

ON SCATTERED SPACES¹

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ABSTRACT. We show that each 0-dimensional Hausdorff space which is scattered can be mapped continuously in a one-to-one way onto a scattered 0-dimensional Hausdorff space of the same weight as its cardinality. This gives an easier and a new proof of the fact that a countable regular space admits a coarser compact Hausdorff topology if and only if it is scattered. We also show that a 0-dimensional, Lindelöf, scattered first-countable Hausdorff space admits a scattered compactification. In particular we give a more direct proof than that of Knaster, Urbanik and Belnov of the fact that a countable scattered metric space is a subspace of $[1, \Omega)$, and deduce a result of W. H. Young as a corollary.

1. A problem of Banach is to characterise all topological spaces which admit a weaker compact Hausdorff topology. Katetov [3] gave a partial solution to this problem by showing that among countable spaces they are precisely the scattered spaces. His approach does not throw enough light on the cardinal invariants attached to such a space. We prove a theorem below, in the second section, a relation between scatteredness and weights of topological spaces. This gives a new proof of Katetov's theorem mentioned above. This relates to a study of scatteredness and the problem of existence of scattered compactifications of scattered completely regular spaces. Many mathematicians are interested in this problem (to mention one see Semadeni [9] and Ryll-Nardzewski and Telgarsky [8]). In the first section of this paper we show that a scattered, 0-dimensional, first-countable, Hausdorff Lindelöf space admits a Hausdorff scattered compactification. This can be achieved by an application of the results of Knaster-Urbanik [4] and Mazurkiewicz-Sierpinski [5]. However our proof is elementary.

DEFINITION 1. A topological space X is called *scattered* if given a nonempty set $A \subset X$ there is an element $x_0 \in A$ which is isolated relative to A .

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DEFINITION 2. Let X be a topological space. Let $A \subset X$. We define $A^0 = A$; $A^1 = \{x \mid x \in A \text{ and } x \in \text{cl}(A - \{x\})\}$. If α is an ordinal and A^α has been defined then we put $A^{\alpha+1} = (A^\alpha)^1$. If α is a limit ordinal and A^β has been defined for all ordinals $\beta < \alpha$ then we put $A^\alpha = \bigcap_{\beta < \alpha} A^\beta$. If α is an ordinal we call A^α the α th derivative of A . A topological space X is said to have a *derived length* if $X^\alpha = \emptyset$ for some ordinal α . In this case the derived length of X is defined as the least ordinal α so that $X^\alpha = \emptyset$. It is noted as $\alpha(X)$.

The following theorem can be deduced from the results of [4], [5], [9] and a theorem of Young (see Corollaries 3, 4). However our approach is easier and hence we feel it worthwhile to present our proof of this result.

THEOREM 1. *A first-countable 0-dimensional Lindelöf Hausdorff space X is scattered if and only if it admits a scattered compactification.*

PROOF. If X admits a scattered compactification then X is clearly scattered. So we assume that X is scattered and we exhibit a scattered compactification for X . For this we note that since X is Lindelöf $X^\alpha = \emptyset$ for a countable ordinal α . Now it is easy to see that such a space is hereditarily Lindelöf. Now two cases arise.

Case (i). α is a limiting ordinal. Then α is a limit of an increasing sequence $\alpha_1 < \alpha_2 < \dots < \alpha_n < \dots$ of countable ordinals. Then each $X - X^{\alpha_i}$ can be written as a countable disjoint union $\bigcup_{j=1}^\infty V_{ij}$ of clopen sets V_{ij} . Putting

$$W_i = \bigcup_{1 \leq j \leq i, 1 \leq k \leq i} V_{jk}$$

we get

$$\bigcup_{i=1}^\infty W_i = X.$$

It is also clear that $\alpha(W_i) \leq \alpha_i \forall i = 1, 2, 3, \dots$, and $X = \bigcup_{i=1}^\infty (W_i - W_{i-1})$ where $W_0 = \emptyset$ and each $W_i - W_{i-1}$ is clopen in X . By induction we can assume that each $W_i - W_{i-1}$ has a scattered compactification K_i . Then the one-point compactification of the disjoint union of the K_i 's is easily verified to be a scattered compactification of X .

