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Reflexive cum Coreflexive Subcategories in Topology*

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The notions of reflexive and coreflexive subcategories in topology have received much attention in the recent past. (See e.g. Kennison [5], Herrlich [2], Herrlich and Strecker [4], Kannan [6–8].) In this paper we are concerned with the following question and its analogues: Let $\mathscr{T}$ be the category of all topological spaces with continuous maps as morphisms. Can a proper subcategory of $\mathscr{T}$ be both reflexive and coreflexive in $\mathscr{T}$? The answer turns out to be in the negative. We show further that almost all nice subcategories of $\mathscr{T}$ have the property that they do not have any proper reflexive cum coreflexive subcategory. Anyhow, examples of subcategories of $\mathscr{T}$ are given which have proper reflexive cum coreflexive subcategories on their own right. The validity of the analogous theorem is discussed in some supercategories of $\mathscr{T}$ also.

An interesting corollary to the proof of Theorem 1 states that a productive intersective divisible topological property (such as e.g., compactness) must fail to be additive.

The proof of the main result is based heavily on topological concepts; it is not known how far the result can be extended to arbitrary categories.

We start with some preliminary definitions. Let $\mathscr{A}$ be a category and $\mathscr{B}$ be a full subcategory of $\mathscr{A}$. Let X be an object in $\mathscr{A}$. Then $\mathrm{Mor}(X, \mathscr{B})$ denotes the family of all morphisms in $\mathscr{A}$ whose domain is X and whose codomain belongs to $\mathscr{B}$. Similarly we have the symbol $\mathrm{Mor}(\mathscr{B}, X)$ with an obvious dual meaning.

If for each object X of $\mathscr{A}$, there exists an $e_X \in \mathrm{Mor}(X, \mathscr{B})$ through which every element of $\mathrm{Mor}(X, \mathscr{B})$ factors uniquely, then $\mathscr{B}$ is called a reflexive subcategory of $\mathscr{A}$. In this case e_X is called a reflection morphism of X (see Diagram 1). In case this e_X is both a monomorphism and an epimorphism for each object X of $\mathscr{A}$, we say that $\mathscr{B}$ is a simple reflexive subcategory of $\mathscr{A}$.

Dually, $\mathscr{B}$ is said to be a coreflexive subcategory of $\mathscr{A}$ if for each object X of $\mathscr{A}$, there exists $f_X \in \mathrm{Mor}(\mathscr{B}, X)$ through which every element of $\mathrm{Mor}(\mathscr{B}, X)$ factors uniquely. f_X is called a coreflection morphism of X (see Diagram 2).

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Diagram 1

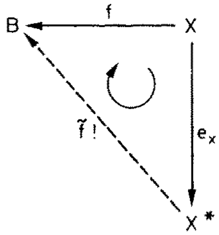
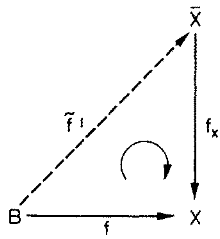


Diagram 2



We are having the following assumptions and conventions:

(1) All families (of any kind of spaces) considered are nonempty.

(2) All subcategories considered are full, unless otherwise stated.

(3) All subcategories considered are replete, i.e., they contain all isomorphic copies of their members.

(4) In all the categories that we are dealing with the morphisms are taken to be "all continuous maps between their objects" unless otherwise stated.

Our proofs of the theorems here are based on several recent results, which are listed below for the sake of convenience:

Definition. A family $\mathscr{A}$ of topological spaces is said to be closed under the formation of intersections, if whenever $X \in \mathscr{A}$ and $\{Y_\alpha / \alpha \in J\}$ is a family of subspaces of X such that each $Y_\alpha \in \mathscr{A}$ and $Y = \bigcap_{\alpha \in J} Y_\alpha$, then $Y \in \mathscr{A}$.

Theorem A (Freyd). *Any reflexive subcategory of a complete category is a complete subcategory. In particular, any reflexive subcategory of $\mathscr{T}$ is closed under the formation of products and intersections (see also* [3]*).*

Theorem B (Freyd). *Any coreflexive subcategory of a category $\mathscr{A}$ preserves limits (see also* [3]*).*

Theorem C (Kennison [5]; Herrlich [2]). *Let $\mathscr{A}$ be a full subcategory of $\mathscr{T}$. Then*

(a) *$\mathscr{A}$ is coreflexive if and only if it is closed under the formation of sums and quotients.*

(b) *$\mathscr{A}$ is simple reflexive if and only if it is closed under the formation of products and subspaces and contains all indiscrete spaces.*

For the definitions of extremal epimorphisms, extremal epi-mono factorisation property and constant-generated categories, refer to [3].

