefficients. A C^0 continuous displacement finite element formulation is presented and the coupled membrane-flexure behaviour of laminated plates is investigated. The nodal unknowns are the three displacements, two rotations and two higher-order functions as the generalized degrees of freedom. The simple isoparametric formulation developed here is capable of evaluating transverse shears and transverse normal stress accurately by using the equilibrium equations. The accuracy of the nine-noded Lagrangian quadrilateral element is then established by comparing the present results with the closed-form, three-dimensional elasticity and other finite element available solutions.

1 INTRODUCTION

It has long been recognized that classical plate theory must be modified to include certain higher-order effects like warping of the cross-section. The first generalization of the classical theory was given by Reissner[1] and Mindlin.[2] Since then, there have been many further generalizations. Perhaps the first higher-order theory, based upon the principle of stationary potential energy, resulting in eleven second-order partial differential

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equations to determine the eleven functions in the assumed displacement model, was given by Lo *et al.*[3,4] A sub-set of the displacement model used in Ref. 3, which neglects the strain energy due to transverse normal stress, has been adopted by Levinson,[5] Murthy[6] and Reddy.[7] Later, Reddy and co-workers presented the displacement[8] and the mixed[9] finite element models of the theory developed earlier.[7] In these,[7–9] Reddy has modified the displacement model of Lo *et al.*, by neglecting the transverse normal stress effect and satisfying the zero transverse shear stress conditions on the bounding planes of the plate, thereby expressing the three displacements of a point in plate space in terms of only five physical midplane displacement quantities. With this, the displacement model adopted by Reddy gives rise to second-order derivatives of transverse displacement in the energy expression, and hence displacement based finite element formulation requires C^1 continuous shape functions which are computationally inefficient and are not amenable to the popular and widely used isoparametric formulation in present day finite element technology.

The aim of the present work is to develop a simple isoparametric finite element formulation. The formulation presented here differs from that of Reddy and co-workers[8,9] in three ways:

(i) Only a part of the conditions for vanishing of the transverse shear stresses on the top and bottom bounding planes of the plate as given by eqn (3) is introduced in the assumed displacement model given by eqn (1).
(ii) The remaining conditions given by eqn (4) are introduced later in the shear rigidity matrix as given by eqn (18).
(iii) The general isoparametric displacement finite element formulation is developed.

The validity of the formulation is established by comparing the present numerical results with other finite element,[9] closed-form[10] and three-dimensional elasticity[11] solutions.

2 THEORY

The development of the present higher-order shear deformation theory begins with the assumption of the displacement field in the following form (Fig. 1)

$$
\begin{aligned}
u(x,y,z) &= u_o(x,y) + z\theta_x(x,y) + z^2 u_o^*(x,y) + z^3\theta_x^*(x,y) \\
v(x,y,z) &= v_o(x,y) + z\theta_y(x,y) + z^2 v_o^*(x,y) + z^3\theta_y^*(x,y) \\
w(x,y,z) &= w_o(x,y)
\end{aligned}
\tag{1}
$$

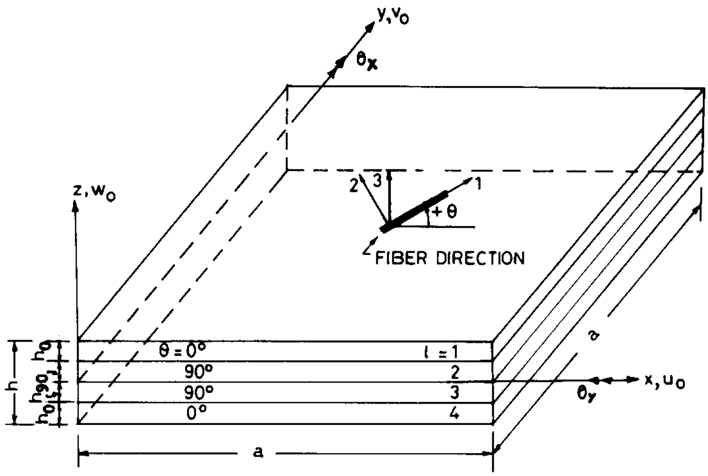

Fig. 1. Geometry of a four-layer symmetric laminate.

where u_o, v_o and w_o denote the displacements of a point (x, y) on the midplane and θ_x and θ_y are the rotations of normals to midplane about the y and x axes respectively. The parameters u_o^*, v_o^*, θ_x^* and θ_y^* are the corresponding higher-order terms in the Taylor's series expansion and are also defined at the midplane.[12,13] The condition that the transverse shear stresses vanish on the plate's top and bottom faces is equivalent to the requirement that the corresponding strains be zero on these surfaces. The transverse shear strains are given by

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \theta_y + 2zv_o^* + 3z^2\theta_y^* + \frac{\partial w_o}{\partial y}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \theta_x + 2zu_o^* + 3z^2\theta_x^* + \frac{\partial w_o}{\partial x} \tag{2}$$

Equating $\gamma_{yz}(x, y, \pm h/2)$ and $\gamma_{xz}(x, y, \pm h/2)$ to zero, we obtain

$$v_o^* = u_o^* = 0 \tag{3}$$

and

$$\theta_y^* = -\frac{4}{3h^2}\left(\theta_y + \frac{\partial w_o}{\partial y}\right); \; \theta_x^* = -\frac{4}{3h^2}\left(\theta_x + \frac{\partial w_o}{\partial x}\right) \tag{4}$$

Introduction of condition (3) in eqn (1) yields a compact displacement form:

$$u = u_o + z\theta_x + z^3\theta_x^*$$

$$v = v_o + z\theta_y + z^3\theta_y^*$$

$$w = w_o \tag{5}$$

Murthy[6] and more recently Reddy and co-workers[7–9] have used conditions (4) to eliminate θ_x^* and θ_y^* from eqn (5) and have obtained a modified displacement model

$$u = u_o + z\left[\theta_x - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\theta_x + \frac{\partial w_o}{\partial x}\right)\right]$$

$$v = v_o + z\left[\theta_y - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\theta_y + \frac{\partial w_o}{\partial y}\right)\right]$$

$$w = w_o \tag{6}$$

In the present formulation, we proceed with the displacement field given by eqn (5) and conditions (4) are introduced later in the shear rigidity matrix.

The strains associated with the displacements in eqn (5) are:

$$\begin{aligned}
\epsilon_x &= \epsilon_{xo} + zk_x + z^3 k_x^* \\
\epsilon_y &= \epsilon_{yo} + zk_y + z^3 k_y^* \\
\epsilon_z &= 0 \\
\gamma_{xy} &= \epsilon_{xyo} + zk_{xy} + z^3 k_{xy}^* \\
\gamma_{yz} &= \phi_y + z^2 \phi_y^* \\
\gamma_{xz} &= \phi_x + z^2 \phi_x^*
\end{aligned} \tag{7}$$

where

$$\epsilon_{xo} = \frac{\partial u_o}{\partial x},\ \epsilon_{yo} = \frac{\partial v_o}{\partial y},\ \epsilon_{xyo} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}$$

$$k_x = \frac{\partial \theta_x}{\partial x},\quad k_y = \frac{\partial \theta_y}{\partial y},\quad k_{xy} = \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}$$

$$k_x^* = \frac{\partial \theta_x^*}{\partial x},\quad k_y^* = \frac{\partial \theta_y^*}{\partial y},\quad k_{xy}^* = \frac{\partial \theta_x^*}{\partial y} + \frac{\partial \theta_y^*}{\partial x}$$

$$\phi_y = \theta_y + \frac{\partial w_o}{\partial y},\ \phi_x = \theta_x + \frac{\partial w_o}{\partial x}$$

$$\phi_y^* = 3\theta_y^*,\ \phi_x^* = 3\theta_x^* \tag{8}$$

The constitutive equations for the L^{th} layer can be written as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}^L = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}^L \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix}^L$$

$$\begin{Bmatrix} \tau_{23} \\ \tau_{13} \end{Bmatrix}^L = \begin{bmatrix} C_{44} & 0 \\ 0 & C_{55} \end{bmatrix}^L \begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix}^L \tag{9}$$

where $(\sigma_1, \sigma_2, \tau_{12}, \tau_{23}, \tau_{13})$ are the stress and $(\epsilon_1, \epsilon_2, \gamma_{12}, \gamma_{23}, \gamma_{13})$ are the linear strain components referred to as the lamina co-ordinates $(1, 2, 3)$, as shown in Fig. 1, and C_{ij}s are the plane stress reduced elastic constants of the L^{th} lamina. The following relations hold between these and the engineering elastic constants:

$$C_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}},\ C_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}},\ C_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}$$

$$C_{33} = G_{12},\ C_{44} = G_{23},\ C_{55} = G_{13} \tag{10}$$

The stress–strain relations for the L^{th} lamina in the laminate co-ordinates (x, y, z) are written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^L = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix}^L \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}^L$$