Case (ii). α is a discrete ordinal > 1 and hence $\alpha = \beta + 1$ for some ordinal β . Then X^β is countable and discrete. So there exists a countable disjoint collection $V_1, V_2, \dots, V_n, \dots$ of clopen sets in X so that $V_i \cap X^\beta$ is a singleton for each i and $X^\beta \subset \bigcup_{i=1}^\infty V_i$. Now putting $V = \bigcup_{i=1}^\infty V_i$ we can separate the two disjoint closed sets $X - V$ and X^β by open sets U and W , where $W \supset X - V$. Write W as a disjoint union $\bigcup_{i=1}^\infty W_i$ of clopen sets W_i

and put

$$F_n = W_{n/2} - \bigcup_{i=1}^{n/2} V_i \quad \text{if } n \text{ is even}$$

and

$$F_n = V_{(n+1)/2} - \bigcup_{i=1}^{(n-1)/2} W_i \quad \text{if } n \text{ is odd}$$

taking $\bigcup_{i=1}^{n-1} W_i = \emptyset$ when $n=1$. Then $X = \bigcup_{n=1}^{\infty} F_n$ and the F_n 's form a pairwise disjoint collection of clopen sets. By the induction assumption each F_{2n} admits a scattered compactification G_{2n} . Now consider some F_i where i is odd. Let $F_i \cap X^\beta = \{a_i\}$. Then the first axiom is satisfied at a_i . So F_i can be written as a disjoint union $\{a_i\} \cup \bigcup_{k=1}^{\infty} S_k$ where each $S_k \subset F_i$ is clopen in F_i . By the induction assumption each S_k admits a scattered compactification Y_k . Then taking Y to be the one-point compactification of the free union of the Y_k 's with a_i as the point at ∞ we get a scattered compactification of G_i of F_i . Then take Y to be the one-point compactification of the free union of the G_n 's where $n=1, 2, \dots$. Then Y is the required compactification.

It is clear that if $\alpha=1$ then X admits a scattered compactification. So the theorem is true.

COROLLARY 2. *Every countable metric space X can be written as a disjoint union $A \cup B$ of two of its subspaces where B is homeomorphic to a subspace of $[1, \Omega]$ where Ω is the first uncountable ordinal and A is a closed subspace which is homeomorphic to the space of all rationals or empty.*

PROOF. Take A to be $\bigcap_{|\alpha| \leq |X|} X^\alpha$ and $B = X - A$. Then $A = \emptyset$ or is homeomorphic to the rationals (see [2] and [10]). B is scattered and hence admits a scattered compactification by Theorem 1 which is, in fact, a countable compactification by following the proof of Theorem 1 carefully. So by results of [5] B is a subspace of $[1, \Omega]$.

REMARK. The study of how A and B fit together to make up the space X is difficult. For some study on this see [7].

COROLLARY 3 (KNASTER AND URBANIK AND BELNOV). *For a countable space X the following are equivalent:*

- (a) X is scattered and metrizable.
- (b) X is a subspace of $[1, \Omega]$.
- (c) X admits a complete metric.
- (d) X admits a countable compactification.

We omit the proof since it follows easily from Corollary 1.

COROLLARY 4 (YOUNG). *A scattered subset of the real line is a G_δ .*

2. Topologies coarser than scattered topologies. It is an unsolved problem of Banach to characterise all topologies which admit a coarser compact Hausdorff topology. Katetov [3] gave a partial solution to this problem by proving that among countable regular spaces, these are precisely the scattered topologies. Here we prove two theorems concerning the weakening of scattered topologies which, together with Theorem 1, give a different proof of the result of [3].

Let \mathcal{L} denote the class of all zero-dimensional scattered Hausdorff spaces. The following theorem asserts that each member of \mathcal{L} admits a weaker topology belonging to \mathcal{L} and satisfying some more conditions.

THEOREM 5. *For each $X \in \mathcal{L}$, there exists $Y \in \mathcal{L}$ and a continuous bijection $e: X \rightarrow Y$ such that*

- (i) $e(X^\alpha) = Y^\alpha$ for each ordinal α .
- (ii) *The weight of Y does not exceed its cardinality.*

PROOF. Since X is scattered, for each x in X there is a unique ordinal α such that $x \in X^\alpha \setminus X^{\alpha+1}$, that is, such that x is isolated in X^α . Since X is zero-dimensional, there exists then a clopen neighborhood V_x of x such that $V_x \cap X^\alpha = \{x\}$. Define a function f_x from X onto the two-element discrete space $\{0, 1\}$ by $f_x(V_x) = \{1\}$ and $f_x(X \setminus V_x) = \{0\}$. Thus to each x in X we have defined a two-valued continuous function f_x on X . Let $F = \{f_x | x \in X\}$.

