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THE VELOCITY DISPERSION OF GIANT MOLECULAR CLOUDS. II. MATHEMATICAL AND NUMERICAL REFINEMENTS

CHARLES F. GAMMIE AND JEREMIAH P. OSTRIKER
Princeton University Observatory, Peyton Hall, Princeton, NJ 08544

AND

CHANDA J. JOG
Department of Physics, Indian Institute of Science, Bangalore 560012, India
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ABSTRACT

It has been proposed by Fukunaga and Jog & Ostriker that gravitational scattering of clouds off one another in encounters caused by differential rotation can account for the velocity dispersion of the largest molecular clouds in our Galaxy. Angular momentum transfer in cloud-cloud scatterings increases the eccentricity, or epicyclic amplitude, of the clouds. This input of random energy is ultimately balanced by dissipative cloud collisions, leading to equilibrium.

Here we recalculate the energy input to the clouds using the proper linearized equations of motion, including the Coriolis force and allowing for changes in the guiding center. Perturbation theory gives a result in the limit of distant encounters and small initial epicyclic amplitudes. Direct integration of the equations of motion allows us to study the strong encounter regime.

Our perturbation theory result differs by a factor of order unity from that of Jog & Ostriker. The result of our numerical integrations for the two-dimensional (planar) velocity dispersion, adopting the same model for energy loss as Jog & Ostriker, is

$$\sigma \simeq 0.94(GM_{\rm cl}\kappa)^{1/3} = 5.1 \text{ km s}^{-1}$$

for our fiducial cloud of mass $5 \times 10^5\ M_\odot$ at $R_0/2$, slightly smaller than the value obtained by Jog & Ostriker.

In an Appendix we calculate the accretion rate for a molecular cloud in the galactic disk.

Subject headings: interstellar: molecules — nebulae: internal motions

1. INTRODUCTION

The motivation for considering gravitational scattering as a mechanism for increasing the velocity dispersion of giant molecular clouds has been discussed by Jog & Ostriker (1988, hereafter JO), and by Fukunaga and collaborators (Fukunaga 1984; Fukunaga & Tosa 1989, and references therein). Briefly, since the velocity dispersion of interstellar clouds changes little over more than three orders of magnitude in mass, we know that the most massive clouds are not in equipartition with the smaller clouds whose kinetic energy is supplied mainly by interactions with supernovae. There must then be some other mechanism accelerating the largest clouds. Gravitation, which acts as a local force in a nearly two-dimensional system like the Galactic disk, can provide a viscous couple capable of transferring energy from ordered rotational motion to random motions and hence accelerating the giant molecular clouds (GMCs).

Our physical approach is motivated by the analysis of random velocities of particles within Saturn's rings by Goldreich & Tremaine (1978). In their model, mildly inelastic collisions between particles set up a viscous stress that transfers energy from rotational to random motion. The requirement that energy gains and losses balance one another then fixes the coefficient of restitution for particle-particle collisions as a function of the optical depth through the ring. Since the coefficient of restitution is a monotonically decreasing function of impact velocity for the materials that make up Saturn's rings, the coefficient of restitution uniquely determines the velocity dispersion.

The Galactic disk obviously differs from planetary rings in important respects. First, there are other forces, such as acceleration by supernovae, ram pressure, and magnetic fields that may act on the clouds. Second, because the gas that makes up the disk is highly dissipative, it hovers on the brink of instability and collective effects (spiral arms) can be important. We work with an idealized model of the Galactic disk that ignores these effects, treating the molecular cloud distribution as uniform and ignoring physical processes other than two-body gravitational scattering and cloud-cloud collisions. In our model, the collisions between clouds are almost completely inelastic and clouds typically undergo only one strong collision at a time. We assume that all the clouds are of the same mass, that there are no destructive or sticking collisions, and that the clouds are indefinitely long-lived. Within the context of this admittedly simplistic model the equilibrium velocity dispersion of GMCs in the disk can be calculated and the dynamics of gravitational scattering in the disk thoroughly understood.