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^L = \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{45} & Q_{55} \end{bmatrix}^L \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^L \tag{11}$$

in which

$$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \tau_{xy}, \tau_{yz}, \tau_{xz})^t$$

and

$$\boldsymbol{\epsilon} = (\epsilon_x, \epsilon_y, \gamma_{xy}, \gamma_{yz}, \gamma_{xz})^{\mathrm{t}} \tag{12}$$

are the stress and linear strain vectors with respect to the laminate axes and Q_{ij}s are the plane stress reduced elastic constants in the plate (laminate) axes of the L^{th} lamina. The transformation of the stresses/strains between the lamina and laminate co-ordinate systems follows the usual transformation rule.[14]

The total potential energy π of the plate is given by

$$\pi = \frac{1}{2}\int_V \boldsymbol{\epsilon}^{\mathrm{t}}\boldsymbol{\sigma}\,\mathrm{d}V - \int_A \mathbf{d}^{\mathrm{t}}\mathbf{P}\,\mathrm{d}A \tag{13}$$

in which A is the mid-surface area of the plate, V is the plate volume, $\mathbf{P}$ is the equivalent load vector corresponding to the seven degrees of freedom and $\mathbf{d}$ is defined as

$$\mathbf{d} = (u_{\mathrm{o}}, v_{\mathrm{o}}, w_{\mathrm{o}}, \theta_x, \theta_y, \theta_x^*, \theta_y^*)^{\mathrm{t}} \tag{14}$$

The expressions for the strain components given by eqn (7) are substituted in eqn (13). The functional given by eqn (13) is then minimized while carrying out explicit integration through the plate thickness. This leads to the following thirteen stress resultants for the n-layered laminate.

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{\mathrm{L}=1}^{n} \int_{h_{\mathrm{L}+1}}^{h_{\mathrm{L}}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \mathrm{d}z$$

$$\left\{\begin{array}{c:c} M_x & M_x^* \\ M_y & M_y^* \\ M_{xy} & M_{xy}^* \end{array}\right\} = \sum_{\mathrm{L}=1}^{n} \int_{h_{\mathrm{L}+1}}^{h_{\mathrm{L}}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} [z \,\vdots\, z^3]\,\mathrm{d}z$$

$$\left\{\begin{array}{c:c} Q_x & Q_x^* \\ Q_y & Q_y^* \end{array}\right\} = \sum_{\mathrm{L}=1}^{n} \int_{h_{\mathrm{L}+1}}^{h_{\mathrm{L}}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} [1 \,\vdots\, z^2]\,\mathrm{d}z \tag{15}$$

After integration, these relations are written in a matrix form which defines the stress resultant/strain relations of the laminate and is given by

$$\left\{\begin{array}{c} \mathbf{N} \\ \hline \mathbf{M} \\ \\ \mathbf{M}^* \\ \hline \mathbf{Q} \\ \\ \mathbf{Q}^* \end{array}\right\} = \left[\begin{array}{c|c|c} \mathbf{A} & \mathbf{B} & 0 \\ \hline \mathbf{B}^{\mathrm{t}} & \mathbf{D}_{\mathrm{B}} & 0 \\ & & \\ \hline & & \\ 0 & 0 & \mathbf{D}_{\mathrm{S}} \\ & & \end{array}\right] \left\{\begin{array}{c} \boldsymbol{\epsilon}_{\mathrm{o}} \\ \hline \mathbf{k} \\ \\ \mathbf{k}^* \\ \hline \boldsymbol{\phi} \\ \\ \boldsymbol{\phi}^* \end{array}\right\} \tag{16}$$

in which

$$\begin{aligned} \mathbf{N} &= \{N_x, N_y, N_{xy}\}^{\mathrm{t}} & \boldsymbol{\epsilon}_{\mathrm{o}} &= \{\epsilon_{xo}, \epsilon_{yo}, \epsilon_{xyo}\}^{\mathrm{t}} \\ \mathbf{M} &= \{M_x, M_y, M_{xy}\}^{\mathrm{t}} & \mathbf{k} &= \{k_x, k_y, k_{xy}\}^{\mathrm{t}} \\ \mathbf{M}^* &= \{M_x^*, M_y^*, M_{xy}^*\}^{\mathrm{t}} & \mathbf{k}^* &= \{k_x^*, k_y^*, k_{xy}^*\}^{\mathrm{t}} \\ \mathbf{Q} &= \{Q_x, Q_y\}^{\mathrm{t}} & \boldsymbol{\phi} &= \{\phi_x, \phi_y\}^{\mathrm{t}} \\ \mathbf{Q}^* &= \{Q_x^*, Q_y^*\}^{\mathrm{t}} & \boldsymbol{\phi}^* &= \{\phi_x^*, \phi_y^*\}^{\mathrm{t}} \end{aligned} \tag{17}$$

The superscript t denotes the transpose of a vector/matrix and

$$\mathbf{A} = \sum_{\mathrm{L}=1}^{n} \begin{array}{l} \begin{bmatrix} Q_{11}H_1 & Q_{12}H_1 & Q_{13}H_1 \\ & Q_{22}H_1 & Q_{23}H_1 \\ \text{Symmetric} & & Q_{33}H_1 \end{bmatrix} \end{array} \text{L}^{\text{th}} \text{ Layer}$$

$$\mathbf{B} = \sum_{\mathrm{L}=1}^{n} \begin{bmatrix} Q_{11}H_2 & Q_{12}H_2 & Q_{13}H_2 & Q_{11}H_4 & Q_{12}H_4 & Q_{13}H_4 \\ Q_{12}H_2 & Q_{22}H_2 & Q_{23}H_2 & Q_{12}H_4 & Q_{22}H_4 & Q_{23}H_4 \\ Q_{13}H_2 & Q_{23}H_2 & Q_{33}H_2 & Q_{13}H_4 & Q_{23}H_4 & Q_{33}H_4 \end{bmatrix} \text{L}^{\text{th}} \text{ Layer}$$

$$\mathbf{D}_{\mathrm{B}} = \sum_{\mathrm{L}=1}^{n} \begin{bmatrix} Q_{11}H_3 & Q_{12}H_3 & Q_{13}H_3 & Q_{11}H_5 & Q_{12}H_5 & Q_{13}H_5 \\ & Q_{22}H_3 & Q_{23}H_3 & Q_{12}H_5 & Q_{22}H_5 & Q_{23}H_5 \\ & & Q_{33}H_3 & Q_{13}H_5 & Q_{23}H_5 & Q_{33}H_5 \\ & & & Q_{11}H_7 & Q_{12}H_7 & Q_{13}H_7 \\ \text{Symmetric} & & & & Q_{22}H_7 & Q_{23}H_7 \\ & & & & & Q_{33}H_7 \end{bmatrix} \text{L}^{\text{th}} \text{ Layer}$$

$$
\mathbf{D}_S = \sum_{L=1}^{n} \begin{bmatrix} Q_{55}H & Q_{45}H & 0 & 0 \\ & Q_{44}H & 0 & 0 \\ & & Q_{55}H^* & Q_{45}H^* \\ \text{Symmetric} & & & Q_{44}H^* \end{bmatrix}_{\text{L}^{\text{th}}\ \text{Layer}} \tag{18}
$$

where

$$
\begin{aligned}
H_1 &= (h_L - h_{L+1}), & H_2 &= \frac{1}{2}(h_L^2 - h_{L+1}^2) \\
H_3 &= \frac{1}{3}(h_L^3 - h_{L+1}^3), & H_4 &= \frac{1}{4}(h_L^4 - h_{L+1}^4) \\
H_5 &= \frac{1}{5}(h_L^5 - h_{L+1}^5), & H_7 &= \frac{1}{7}(h_L^7 - h_{L+1}^7)
\end{aligned} \tag{19}
$$

$$
H = \left(H_1 - H_3 \times \frac{4}{h^2}\right), \quad H^* = \left(H_5 - H_3 \times \frac{h^2}{4}\right)
$$

The shear rigidity matrix $\mathbf{D}_S$ in eqn (18) is evolved by incorporating an alternate form of the conditions (4), viz.