We now show that this family F of functions on X separates points of X . Let x and y be any two distinct elements of X . Now two cases are possible. There may exist an α such that both x and y are isolated in X^α or there may not. If the first case holds then the function f_x separates x and y . For, $f_x(x) = 1$ and $f_x(y) = 0$. In the latter case, there exists a least ordinal α such that only one of the two elements belongs to $X^{\alpha+1}$, say x . Then $y \in X^\alpha \setminus X^{\alpha+1}$ and $x \in X^{\alpha+1}$. Now $V_y \cap X^\alpha = \{y\}$. Therefore $x \notin V_y$ and so $f_y(x) = 1$, but $f_y(y) = 0$. Thus f_y separates x and y . Thus we have proved that F separates elements of X .

Define a map $e: X \rightarrow \{0, 1\}^F$ by $(e(x))_{f_y} = f_y(x) \forall x \in X$ and $\forall f_y \in F$. Clearly the map e is continuous. It is one-to-one since F separates points of X . Let Y be the range of e . Then Y is obviously zero-dimensional and Hausdorff. Also,

$$\begin{aligned} \text{the weight of } Y \text{ is } &\leq \text{the weight of } \{0, 1\}^F \\ &= \text{cardinality of } F \\ &= \text{cardinality of } X \\ &= \text{cardinality of } Y. \end{aligned}$$

We complete the proof by showing that this map e has the property that $e(X^\alpha) = Y^\alpha$ for each ordinal α .

We use transfinite induction on α . To prove the result for $\alpha=1$, we show that e takes isolated points to isolated points.

If $x \in X$ is isolated, then V_x must be $\{x\}$ and so f_x vanishes at all points other than x . It follows that $e(x)$ has nonzero entry only in the coordinate corresponding to f_x and that $e(y)$ has zero entry there for all other elements y . Consequently $e(x)$ is an isolated point of Y . Conversely, if x is a non-isolated point of X , then $e(x)$ is also nonisolated in Y , since e is one-to-one and continuous. Thus $e(X')=Y'$.

Suppose as induction hypothesis that we have proved that $e(X^\beta)=Y^\beta$ for each $\beta < \alpha$. If α is limiting, then

$$\begin{aligned} e(X^\alpha) &= e\left(\bigcap_{\beta < \alpha} X^\beta\right) = \bigcap_{\beta < \alpha} e(X^\beta) \quad (\text{since } e \text{ is 1-1}) \\ &= \bigcap_{\beta < \alpha} Y^\beta \quad (\text{by induction hypothesis}) \\ &= Y^\alpha. \end{aligned}$$

Now let α be nonlimiting. Then since e is 1-1 and continuous, we have

$$e(X^\alpha) = e(X^{\alpha-1+1}) \subseteq (e(X^{\alpha-1}))^1 = Y^{\alpha-1+1} = Y^\alpha.$$

To prove the reverse inequality, we show that $e(X \setminus X^\alpha) \subset Y \setminus Y^\alpha$. Now

$$e(X \setminus X^\alpha) = e(X \setminus X^{\alpha-1}) \cup e(X^{\alpha-1} \setminus X^\alpha).$$

But

$$\begin{aligned} e(X \setminus X^{\alpha-1}) &\subset Y \setminus Y^{\alpha-1} \quad (\text{by the induction hypothesis and since } e \text{ is 1-1}) \\ &\subset Y \setminus Y^\alpha. \end{aligned}$$

If $x \in X^{\alpha-1} \setminus X^\alpha$, then x is isolated in $X^{\alpha-1}$ and so $f_x(x)=1$ and $f_x(y)=0$ for each $y \neq x$ in $X^{\alpha-1}$. Therefore $e(x)$ is isolated in $e(X^{\alpha-1})=Y^{\alpha-1}$. Thus $e(x) \in Y^{\alpha-1} \setminus Y^\alpha$.

Thus we have shown that $e(X^\alpha)=Y^\alpha$ for every ordinal α . This in particular proves that Y is scattered and the proof of the theorem is complete.