Theorem D (Herrlich and Strecker [3]). *Let $\mathscr{A}$ be a constant-generated locally small category having sums and extremal epi-mono-factorisation property. Then a full subcategory $\mathscr{B}$ of $\mathscr{A}$ is coreflexive if and only if it is closed under the formation of sums and extremal quotients.*

Theorem E (Kannan [7]). *A continuous map $f: X \to Y$ between two topological spaces is pseudo-open if and only if it is an extremal epimorphism when viewed as a morphism in the category of closure spaces.*

Theorem F (Kannan [10]). *Let X and Y be any two topological spaces, where X is not indiscrete. Then there exists a cardinal m such that Y is a quotient of a subspace of X^m.*

Now we are ready to state and prove our theorems:

Theorem 1. *Let $\mathscr{A}$ be a reflexive cum coreflexive subcategory of $\mathscr{T}$. Then $\mathscr{A} = \mathscr{T}$.*

Proof. The proof is divided into four steps.

First we show that $\mathscr{A}$ contains all products of discrete spaces. Since $\mathscr{A}$ is reflexive and since the reflection of a nonempty space cannot be empty, we get that $\mathscr{A}$ contains at least one nonempty space. Since $\mathscr{A}$ is coreflexive, it follows from Theorem C (a), that $\mathscr{A}$ is closed under the formation of quotients and sums, so that $\mathscr{A}$ contains an one-element space and therefore all discrete spaces. Again since $\mathscr{A}$ is reflexive, it follows from Theorem A that $\mathscr{A}$ is closed under the formation of products, so that $\mathscr{A}$ contains all products of discrete spaces.

Secondly, we show that $\mathscr{A}$ contains all open subspaces of X^m where X is any discrete space and m is any cardinal. Let $U \subset X^m$ be open. Then U is a set union of open rectangles $\{U_\alpha/\alpha \in J\}$. If $i_\alpha: U_\alpha \to U$ is the inclusion map for each $\alpha \in J$ and if $i = \sum_{\alpha \in J} i_\alpha$ is the sum of these maps, then it can be verified that the map $i: \sum_{\alpha \in J} U_\alpha \to U$ is an open continuous map. Thus U is a quotient of a sum of open rectangles U_α. But each open rectangle U_α is a product of discrete spaces, and hence by the first step, it is a member of $\mathscr{A}$. Also by Theorem C (a), $\mathscr{A}$ is closed under the formation of sums and quotients. Hence it follows that U belongs to $\mathscr{A}$.

Next, we show that $\mathscr{A}$ contains all subspaces of powers of discrete spaces (i.e., all zerodimensional Hausdorff spaces). Let Z be a subspace of X^m where X is a discrete space and m is a cardinal. Then Z is the intersection of all open subspaces of X^m, containing Z. But by the second step, any open subspace of X^m belongs to $\mathscr{A}$; and since A is reflexive it is by Theorem A, closed under the formation of intersections. So, Z belongs to $\mathscr{A}$.

Finally, we show that every topological space belongs to $\mathscr{A}$. If X is a discrete space with two elements and if Y is any topological space, then by Theorem F, Y is a quotient of a subspace of X^m for some cardinal m. Hence by the third step, we get that Y is a quotient of a member of $\mathscr{A}$. But since $\mathscr{A}$ is coreflexive, it follows that Y belongs to $\mathscr{A}$.

This proves that $\mathscr{A} = \mathscr{T}$.

We have actually proved the following stronger results:

Result 1. Let $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \cdots \subset \mathscr{A}_n \subset \cdots$ be an increasing sequence of subcategories of $\mathscr{T}$ such that $\mathscr{A}_n$ is reflexive in $\mathscr{T}$ for even n and coreflexive in $\mathscr{T}$ for odd n. Then $\mathscr{A}_n = \mathscr{T}$ for each $n > 4$.

Let us define that a topological property is a family of topological spaces containing all homeomorphic images of its members (i.e., it is a replete subcategory of $\mathscr{T}$). A topological property is said to be productive (respectively intersective, additive or divisible) if it is closed under the formation of products (respectively intersections, sums or quotients).

We have proved above the following:

Result 2. If a topological property is productive, intersective, additive and divisible, then it must be improper (viz. all topological spaces).

We do not know whether there exist proper topological properties which are productive additive and divisible. See Problem 2 of [9].

We now pass on to discuss the validity of analogous theorems in some subcategories of $\mathscr{T}$. Let $\mathscr{H}$ be the full subcategory of $\mathscr{T}$ generated by all Hausdorff spaces. Then the proof of Theorem 1 proves also the following:

Corollary. *Let $\mathscr{A}$ be a reflexive cum coreflexive subcategory of $\mathscr{H}$. Then $\mathscr{A} = \mathscr{H}$.*

We have only to note that any coreflexive subcategory of $\mathscr{H}$ is closed under the formation of sums and Hausdorff quotients.