$$
\begin{aligned}
\phi_y + \frac{h^2}{4}\phi_y^* &= 0 \\
\phi_x + \frac{h^2}{4}\phi_x^* &= 0
\end{aligned} \tag{20}
$$

and the resulting theory becomes consistent in the sense that it satisfies zero transverse shear stress conditions on the bounding planes of the plate. If the conditions given by eqn (20) are not incorporated, then the resulting non-consistent theory does not satisfy the zero transverse shear stress conditions on the bounding planes of the plate. In this case, the shear rigidity matrix $\mathbf{D}_S^*$ is defined as

$$
D_S^* = \sum_{L=1}^{n} \begin{bmatrix} Q_{55}H_1 & Q_{45}H_1 & Q_{55}H_3 & Q_{45}H_3 \\ & Q_{44}H_1 & Q_{45}H_3 & Q_{44}H_3 \\ & & Q_{55}H_5 & Q_{45}H_5 \\ \text{Symmetric} & & & Q_{44}H_5 \end{bmatrix}_{\text{L}^{\text{th}}\ \text{layer}} \tag{21}
$$

The transverse shear stresses τ_{xz}^{L} and τ_{yz}^{L} cannot accurately be given by eqn (11) as the continuity condition at the interfaces of any two layers is not satisfied for laminated plates. For this reason, the interlaminar shear and normal stresses $(\tau_{xz}^{L}, \tau_{yz}^{L}, \sigma_{z}^{L})$ between layer (L) and (L + 1) at $z = h_{L+1}$ are obtained by integrating the three equilibrium equations of elasticity for each layer over the lamina thickness and summing over layers L through n as follows:

$$\tau_{xz}^{L}\Big|_{z=h_{L+1}} = -\sum_{L=1}^{n}\int_{h_{L+1}}^{h_L}\left(\frac{\partial\sigma_x^L}{\partial x}+\frac{\partial\tau_{xy}^L}{\partial y}\right)dz$$

$$\tau_{yz}^{L}\Big|_{z=h_{L+1}} = -\sum_{L=1}^{n}\int_{h_{L+1}}^{h_L}\left(\frac{\partial\sigma_y^L}{\partial y}+\frac{\partial\tau_{xy}^L}{\partial x}\right)dz$$

$$\sigma_{z}^{L}\Big|_{z=h_{L+1}} = -\sum_{L=1}^{n}\int_{h_{L+1}}^{h_L}\left(\frac{\partial\tau_{xz}^L}{\partial x}+\frac{\partial\tau_{yz}^L}{\partial y}\right)dz \tag{22}$$

3 FINITE ELEMENT FORMULATION

We follow the standard finite element technique in which the total solution domain is discretized into NE sub-domains (elements) such that

$$\pi(\mathbf{d}) = \sum_{e=1}^{NE}\pi^{e}(\mathbf{d}) \tag{23}$$

where π and π^e are the total potential of the system and the element respectively. The element potential can be expressed in terms of internal strain energy U^e and the external work done W^e for an element e as,

$$\pi^{e}(\mathbf{d}) = U^{e} - W^{e} \tag{24}$$

in which $\mathbf{d}$ is the vector of unknown displacement variables in the problem and is defined by eqn (14). If the same interpolation function is used to define all the components of the generalized displacement vector $\mathbf{d}$, we can write

$$\mathbf{d} = \sum_{i=1}^{NN} N_i\mathbf{d}_i \tag{25}$$

where N_i is the interpolating (shape) function associated with node i, $\mathbf{d}_i$ is the value of $\mathbf{d}$ corresponding to node i and NN is the number of nodes in the element.

The extensional strains $\boldsymbol{\epsilon}_o$, the bending curvatures $(\mathbf{k}, \mathbf{k}^*)$ and the transverse shear strains $(\boldsymbol{\phi}, \boldsymbol{\phi}^*)$ can be written in terms of the nodal displacements $\mathbf{d}$ by referring to eqn (8). The result can be written in matrix form as follows:

$$\boldsymbol{\epsilon}_o = \mathbf{L}_E \mathbf{d}$$

$$\begin{Bmatrix} \mathbf{k} \\ \mathbf{k}^* \end{Bmatrix} = \mathbf{L}_B \mathbf{d}$$

$$\begin{Bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\phi}^* \end{Bmatrix} = \mathbf{L}_S \mathbf{d} \tag{26}$$

The subscripts E, B and S refer to extension, bending and shear respectively and the vector of element nodal displacements is given by eqn (14). The matrices $\mathbf{L}_E$, $\mathbf{L}_B$ and $\mathbf{L}_S$ attain the following form

$$\mathbf{L}_E = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{L}_B = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

$$\mathbf{L}_{\mathrm{S}} = \begin{bmatrix} 0 & 0 & \dfrac{\partial}{\partial x} & 1 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\partial}{\partial y} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \tag{27}$$

With the generalized displacement vector $\mathbf{d}$ known at all points within the element, the generalized strain vectors at any point are determined with the aid of eqns (27) and (25), as follows:

$$\boldsymbol{\epsilon}_{o} = \mathbf{L}_{\mathrm{E}}\,\mathbf{d} = \mathbf{L}_{\mathrm{E}} \sum_{i=1}^{NN} N_i \mathbf{d}_i = \sum_{i=1}^{NN} \mathbf{B}_{i\mathrm{E}}\,\mathbf{d}_i = \mathbf{B}_{\mathrm{E}}\,\mathbf{a}$$

$$\begin{Bmatrix} \mathbf{k} \\ \mathbf{k}^* \end{Bmatrix} = \mathbf{L}_{\mathrm{B}}\,\mathbf{d} = \mathbf{L}_{\mathrm{B}} \sum_{i=1}^{NN} N_i \mathbf{d}_i = \sum_{i=1}^{NN} \mathbf{B}_{i\mathrm{B}}\,\mathbf{d}_i = \mathbf{B}_{\mathrm{B}}\,\mathbf{a}$$

$$\begin{Bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\phi}^* \end{Bmatrix} = \mathbf{L}_{\mathrm{S}}\,\mathbf{d} = \mathbf{L}_{\mathrm{S}} \sum_{i=1}^{NN} N_i \mathbf{d}_i = \sum_{i=1}^{NN} \mathbf{B}_{i\mathrm{S}}\,\mathbf{d}_i = \mathbf{B}_{\mathrm{S}}\,\mathbf{a} \tag{28}$$

where

$$\mathbf{B}_{i\mathrm{E}} = \mathbf{L}_{\mathrm{E}} N_i \quad , \quad \mathbf{B}_{\mathrm{E}} = \sum_{i=1}^{NN} \mathbf{B}_{i\mathrm{E}}$$

$$\mathbf{B}_{i\mathrm{B}} = \mathbf{L}_{\mathrm{B}} N_i \quad , \quad \mathbf{B}_{\mathrm{B}} = \sum_{i=1}^{NN} \mathbf{B}_{i\mathrm{B}}$$

$$\mathbf{B}_{i\mathrm{S}} = \mathbf{L}_{\mathrm{S}} N_i \quad , \quad \mathbf{B}_{\mathrm{S}} = \sum_{i=1}^{NN} \mathbf{B}_{i\mathrm{S}}$$

and

$$\mathbf{a} = \{\mathbf{d}_1^{\mathrm{t}}, \mathbf{d}_2^{\mathrm{t}}, \ldots, \mathbf{d}_{NN}^{\mathrm{t}}\}^{\mathrm{t}} \tag{29}$$

For the elastostatic analysis, the internal strain energy of an element due to extension, bending and shear can be determined by integrating the products of inplane stress resultants and extensional strains, moment stress resultants and bending curvatures and shear stress resultants and shear strains over the area of an element.

$$U^e = \frac{1}{2}\int_A \left[\boldsymbol{\epsilon}_o^t \mathbf{N} + \begin{Bmatrix} \mathbf{k} \\ \mathbf{k}^* \end{Bmatrix}^t \{\mathbf{M}, \mathbf{M}^*\} + \begin{Bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\phi}^* \end{Bmatrix}^t \{\mathbf{Q}, \mathbf{Q}^*\} \right] \mathrm{d}A \tag{30}$$

Implementing the stress resultants given by eqn (16) in the strain energy expression (30) gives

$$U^e = \frac{1}{2}\int_A \left[\boldsymbol{\epsilon}_o^t \mathbf{A} \boldsymbol{\epsilon}_o + \begin{Bmatrix} \mathbf{k} \\ \mathbf{k}^* \end{Bmatrix}^t \mathbf{B}^t \boldsymbol{\epsilon}_o + \boldsymbol{\epsilon}_o^t \mathbf{B} \begin{Bmatrix} \mathbf{k} \\ \mathbf{k}^* \end{Bmatrix} + \begin{Bmatrix} \mathbf{k} \\ \mathbf{k}^* \end{Bmatrix}^t \mathbf{D}_B \begin{Bmatrix} \mathbf{k} \\ \mathbf{k}^* \end{Bmatrix} \right.$$
$$\left. + \begin{Bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\phi}^* \end{Bmatrix}^t \mathbf{D}_S \begin{Bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\phi}^* \end{Bmatrix} \right] \mathrm{d}A \tag{31}$$