Before working through the details, we can clarify the problem by making a few simple estimates for the rate at which clouds gain and lose random energy due to gravitational scattering and physical collisions.

First, a cloud will typically lose some fraction of order unity of its total random energy ($= \frac{1}{2}M_{\rm cl}\kappa^2 a^2$, where $M_{\rm cl}$ is the cloud mass, κ is the epicyclic frequency, and a is the epicyclic amplitude) in physical collisions that occur with a frequency

$\omega_{\rm enc}$. Denoting an ensemble average by angle brackets, we have

$$\left(\frac{d\langle a^2\rangle}{dt}\right)_{\rm loss} \sim -\omega_{\rm enc}\langle a^2\rangle . \quad (1)$$

Next, we note that for gravitational scattering in two dimensions, the most important encounters are the strong, close ones. Strong encounters occur when two clouds approach more closely than a few times the tidal radius of the cloud; since molecular clouds are observed to have a radius that is of the same order as the tidal radius (Stark & Blitz 1978), strong encounters will occur with about the same frequency as physical collisions. The typical velocity change in these encounters is essentially the shear velocity at the tidal radius $\sim 2A(GM_{\rm cl}/\kappa^2)^{1/3} \sim (GM_{\rm cl}\kappa)^{1/3}$. This estimate holds for systems where the velocity dispersion is less than about the escape velocity from the surface of the clouds, a condition that is satisfied by the molecular cloud population of our Galaxy. Thus

$$\left(\frac{d\langle a^2\rangle}{dt}\right)_{\rm gain} \sim \omega_{\rm enc}(GM_{\rm cl}\kappa)^{2/3} . \quad (2)$$

Equating gains and losses gives for the steady state

$$\langle a^2\rangle = \alpha(GM_{\rm cl}\kappa)^{2/3} , \quad (3)$$

where $\alpha \sim 1$. Our final numerical results (§ 3) yield $\alpha = 1.19$.

JO have estimated the heating and cooling rates (eqs. [1] and [2]) for the cloud distribution function using perturbation theory. In this paper we recalculate the energy exchange in the limit of large impact parameter, starting with the proper linearized equations of motion rather than the approximations introduced by JO (see their eqs. [34] and [35]). These equations of motion include the Coriolis force and allow for changes in the guiding center over the course of an encounter. In addition, we retain only the lowest order term in the expansion of the postencounter epicyclic energy and simplify considerably the mathematical treatment. Since the most important encounters occur in the region where perturbation theory is inapplicable, we numerically integrate orbits to obtain the exact heating and cooling terms under certain assumptions about the cloud-cloud collision process.

The perturbation theory aspect of this problem is formally analogous to the classical problem of the perturbation of one planet by another in the Sun's gravitational potential, although the galactic potential is not Keplerian. In the limit of distant encounters where the finite size of the clouds can be neglected, cloud-cloud scattering is similar to cloud-star scattering, a problem first considered by Spitzer & Schwarzschild (1951, 1953), who concluded that the existence of massive ($10^5\ M_\odot$) objects in the disk could account for the observed stellar velocity dispersions. The principle difference between cloud-star scattering and cloud-cloud scattering is that the epicyclic amplitude of stars is large compared to the tidal radius of clouds in the Galactic disk (the dispersion-dominated regime), whereas giant molecular clouds have epicyclic amplitudes roughly of the same order as the tidal radius of the clouds (the shear-dominated regime). This prevents the direct application of some of the stellar results to the GMC population. More recently, the evolution of stellar velocity dispersions has been considered by Icke (1982), Lacey & Ostriker (1985), and Binney & Lacey (1988), and it is discussed in Binney & Tremaine (1987, see Ch. 7, exercise 12). The velocity dispersion of molecular clouds has also been considered in a series of papers by Fukunaga (see Fukunaga & Tosa 1989, and references therein) using both numerical and analytic methods. The present work is principally concerned with the problem of two-body scattering with dissipational collisions in a disk potential, which has not been completely treated in any of the above works.