Theorem 2. *Let $\mathscr{A}$ be a simple reflexive subcategory of $\mathscr{T}$. Let $\mathscr{B}$ be a reflexive cum coreflexive subcategory of $\mathscr{A}$. Then $\mathscr{B} = \mathscr{A}$.*

Proof. Since $\mathscr{A}$ is reflexive in $\mathscr{T}$, $\mathscr{A}$ must contain at least one non-empty space. The following are the only three possibilities: (1) $\mathscr{A}$ contains only empty spaces and singleton spaces (2) $\mathscr{A}$ consists precisely of the indiscrete spaces. (3) $\mathscr{A}$ contains at least one non-indiscrete space.

Case (1). If every member of $\mathscr{A}$ has at most one element, it is trivially proved that $\mathscr{B} = \mathscr{A}$.

Case (2). If $\mathscr{A}$ is the category of all indiscrete spaces, then $\mathscr{A}$ is equivalent to the category of all sets and functions. The reflexivity of $\mathscr{B}$ shows that $\mathscr{B}$ has at least one nonempty set as a member. This guarantees that the coreflection of a nonempty space X must be nonempty. Further if f_X denotes the coreflection morphism of X, then f_X must be onto. For otherwise some constant functions in $\mathrm{Mor}(\mathscr{B}, X)$ cannot be factored. Also f_X must be one-to-one. For otherwise some members of $\mathrm{Mor}(\mathscr{B}, X)$ can be factored in two different ways. Thus f_X is an isomorphism. Since $\mathscr{B}$ is replete, this implies that $\mathscr{B} = \mathscr{A}$.

Case (3). Let $\mathscr{A}$ contain at least one non-indiscrete space. We split the proof in this case into three steps.

We first show that $\mathscr{A}$ has extremal epi-monofactorisation property. Let $f: X \to Y$ be a morphism in $\mathscr{A}$. Now those topologies on the set $\varphi(Y)$ of Y which belong to $\mathscr{A}$ form a complete lattice, where the lattice-union coincides with that in the lattice of all topologies on Y. For, the union of topologies belonging to $\mathscr{A}$ is homeomorphic to a subspace of a product of members of $\mathscr{A}$ and so by Theorem C (b) belongs to $\mathscr{A}$. Further the indiscrete topology belongs to $\mathscr{A}$ by Theorem C (b). So among the different topologies on $\varphi(Y)$ which allow $f: X \to \varphi(Y)$ to be continuous, there exists a strongest one. Let Y' denote the set $\varphi(Y)$ with this topology and let $f': X \to Y'$ be the same function as $f: X \to Y$. Then it is evident that $f' = i \cdot f$ (where i is the inclusion map $Y' \to Y$) is an extremal epi-mono factorisation of f and that it is unique.

Next, we show that $\mathscr{B}$ contains all zero-dimensional Hausdorff spaces. Since $\mathscr{A}$ is productive and hereditary, it can be proved that $\mathscr{A}$ is additive also. Since $\mathscr{B}$ is coreflexive in $\mathscr{A}$, it follows that $\mathscr{B}$ is also additive. Already $\mathscr{B}$ contains at least one nonempty space since it is reflexive in $\mathscr{A}$. Every ordinary quotient map which is a morphism in $\mathscr{A}$ is easily seen to be an extremal epimorphism in $\mathscr{A}$. By Theorem D, it follows that $\mathscr{B}$ is closed under these maps. So, $\mathscr{B}$ contains a singleton space. So $\mathscr{B}$ contains all discrete spaces. But already $\mathscr{B}$ is reflexive in $\mathscr{A}$ and $\mathscr{A}$ is reflexive in $\mathscr{T}$ so that $\mathscr{B}$ is reflexive in $\mathscr{T}$. So, $\mathscr{B}$ is closed under the formation of products and intersections. Now a proof along the lines of the proof of Theorem 1 shows that $\mathscr{B}$ contains all zerodimensional Hausdorff spaces.

It is seen that every member of $\mathscr{A}$ is an extremal quotient in $\mathscr{A}$, of a zero-dimensional Hausdorff space. This completes the proof that $\mathscr{B} = \mathscr{A}$.

Corollary. *Let $\mathscr{A}$ be the category of all regular spaces (completely regular spaces). Let $\mathscr{B}$ be a reflexive cum coreflexive subcategory of $\mathscr{A}$. Then $\mathscr{B} = \mathscr{A}$.*

Theorem 3. *Let $\mathscr{A}$ be a coreflexive subcategory of $\mathscr{T}$ generated by a family of zerodimensional Hausdorff spaces. Further, let $\mathscr{A}$ be open-hereditary. Then $\mathscr{A}$ admits no proper reflexive cum coreflexive subcategory.*

Proof. Let $\mathscr{B}$ be a reflexive cum coreflexive subcategory of $\mathscr{A}$. Then $\mathscr{B}$ is coreflexive in $\mathscr{T}$ as well. So, $\mathscr{B}$ is closed under the formation of sums and quotients. In particular $\mathscr{B}$ contains all discrete spaces. To show that $\mathscr{B} = \mathscr{A}$, it suffices to show that every zerodimensional Hausdorff member of $\mathscr{A}$ belongs to $\mathscr{B}$.