Substituting eqn (28) for extension, bending and shear strains into eqn (31) leads to the internal strain energy expression in terms of the nodal displacements as follows:

$$U^e = \frac{1}{2}\int_A \{ \mathbf{a}^t B_E^t A B_E \mathbf{a} + \mathbf{a}^t B_B^t B^t B_E \mathbf{a} +$$
$$\mathbf{a}^t B_E^t B B_B \mathbf{a} + \mathbf{a}^t B_B^t D_B B_B \mathbf{a} +$$
$$\mathbf{a}^t B_S^t D_S B_S \mathbf{a} \} \, \mathrm{d}A \tag{32}$$

Expression (32) can be written in concise form as

$$U^e = \frac{1}{2}[\mathbf{a}^t K^e \mathbf{a}] \tag{33}$$

where K^e is the stiffness matrix for an element e and includes extension, bending and the transverse shear effects and is given by

$$K^e = \int_A \{ \mathbf{B}_E^t \mathbf{A} \mathbf{B}_E + \mathbf{B}_B^t \mathbf{B}^t \mathbf{B}_E + \mathbf{B}_E^t \mathbf{B} \mathbf{B}_B + \mathbf{B}_B^t \mathbf{D}_B \mathbf{B}_B + \mathbf{B}_S^t \mathbf{D}_S \mathbf{B}_S \} \, \mathrm{d}A \tag{34}$$

The computation of the element stiffness matrix from eqn (34) is economized by explicit multiplication of the $\mathbf{B}_i$, $\mathbf{D}$ and $\mathbf{B}_j$ matrices instead of carrying out the full matrix multiplication of the triple product. In addition, due to the symmetry of the stiffness matrix, only the blocks $\mathbf{K}_{ij}$ lying on one side of the main diagonal are formed. The integral is evaluated using the Gauss quadrature

$$K_{ij}^{e} = \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}_i^{t} D \mathbf{B}_j |\mathbf{J}| \mathrm{d}\epsilon \, \mathrm{d}\eta$$

$$K_{ij}^{e} = \sum_{a=1}^{g} \sum_{b=1}^{g} W_a W_b |\mathbf{J}| \mathbf{B}_i^{t} \mathbf{D} \mathbf{B}_j \tag{35}$$

where W_a and W_b are weighting coefficients, g is the number of numerical quadrature points in each of the two directions (x and y) and $|J|$ is the determinant of the standard Jacobian matrix. The subscripts i and j vary from one to a number of nodes per element. The matrices B_i and D are defined as

$$B_i = \begin{bmatrix} B_{i\mathrm{E}} \\ \text{----} \\ B_{i\mathrm{B}} \\ \text{----} \\ B_{i\mathrm{S}} \end{bmatrix}, \quad D = \begin{bmatrix} A & B & 0 \\ B^{t} & D_{\mathrm{B}} & 0 \\ 0 & 0 & D_{\mathrm{S}} \end{bmatrix} \tag{36}$$

and B_j is obtained by replacing i by j.

For the problem of bending of laminated anisotropic plates, the applied external forces F consist of concentrated nodal loads F_c, each corresponding to a nodal degree of freedom, a distributed load q acting over the element in the z direction and a sinusoidally distributed load P_{mn} acting over the element in the z direction. The total external work done by these forces may be expressed as follows:

$$W^{e} = \frac{1}{2} \mathbf{a}^{t} \mathbf{F}_c + \frac{1}{2} \mathbf{a}^{t} \int_A \{N^{t} q + N^{t} P_{mn}\} \, \mathrm{d}A \tag{37}$$

The integral of eqn (37) is evaluated numerically using the Gauss quadrature. The result is

$$\mathbf{P} = \sum_{a=1}^{g} \sum_{b=1}^{g} W_a W_b |J| N_i \{0010000\}^{t} \left\{ q + P_{mn} \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} \right\} \tag{38}$$

where a and b are the plate dimensions, x and y are the Gauss point co-ordinates and m and n are the usual harmonic numbers.

4 NUMERICAL EXAMPLES AND DISCUSSION

Performance of the present finite element formulation is demonstrated by comparing results for various laminate geometries with those obtained using elasticity, closed-form and other finite element formulations. The selective integration scheme, namely 3×3 for membrane and flexure and 2×2 for shear contributions, has been employed. In all the examples considered, the individual laminae are taken to be of equal thickness and the following set of material properties is used for each lamina:

$$\frac{E_1}{E_2} = 25, \frac{G_{12}}{E_2} = 0{\cdot}5, \frac{G_{23}}{E_2} = 0{\cdot}2, \nu_{12} = 0{\cdot}25 \tag{39}$$

It is assumed that $G_{13} = G_{12}$ and $\nu_{21} = (E_2/E_1)\nu_{12}$. The simply supported square plate is discretized with four nine-noded quadrilateral elements in a quarter plate except for the convergence study. The values of stress resultants and stresses are at the nearest Gauss points. The deflection, stresses and stress resultants are presented here in the non-dimensional form using the following multipliers.

$$m_1 = \frac{100h^3E_2}{qa^4}, m_2 = \frac{1}{qa^2}, m_3 = \frac{1}{qa}$$

$$m_4 = \frac{h^2}{qa^2}, m_5 = \frac{h}{qa}, m_6 = \frac{100h^2}{qa^2} \tag{40}$$

The superscripts e and c used in the various tables that follow, represent values of stresses obtained from equilibrium and constitutive relations, respectively.

The following three unsymmetrically laminated simply supported square plate problems are considered.

4.1 Example 1

This example is considered to bring out the effect of mesh refinement on the deflection and stress predictions with the present element. The numerical results are compared with the exact solutions. The following three cases with different loading conditions and lamination schemes are considered for this purpose.

(i) A two layer unsymmetric cross-ply (0°/90°) square plate subjected to sinusoidal transverse load.

(ii) A two layer unsymmetric cross-ply (0°/90°) square plate subjected to uniform transverse load.
(iii) An eight layer unsymmetric angle-ply (45°/−45°/. . . 8 layers) square plate subjected to sinusoidal transverse load.

The numerical results for the above three cases showing convergence of deflection and stresses are given in Tables 1–3 respectively. The convergence of maximum transverse shear stress with the number of elements is shown graphically for the first case in Fig. 2. The following points are noted from this study:

—Maximum transverse deflection and inplane stresses predicted with 2×2 mesh are reasonably accurate and little further improvement is observed with mesh refinement.
—An accurate estimation of transverse shear stresses through equilibrium equations needs a more refined mesh, as is evident from Fig. 2.

4.2 Example 2

This example is selected to establish the accuracy of stress predictions through the thickness in the present development. The numerical results are compared with three-dimensional elasticity solutions. A two-layer unsymmetric cross-ply (0°/90°) square plate under sinusoidal transverse load is considered for this purpose. The numerical results for a square plate with side-to-thickness ratios of 4, 10, 50 and 100 are given respectively in Tables 4–7. The maximum stress resultant and deflection values are given in Table 8. The variation of maximum transverse deflection with different side-to-thickness ratios is shown in Fig. 3. The normal stress (σ_x) and the transverse shear stress variation (τ_{xz}) through the plate thickness is shown graphically in Figs 4 and 5 respectively.

The following important observations are made from the numerical results presented in Tables 4–8 and Figs 3–5.

—For a thick plate ($a/h = 4$), errors in the inplane normal stress (σ_x) computations are higher in comparison with the inplane shear (τ_{xy}) and the transverse shear stress (τ_{xz}).
—All stress components evaluated are reasonably accurate for moderately thick-to-thin plates ($a/h \geq 10$).
—The results for stress resultants presented in Table 8 should serve as a bench-mark for future comparative studies. They will also be useful to designers of composite laminates.
—The curve in Fig. 3 shows the inadequacy of the classical lamination theory to predict the displacements accurately.