COROLLARY 6. *Let (X, τ) be minimal (in the lattice of all topologies on X) with respect to the property of being in \mathcal{L} . Then the weight of (X, τ) is \leq the cardinality of X .*

COROLLARY 7. *For each ordinal α , let \mathcal{L}_α denote the class of all members (X, τ) of \mathcal{L} such that $w(X)=\alpha$. If (X, τ) is minimal with respect to the property of being in \mathcal{L}_α , then the weight of X does not exceed the cardinality of X .*

COROLLARY 8. *Let X be a countable space in \mathcal{L} . Then there is a coarser topology on X which is metrisable and scattered.*

The last corollary follows from Theorem 5 and the fact that every regular second-countable space is metrisable.

THEOREM 9. *Let X be a subspace of a well-ordered space. Then X admits a weaker compact Hausdorff topology.*

PROOF. Let $\alpha = \sup X$. Define a map $g: X \rightarrow [1, \alpha]$ as follows:

$$\begin{aligned} g(x) &= y && \text{if } \exists y \in \bar{X} \setminus X \text{ such that } y < x \text{ and } [y, x] \cap X = \{x\}, \\ &= \alpha && \text{if } \alpha \notin X \text{ and if } x \text{ is the least element of } X, \\ &= x && \text{otherwise.} \end{aligned}$$

If y_1 and y_2 are two elements of $\bar{X} \setminus X$ such that $y_1 < y_2 < x$, it follows that $[y_1, x] \cap X$ must contain y_2 in its closure. So, the above function g is well defined.

We claim that g is one-to-one. Let $g(x_1) = g(x_2)$. If this common value is α , then either $\alpha \in X$ in which case $x_1 = x_2 = \alpha$, or $\alpha \notin X$ in which case $x_1 = x_2 =$ the least element of X . If this common value is $y \neq \alpha$, then either $y \in X$ in which case $x_1 = x_2 = y$, or $y \notin X$ in which case

$$x_1 = x_2 = \text{glb}\{x \in X \mid x \geq y\}.$$

Next we assert that if $x_1 \leq x_2$ in X , then $x_1 \leq g(x_2) \leq x_2$, provided none of these points is the least element of X . This is immediate from the definition of g . This observation leads to two important consequences: First, except for the least element of X , the function g is strictly order preserving. Secondly, if (x_β) is an increasing transfinite well-ordered sequence in X converging to x in X , then $x_{\beta-1} \leq g(x_\beta) \leq x_\beta$ for each non-limiting ordinal β and so $(g(x_\beta)) \rightarrow x$. This shows that the map g is continuous.

Finally, we show that the range of g is closed in $[1, \alpha]$. Let $(g(x_\beta)) \rightarrow t$ be a well-ordered strictly increasing transfinite sequence in $g(X)$ converging to t in $[1, \alpha]$. We want to show that $t \in g(X)$. If $g(x_{\beta_1}) \leq g(x_{\beta_2})$, we easily see that $g(x_{\beta_1}) \leq x_{\beta_1} \leq g(x_{\beta_2})$. Therefore the transfinite sequence (x_β) also converges to t in $[1, \alpha]$. Therefore $t \in \bar{X}$.

If $t = \alpha$ and if $\alpha \in X$, then $g(\alpha) = t$ so that $t \in g(X)$; if $t = \alpha$ and if $\alpha \notin X$, then $g(x_0) = t$ where x_0 is the least element of X , so that $t \in g(X)$.

Let $t \neq \alpha$. If $t \in X$ and if $x_1 < t$ is another element of X , then there exists $x_2 \in X$ such that $x_1 < x_2 < t$ (since t belongs to \bar{X}). This implies that $g(t) = t$ so that $t \in g(X)$. If $t \notin X$, then let $x = \text{glb}\{y \in X \mid y \geq t\}$. Then $t = g(x)$, so that $t \in g(X)$.

Thus $g(X)$ is a closed subset of $[1, \alpha]$.

Thus g is a 1-1 continuous map from X onto a compact Hausdorff space.

COROLLARY 10 (KATETOV [3]). *A countable regular space admits a weaker compact Hausdorff topology if and only if it is scattered.*

PROOF. One part follows easily from the facts that every countable compact Hausdorff space is scattered and that any topology finer than a scattered topology must be scattered. To prove the converse, let X be a countable regular scattered space. Then $X \in \mathcal{L}$. Therefore by Corollary 8, there is a weaker topology which is metrisable and scattered. Call this new space X_1 . Then by Corollary 3, X_1 is homeomorphic to a subspace of $[1, \Omega]$. Again by Theorem 9, this subspace of $[1, \Omega]$ admits a weaker compact Hausdorff topology. It follows that X admits a weaker compact Hausdorff topology.

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