Let $Z \in \mathscr{A}$ be zerodimensional Hausdorff.

Then Z is an intersection of open subspaces of products of discrete spaces. Since intersection can be viewed as a limit, it follows from Theorems A and B, that it suffices to prove that $\mathscr{B}$ contains the $\mathscr{A}$-coreflections of open subspaces of products of discrete spaces. But using the open-heredity of $\mathscr{A}$ and the characterisation of coreflection as the topology of $\mathscr{A}$-open sets (as defined in [6]), it can be shown that these are quotients of sums of the $\mathscr{A}$-coreflections of the rectangles made up of discrete spaces. Now the proof is complete, by the observation that $\mathscr{B}$ is closed under sums, quotients and the products in $\mathscr{A}$ (viz., the $\mathscr{A}$-coreflections of usual products).

Corollary. *Let $\mathscr{A}$ be any one of the following categories:*

(1) *Sequential spaces.*

(2) *m-sequential spaces (where m is an infinite cardinal).*

(3) *Quotients of orderable spaces.*

(4) *P_α-spaces.*

(5) *'α-sequential spaces (where α is a regular ordinal).*

Then $\mathscr{A}$ has no proper reflexive cum coreflexive subcategory.

Now let us consider some examples, where the analogous theorem fails.

Example 1. Let $\mathscr{A}$ be any complete lattice as defined in [1]. Then any complete sublattice of $\mathscr{A}$ containing the two bounds of $\mathscr{A}$ is a reflexive cum coreflexive subcategory of $\mathscr{A}$. This example shows in particular that the analogue of the above theorem may fail in some subcategories of $\mathscr{T}$ also. But this can be seen by a less trivial example:

Example 2. Let $\mathscr{T}_0$ be the category of all topological spaces with base points, where the morphisms are the base-point-preserving continuous maps. Then it can be shown that the singleton spaces constitute a reflexive cum coreflexive subcategory of $\mathscr{T}_0$.

Both of the above examples are however non-full subcategories of $\mathscr{T}$. We can also get such full subcategories of $\mathscr{T}$:

Theorem 4. *There exists a full subcategory $\mathscr{A}$ of $\mathscr{T}$ which admits a proper reflexive cum coreflexive subcategory of itself.*

Proof. Let $\mathscr{D}$ be the category of all discrete spaces, $\mathscr{I}$ be the category of all indiscrete spaces and let $\mathscr{A}$ be a category of connected T_1 spaces. Then we claim that $\mathscr{D} \cup \mathscr{I}$ is both reflexive and coreflexive in $\mathscr{A} \cup \mathscr{D} \cup \mathscr{I}$. It is in fact simple reflexive also. The coreflection of a non-indiscrete space is the discrete space on the same set. The reflection of a nondiscrete space is the indiscrete space on the same set. The reflection morphisms and the coreflection morphisms are the identity maps. Noting that any map with connected domain and discrete range must be a constant map, and that any map with an indiscrete domain and T_1 range must also be a constant map, we omit the other details of the straightforward proof of our assertion.

Lastly, we consider some supercategories of $\mathcal{T}$.

Definition. A closure space is a set X together with a map $u = 2^X \to 2^X$ satisfying

(i) $u(A) \supset A$ for each $A \subset X$.

(ii) $u(\emptyset) = \emptyset$ and

(iii) $u(A \cup B) = u(A) \cup u(B)$ for each $A, B \subset X$.

A continuous map f from a closure space (X_1, v_1) to a closure space (X_2, v_2) is defined as a map $f: X_1 \to X_2$ with the property that

$$f(v_1(A)) \subset v_2(f(A)) \quad \text{for each} \quad A \subset X.$$

It can be seen that the category $\mathcal{C}$ of closure spaces is a constant-generated locally small category with extremal epi-mono factorisation property so that Theorem D can be applied. It can also be proved easily that $\mathcal{T}$ is equivalent to a full reflexive subcategory of $\mathcal{C}$. These facts together with theorem E and some modifications in the proof of Theorem 1 give the following result:

Theorem 5. *The category $\mathcal{C}$ of closure spaces has no proper reflexive cum coreflexive subcategory.*

We conclude with the following remark:

Theorem 6. *Let m be any cardinal. Then there exists a supercategory of $\mathcal{T}$, which has exactly m reflexive cum coreflexive subcategories.*

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