TABLE 1

Convergence of Maximum Deflection and Stresses for a Simply Supported Unsymmetrically Laminated Cross-ply (0°/90°) Square Plate under Sinusoidal Transverse Load ($a/h = 10$)

Source	*Mesh size in a quarter plate*	z/h	$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}, \pm 0{\cdot}05\right)$	$\tau_{xy} \times m_4$ $(0, 0, \pm 0{\cdot}05)$	$\tau_{xz}^e \times m_5$ $\left(0, \frac{a}{2}, 0{\cdot}02\right)$	$\tau_{xz}^c \times m_5$ $\left(0, \frac{a}{2}, 0\right)$	$w_o \times m_1$ $\left(\frac{a}{2}, \frac{a}{2}, 0\right)$
Present	2 × 2	0·5	0·764 4	−0·054 2	0·296 3	0·331 0	1·212 8
		−0·5	−0·086 2	0·054 2			
	3 × 3	0·5	0·763 3	−0·053 7	0·316 1	0·336 9	1·212 5
		−0·5	−0·085 9	0·053 7			
	4 × 4	0·5	0·760 3	−0·053 7	0·323 4	0·338 7	1·212 5
		−0·5	−0·085 7	0·053 7			
	5 × 5	0·5	0·759 3	−0·053 7	0·326 7	0·339 5	1·212 4
		−0·5	−0·085 5	−0·053 7			
Pagano[11]	—	0·5	0·730 0	−0·053 8	0·331 0	—	—
		−0·5	−0·089 0	0·053 6			
Ren[10]	—	0·5	0·734	−0·054 0	0·332 0	—	—
		−0·5	−0·087 0	0·054 0			

TABLE 2

Convergence of Maximum Deflection and Stresses for a Simply Supported Unsymmetrically Laminated Cross-ply (0°/90°) Square Plate under Uniform Transverse Load ($a/h = 10$)

Source	*Mesh size in a quarter plate*	z/h	$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\tau_{xy} \times m_4$ $(0, 0)$	$\tau^e_{xz} \times m_5$ $\left(0, \frac{a}{2}, 0{\cdot}02\right)$	$\tau^c_{xz} \times m_5$ $\left(0, \frac{a}{2}, 0\right)$	$w_o \times m_l$ $\left(\frac{a}{2}, \frac{a}{2}, 0\right)$
Present	2 × 2	0·5	1·146 0	−0·096 99	0·503 0	0·628 3	1·911 3
		−0·5	−0·129 2	0·096 99			
	3 × 3	0·5	1·060 0	−0·097 33	0·573 3	0·670 3	1·917 9
		−0·5	−0·126 5	0·097 33			
	4 × 4	0·5	1·064 0	−0·098 11	0·607 3	0·690 2	1·916 5
		−0·5	−0·127 1	0·098 11	(z = 0·03)		
	5 × 5	0·5	0·994 8	−0·099 97	0·643 2	0·711 8	1·918 0
		−0·5	−0·125 1	0·099 97	(z = 0·03)		
Putcha and Reddy[9]	2 × 2Q	—	—	—	—	—	1·919 3
	Exact	—	—	—	—	—	1·917 3

TABLE 3

Convergence of Maximum Deflection and Stresses for a Simply Supported Unsymmetrically Laminated Angle-ply (45°/−45° . . . 8 layers) Square Plate under Sinusoidal Transverse Load (a/h = 10)

Source	*Mesh size in a quarter plate*	z/h	$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\tau_{xy} \times m_4$ $(0, 0)$	$\tau^e_{xz} \times m_5$ $\left(0, \frac{a}{2}, 0\right)$	$\tau^c_{xz} \times m_5$ $\left(0, \frac{a}{2}, 0\right)$	$w_o \times m_1$ $\left(\frac{a}{2}, \frac{a}{2}, 0\right)$
Present	2 × 2	0·5	0·163 3	−0·160 1	0·248 8	0·234 7	0·419 3
		−0·5	−0·180 6	0·173 2			
	3 × 3	0·5	0·159 1	−0·154 5	0·241 9	0·237 1	0·419 7
		−0·5	−0·180 7	0·172 4			
	4 × 4	0·5	0·160 2	−0·154 9	0·245 2	0·237 6	0·419 6
		−0·5	0·183 2	0·174 3			
	5 × 5	0·5	0·162 6	−0·141 5	0·246 8	0·237 9	0·419 7
		−0·5	−0·189 5	0·160 6			
Putcha and Reddy[9]	—	—	—	—	—	—	0·42[a]

[a] Interpreted from graph.

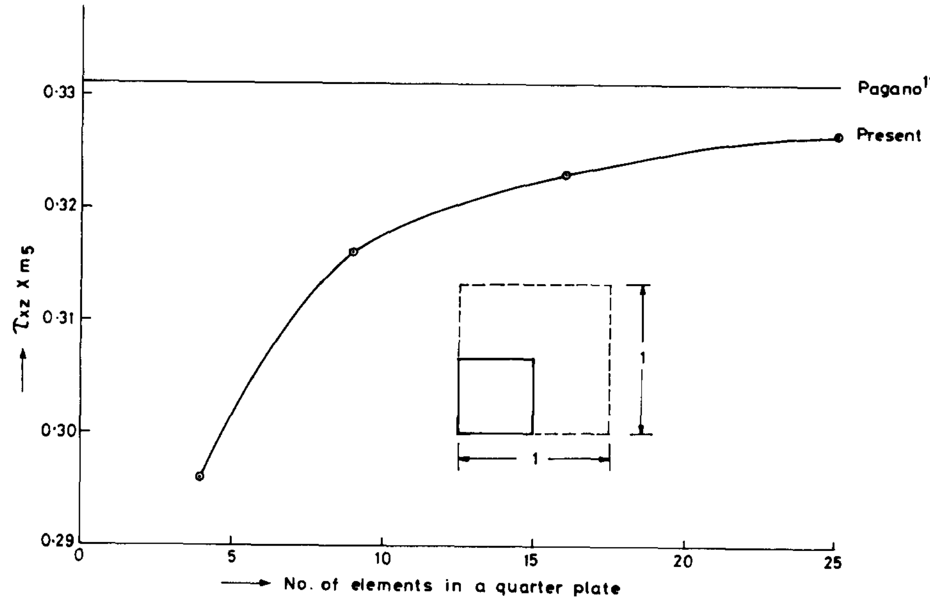

Fig. 2. Convergence of transverse shear stress (τ_{xz}) with the mesh refinement.

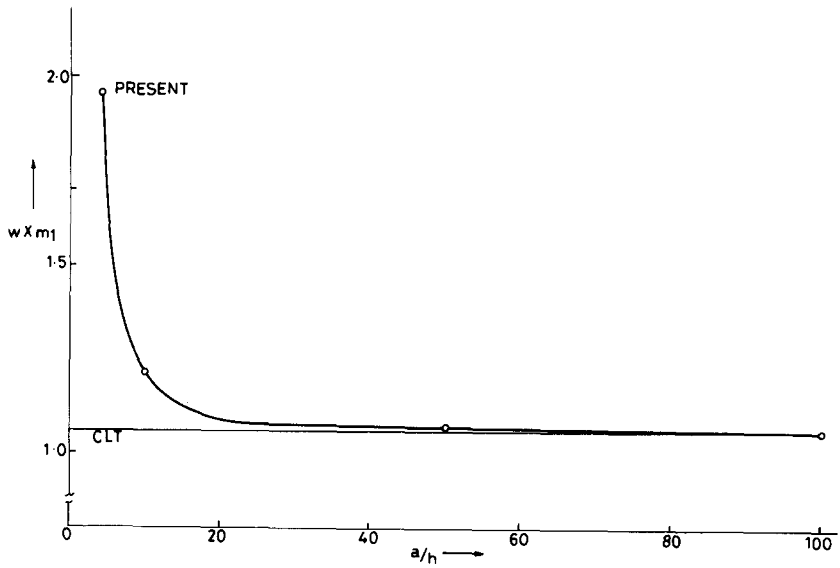

Fig. 3. Effect of plate side-to-thickness ratio (a/h) on the centre deflection ($w \times m_1$) of a simply supported (0°/90°) cross-ply square plate under sinusoidal load.

TABLE 4

Variation of Stresses Through the Thickness for a Simply Supported Unsymmetrically Laminated Cross-ply (0°/90°) Square Plate under Sinusoidal Transverse Load (a/h = 4)

	Present				*Pagano*[11]		
z/h	$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\tau_{xy} \times m_4$ $(0, 0)$	$\tau^e_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$	$\tau^c_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$	$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\tau_{xy} \times m_4$ $(0, 0)$	$\tau_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$
0·5	1·018 1	−0·060 0	0·0	0·0	0·780 7	−0·059 1	0·0
0·4	0·478 7	−0·039 5	0·214 5	0·119 4	0·398 8	−0·042 7	0·197 6
0·3	0·092 4	−0·024 7	0·299 5	0·212 3	0·143 7	−0·030 7	0·291 9
0·2	−0·178 8	−0·014 1	0·291 5	0·278 5	−0·059 8	−0·020 9	0·312 7
0·1	−0·373 2	−0·006 3	0·218 2	0·318 5	−0·279 2	−0·011 7	0·265 9
0·0	−0·529 3	0·0	0·095 4	0·331 8	−0·587 2	0·001 2	0·135 3
−0·0	−0·016 0	0·0	—	0·132 7	−0·024 7	0·001 2	—
−0·1	−0·023 8	0·006 3	0·089 4	0·127 4	−0·037 2	0·008 9	0·124 4
−0·2	−0·033 4	0·014 1	0·078 7	0·111 5	−0·050 8	0·018 8	0·106 3
−0·3	−0·046 8	0·024 7	0·062 4	0·084 9	−0·066 7	0·028 2	0·080 7
−0·4	−0·065 9	0·039 5	0·038 3	0·047 8	−0·085 9	0·041 0	0·046 1
−0·5	−0·092 6	0·060 0	0·003 6	0·0	−0·109 8	0·058 8	0·0

TABLE 5

Variation of Stresses Through the Thickness for a Simply Supported Unsymmetrically Laminated Cross-ply (0°/90°) Square Plate under Sinusoidal Transverse Load (a/h = 10)

z/h	*Present*				*Pagano*[11]		
	$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\tau_{xy} \times m_4$ $(0, 0)$	$\tau_{xz}^e \times m_5$ $\left(0, \frac{a}{2}\right)$	$\tau_{xz}^c \times m_5$ $\left(0, \frac{a}{2}\right)$	$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\tau_{xy} \times m_4$ $(0, 0)$	$\tau_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$
0·5	0·764 4	−0·054 2	0·00	0·00	0·730	−0·053 8	0·00
0·4	0·458 4	−0·041 9	0·182 0	0·119 2	0·442	−0·042 2	0·198
0·3	0·173 7	−0·030 5	0·279 0	0·211 8	0·177	−0·031 4	0·307
0·2	−0·094 9	−0·019 9	0·296 3	0·278 0	−0·077	−0·021 0	0·331
0·1	−0·352 9	−0·009 8	0·238 6	0·317 8	−0·335	−0·010 7	0·271
0·0	−0·605 5	0·0	0·108 6	0·331 0	−0·610	0·000 1	0·125
−0·0	−0·018 4	0·0	—	0·132 4	−0·020	0·000 1	—
−0·1	−0·030 9	0·009 8	0·101 1	0·127 1	−0·033	0·010 4	0·115
−0·2	−0·043 6	0·019 9	0·087 4	0·111 2	−0·046	0·020 7	0·098
−0·3	−0·056 9	0·030 5	0·067 1	0·084 7	−0·060	0·031 1	0·073
−0·4	−0·071 0	0·041 9	0·039 8	0·047 7	−0·074	0·042 0	0·041
−0·5	−0·086 2	0·054 2	0·004 6	0·00	−0·089	0·053 6	0·00

TABLE 6

Variation of Stresses Through the Thickness for a Simply Supported Unsymmetrically Laminated Cross-ply (0°/90°) Square Plate under Sinusoidal Transverse Load ($a/h = 50$)

	Present							
z/h	$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\sigma_y \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\tau_{xy} \times m_4$ $(0, 0)$	$\tau^e_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$	$\tau^c_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$	$\tau^e_{yz} \times m_5$ $\left(\frac{a}{2}, 0\right)$	$\tau^c_{yz} \times m_5$ $\left(\frac{a}{2}, 0\right)$	$\sigma^e_z \times m_6$ $\left(\frac{a}{2}, \frac{a}{2}\right)$
0·5	0·724 4	0·085 2	−0·053 0	0·0	0·0	0·0	0·0	1·00
0·4	0·454 8	0·071 8	−0·042 3	0·175 4	0·118 8	0·040 1	0·047 5	0·910 0
0·3	0·185 8	0·058 5	−0·031 7	0·275 0	0·211 2	0·068 4	0·084 5	0·822 2
0·2	−0·082 5	0·045 2	−0·021 1	0·297 6	0·277 2	0·090 0	0·110 9	0·736 0
0·1	−0·350 4	0·032 0	−0·010 6	0·243 2	0·316 8	0·104 7	0·126 7	0·650 8
0·0	−0·618 0	0·018 7	0·0	0·111 9	0·330 0	0·112 8	0·132 0	0·566 4
−0·0	−0·018 7	0·618 0	0·0	—	0·132 0	—	0·330 0	—
−0·1	−0·032 0	0·350 4	0·010 6	0·103 8	0·126 7	0·244 2	0·316 8	0·481 6
−0·2	−0·045 2	0·082 5	0·021 1	0·089 1	0·110 9	0·298 6	0·277 2	0·396 4
−0·3	−0·058 5	−0·185 8	0·031 7	0·067 5	0·084 5	0·276 0	0·211 2	0·310 0
−0·4	−0·071 8	−0·454 8	0·042 3	0·039 2	0·047 5	0·176 3	0·118 8	0·222 4
−0·5	−0·085 2	−0·724 4	0·053 0	0·004 0	0·0	−0·000 5	0·0	0·132 4

TABLE 7
Variation of Stresses Through the Thickness for a Simply Supported Unsymmetrically Laminated Cross-ply (0°/90°) Square Plate under Sinusoidal Transverse Load (a/h = 100)

z/h	Present							
	$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\sigma_y \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\tau_{xy} \times m_4$ $(0, 0)$	$\tau_{xz}^e \times m_5$ $\left(0, \frac{a}{2}\right)$	$\tau_{xz}^c \times m_5$ $\left(0, \frac{a}{2}\right)$	$\tau_{yz}^e \times m_5$ $\left(\frac{a}{2}, 0\right)$	$\tau_{yz}^c \times m_{45}$ $\left(\frac{a}{2}, 0\right)$	$\sigma_z^e \times m_6$ $\left(\frac{a}{2}, \frac{a}{2}\right)$
0·5	0·722 9	0·085 2	−0·053 0	0·0	0·0	0·0	0·0	1·00
0·4	0·454 3	0·071 9	−0·042 4	0·175 4	0·118 7	0·040 2	0·047 5	0·910 2
0·3	0·186 0	0·058 6	−0·031 8	0·275 1	0·211 0	0·068 6	0·084 4	0·820 9
0·2	−0·082 3	0·045 3	−0·021 2	0·297 8	0·276 9	0·090 2	0·110 8	0·732 1
0·1	−0·350 4	0·032 0	−0·010 6	0·243 4	0·316 5	0·105 1	0·126 6	0·643 5
0·0	−0·618 5	0·018 7	0·0	0·112 0	0·329 6	0·113 1	0·131 9	0·555 1
−0·0	−0·018 7	0·618 5	0·0	—	0·131 9	—	0·329 6	—
−0·1	−0·032 0	0·350 4	0·010 6	0·104 0	0·126 6	0·244 5	0·316 5	0·466 7
−0·2	−0·045 3	0·082 3	0·021 2	0·089 2	0·110 8	0·298 9	0·276 9	0·378 1
−0·3	−0·058 6	−0·186 0	0·031 8	0·067 5	0·084 4	0·276 2	0·211 0	0·289 3
−0·4	−0·071 9	−0·454 3	0·042 4	0·039 1	0·047 5	0·176 5	0·118 7	0·200 0
−0·5	−0·085 2	−0·722 9	0·053 0	0·003 9	0·0	−0·000 4	0·0	0·110 2

TABLE 8

Maximum Deflections and Stress Resultants for a Simply Supported Unsymmetrically Laminated Cross-ply (0°/90°) Square Plate under Sinusoidal Transverse Load

Source	a/h	$w_o \times m_1$ $\left(\frac{a}{2}, \frac{a}{2}, 0\right)$	$M_x \times m_2$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$M_{xy} \times m_2$ $(0, 0)$	$N_x \times m_3$ $\left(\frac{3a}{8}, \frac{a}{2}\right)$	$N_{xy} \times m_3$ $\left(\frac{a}{8}, 0\right)$	$Q_x \times m_3$ $\left(0, \frac{a}{2}\right)$	$N_x \times m_3$ $\left(\frac{a}{2}, \frac{3a}{8}\right)$
Present	4	1·956 3	0·043 2	−0·008 0	0·001 3	0·001 6	0·154 8	−0·001 4
	10	1·212 8	0·042 5	−0·008 7	0·005 6	0·004 6	0·154 5	−0·006 2
	50	1·070 1	0·042 3	−0·008 8	0·014 8	0·024 3	0·154 0	−0·014 6
	100	1·065 6	0·042 3	−0·008 8	0·024 7	0·048 9	0·153 8	−0·022 9

TABLE 9

Variation of Stresses Through the Thickness for a Simply Supported Unsymmetrically Laminated Angle-ply (45°/−45°/ . . . 8 layers) Square Plate under Sinusoidal Transverse Load

Source	z/h	$a/h = 4$				$a/h = 10$			
		$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\tau_{xy} \times m_4$ $(0, 0)$	$\tau^e_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$	$\tau^c_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$	$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\tau_{xy} \times m_4$ $(0, 0)$	$\tau^e_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$	$\tau^c_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$
	0·500	0·276 2	−0·274 3	0·00	0·00	0·163 3	−0·160 1	0·00	0·00
	0·375	0·084 4	−0·083 9	0·150 0	0·099 8	0·102 5	−0·100 7	0·102 0	0·100 2
		0·112 3	−0·110 5		0·102 4	0·118 7	−0·115 9		0·102 7
	0·250	0·020 9	−0·020 3	0·224 0	0·175 5	0·071 4	−0·070 0	0·191 9	0·176 0
		−0·008 7	0·007 7		0·171 0	0·054 9	−0·054 6		0·171 8
	0·125	−0·036 1	0·033 4	0·201 6	0·213 7	0·016 0	−0·017 3	0·217 5	0·214 7
		−0·006 2	0·005 1		0·219 4	0·033 2	−0·033 3		0·220 0
Present	0·0	−0·001 0	−0·002 0	0·228 2	0·234 0	−0·000 4	−0·001 3	0·248 8	0·234 7
		−0·030 6	0·026 1		0·228 0	−0·018 5	0·015 6		0·229 0
	−0·125	−0·025 2	0·018 8	0·214 7	0·213 7	−0·052 9	0·048 5	0·226 8	0·214 7
		0·004 2	−0·009 0		0·219 4	−0·034 0	0·030 7		0·220 0
	−0·250	−0·022 9	0·016 4	0·252 0	0·175 5	−0·072 2	0·067 4	0·206 5	0·176 0
		−0·052 5	0·044 5		0·171 0	−0·091 8	0·085 8		0·171 8
	−0·375	−0·145 7	0·136 1	0·195 9	0·098 8	−0·139 4	0·131 9	0·131 6	0·100 2
		−0·114 4	0·106 6		0·102 4	−0·119 5	0·113 3		0·102 7
	−0·500	−0·302 5	0·294 1	0·069 4	0·00	−0·180 6	0·173 2	0·038 7	0·00

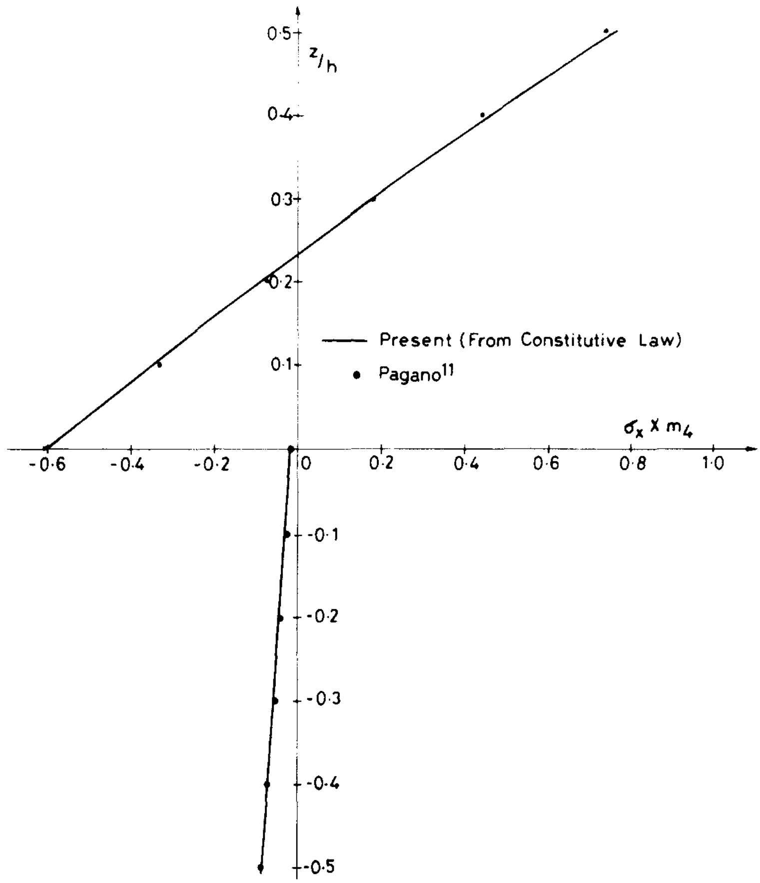

Fig. 4. Variation of the normal stress ($\sigma_x \times m_4$) through the laminate thickness for a simply supported cross-ply (0°/90°) square plate under sinusoidal transverse load ($a/h = 10$).

—It is clear from Fig. 5 that the transverse shear stress variation through the thickness of plate is correctly given by equilibrium equations. As anticipated, the maximum value occurs at the level of neutral surface and the stress is continuous at the interface. But this is not true when transverse shear stress is computed using stress–strain constitutive relations.

4.3 Example 3

This example is an extension of Example 2 for an 8 layer unsymmetric angle-ply (45°/−45°/ . . . 8 layers) square plate under sinusoidal transverse load. The numerical results for stresses are presented in Tables 9 and 10 and

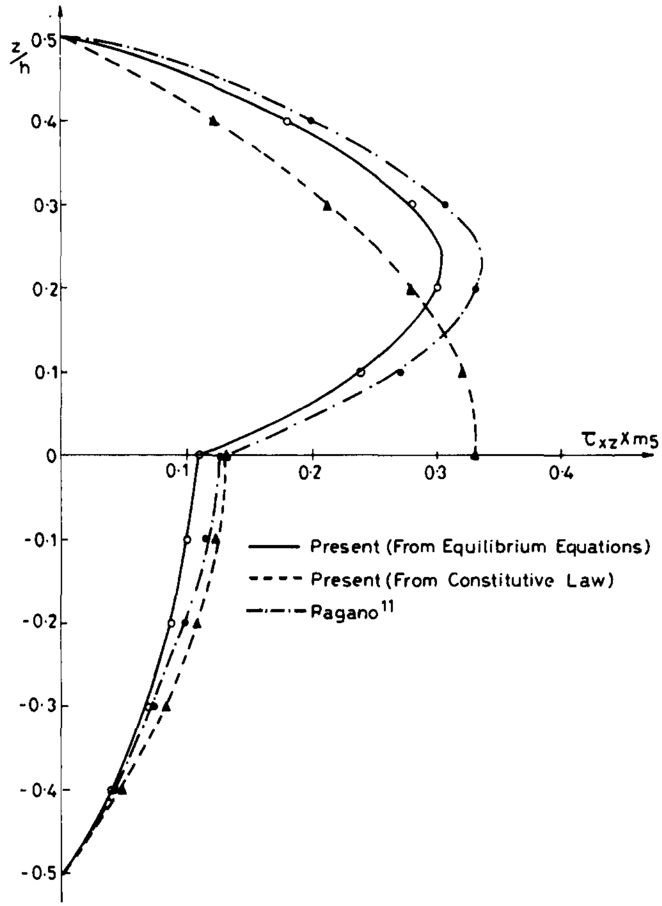

Fig. 5. Variation of the transverse shear stress ($\tau_{xz} \times m_5$) through the laminate thickness for a simply supported cross-ply (0°/90°) square plate under sinusoidal transverse load (a/h = 10).

maximum deflection and stress resultant values are given in Table 11. The variation of maximum transverse deflection with different side-to-thickness ratios is shown in Fig. 6. The normal stress (σ_x) and the transverse shear stress variation (τ_{xz}) through the plate thickness is shown graphically in Figs 7 and 8 respectively. These results should serve as a bench-mark for future investigations.

5 CONCLUSIONS

A simple C^0 isoparametric formulation of a higher-order theory which satisfies the zero transverse shear stress conditions on the top and bottom

TABLE 10

Variation of Stresses Through the Thickness for a Simply Supported Unsymmetrically Laminated Angle-ply (45°/−45°/ . . . 8 layers) Square Plate under Sinusoidal Transverse Load

Source	z/h	$a/h = 50$				$a/h = 100$			
		$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\tau_{xy} \times m_4$ $(0, 0)$	$\tau^e_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$	$\tau^c_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$	$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\tau_{xy} \times m_4$ $(0, 0)$	$\tau^e_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$	$\tau^c_{xz} \times m_5$ $\left(0, \frac{a}{2}\right)$
	0·500	0·146 2	−0·143 0	0·00	0·00	0·146 2	−0·143 0	0·00	0·00
	0·375	0·105 3	−0·103 4	0·096 7	0·100 1	0·105 3	−0·103 5	0·096 2	0·099 9
		0·119 7	−0·116 9		0·102 6	0·119 7	−0·116 9		0·102 5
	0·250	0·079 3	−0·078 0	0·188 0	0·175 8	0·079 5	−0·078 2	0·188 6	0·175 8
		0·060 5	−0·064 1		0·171 5	0·064 5	−0·064 2		0·171 3
	0·125	0·023 9	−0·025 0	0·220 4	0·214 4	0·023 9	−0·025 0	0·219 4	0·214 2
		0·039 4	−0·039 4		0·219 8	0·039 5	−0·039 6		0·219 7
Present	0·0	−0·000 4	−0·001 1	0·252 0	0·234 4	−0·000 5	−0·001 0	0·251 5	0·234 4
		−0·016 7	0·014 1		0·228 6	−0·016 6	0·014 1		0·228 4
	−0·125	−0·057 2	0·053 2	0·228 6	0·214 4	−0·057 2	0·053 2	0·228 6	0·214 2
		−0·040 2	0·037 3		0·219 8	−0·040 5	0·037 6		0·219 7
	−0·250	−0·080 1	0·075 8	0·202 0	0·175 8	−0·080 5	0·076 2	0·200 2	0·175 8
		−0·097 8	0·092 3		0·171 5	−0·097 8	0·092 3		0·171 3
	−0·375	−0·138 6	0·131 6	0·124 2	0·100 1	−0·138 5	0·131 6	0·123 1	0·099 9
		−0·120 5	0·114 8		0·102 6	−0·120 6	0·114 9		0·102 5
	−0·500	−0·161 4	0·154 3	0·034 4	0·00	−0·160 8	0·153 7	0·034 9	0·00

TABLE 11
Maximum Deflections and Stress Resultants for a Simply Supported Unsymmetrically Laminated Angle-ply (45°/−45° . . . 8 layers) Square Plate under Sinusoidal Transverse Load

Source	*a/h*	$w_o \times m_1$ $\left(\frac{a}{2}, \frac{a}{2}, 0\right)$	$M_x \times m_2$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$M_{xy} \times m_2$ $(0, 0)$	$N_x \times m_3$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$N_{xy} \times m_3$ $(0, 0)$	$Q_x \times m_3$ $\left(0, \frac{a}{2}\right)$
Present	4	1·279 1	0·026 6	−0·025 6	−0·062 0	0·047 2	0·154 0
Putcha and Reddy[9]		1·28	—	—	—	—	—
Present	10	0·419 3	0·026 4	−0·025 4	−0·093 3	0·070 8	0·154 6
Putcha and Reddy[9]		0·42	—	—	—	—	—
Present	50	0·252 2	0·026 4	−0·025 5	−0·420 9	0·320 0	0·154 4
Putcha and Reddy[9]		0·26	—	—	—	—	—
Present	100	0·246 9	0·026 4	−0·025 5	−0·839 4	0·638 5	0·154 3

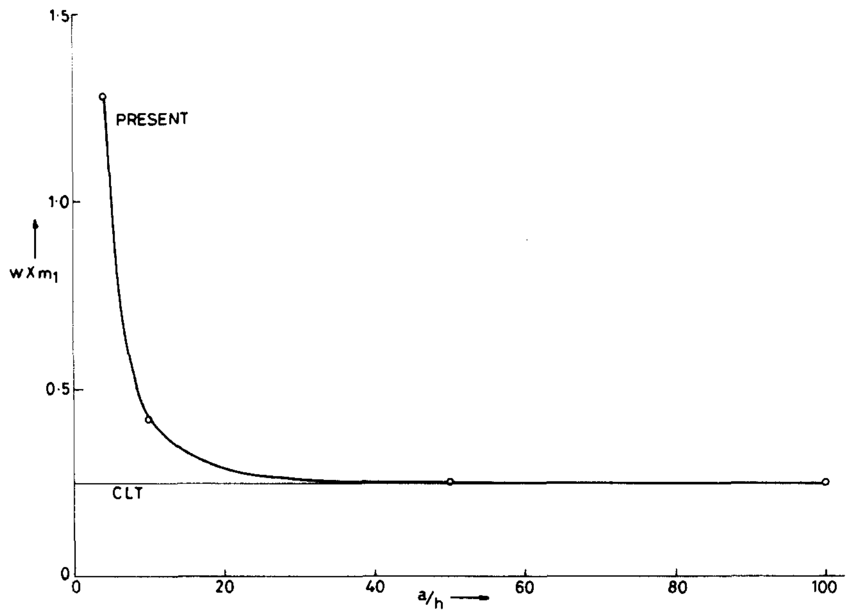

Fig. 6. Effect of plate side-to-thickness ratio (a/h) on the centre deflection ($w \times m_1$) of a simply supported (45°/−45°/ . . . 8 layers) angle-ply square plate under sinusoidal load.

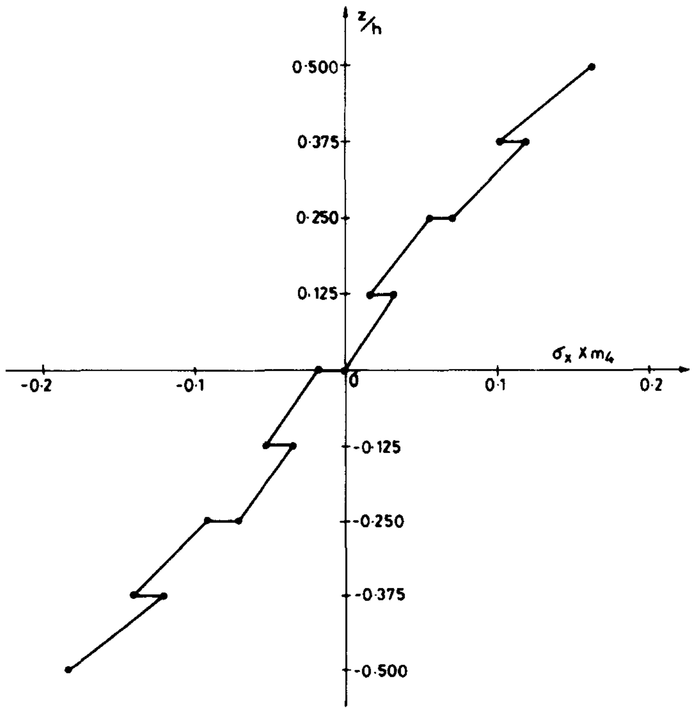

Fig. 7. Variation of the normal stress ($\sigma_x \times m_4$) through the laminate thickness for a simply supported angle-ply (45°/−45°/ . . . 8 layers) square plate under sinusoidal transverse load ($a/h = 10$).

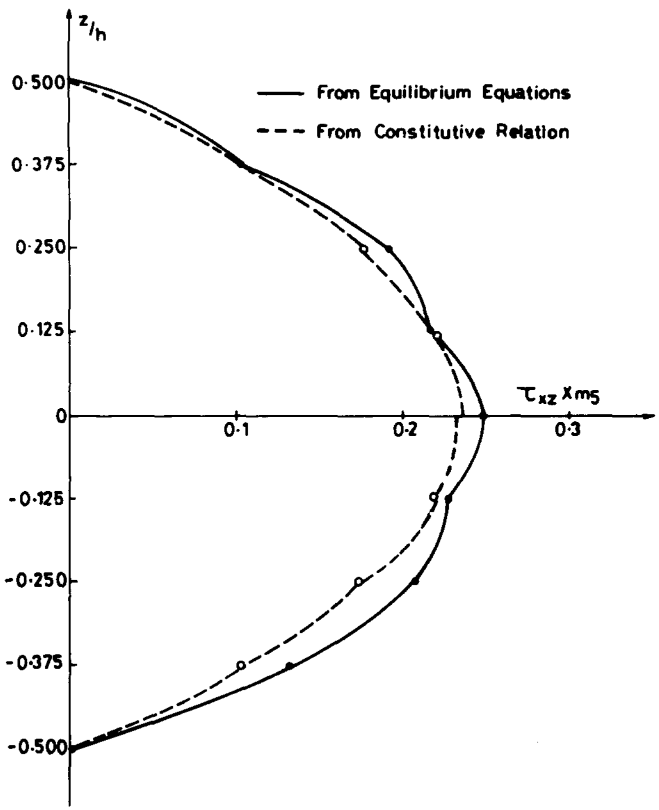

Fig. 8. Variation of the transverse shear stress ($\tau_{xz} \times m_5$) through the laminate thickness for a simply supported angle-ply (45°/−45°/ . . . 8 layers) square plate under sinusoidal transverse load (a/h = 10).

surfaces of a plate is presented. Such an approach is commercially attractive due to ease in software development and implementation in an existing general purpose programme which is usually based on isoparametric formulation. The nine-noded Lagrangian element of the isoparametric quadrilateral family developed here is a fairly simple element. The results obtained here have proved the simplicity and reliability of this element in a general stress analysis. The displacement model used here is the simplest in the family of higher-order models and the theory does not require the usual shear correction coefficient(s) generally associated with first-order shear deformable theories of Reissner and Mindlin.

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