The plan for the remainder of this paper is as follows. In § 2, we discuss the governing equations and, using perturbation theory, obtain an expression for the change in epicyclic energy that is valid in the limit of distant encounters. In § 3, we present the results of our numerical investigation. § 4 contains our conclusions.

2. GRAVITATIONAL SCATTERING IN A DISK POTENTIAL

Gravitational scattering in a disklike potential can be divided into three regimes. At impact parameters small compared to the tidal radius the clouds repel each other on horseshoe orbits and never come close to one another. These encounters are weak and can be treated analytically (see, e.g., the simple discussion in Goldreich & Tremaine 1982; also Dermott & Murray 1981). At moderate impact parameter the encounters are close, the orbits are complex, and we are reduced to solving the full three-body problem. This of course can only be done numerically. At large impact parameter the encounters are again weak and the encounter can be treated perturbatively.

The small impact parameter encounters (horseshoe orbits) are characterized by initial convergence of the clouds in azimuth due to differential rotation. Because of the Coriolis force, the gravitational interaction of the clouds effectively repels them from each other and they slowly turn around and diverge in azimuth, describing a horseshoe shape in a frame rotating with the center of mass of the clouds (see orbit E in Fig. 1, which shows the evolution of the relative coordinate). The contribution of these orbits to the heating of the cloud distribution can be neglected for at least two reasons. First, the encounter frequency is proportional to the shear velocity, which in turn is proportional to the impact parameter, so there are relatively few encounters at small impact parameter. Second, these encounters take place very slowly in comparison

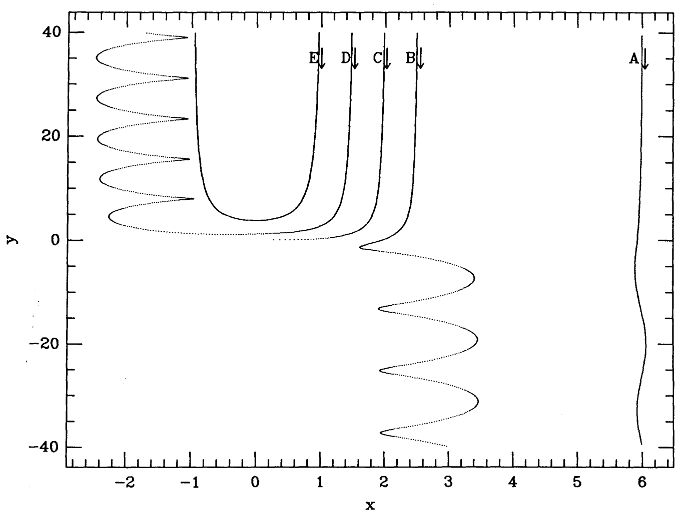

FIG. 1.—Orbits in the relative coordinate, with the Galactic center in the $-x$ direction and rotation locally in the $+y$ direction. The initial epicyclic amplitude is 0 for all the orbits, and the effective radius is 0.4.

to an epicyclic period, so that the epicyclic energy is protected by adiabatic invariance and thus changes little.

The large impact parameter encounters will also turn out to be negligible, but this is less obvious, so we carry out the perturbation calculation below. There has been considerable attention to the problem of gravitational scattering of two bodies in the presence of a much more massive third body in connection with planetary rings (Goldreich & Tremaine 1980; Petit & Hénon 1986; Hénon & Petit 1986). A similar problem arises in the theory of mass condensations in the protosolar nebula (Safronov 1972, § 21; more recently Nishida 1983; Hasegawa & Nakazawa 1990; Greenzweig & Lissauer 1990).

2.1. *Governing Equations*

We wish to calculate the change in epicyclic energy of a cloud of mass M_1 due to an encounter with a second cloud of mass M_2. We shall define the epicyclic energy as

$$E_{\mathrm{epi},i} \equiv \frac{1}{2} M_i \kappa^2 a^2 , \tag{4}$$

where κ is the epicyclic frequency, a is the radial amplitude of the epicyclic oscillation, and ($i = 1, 2$) denotes the ith cloud. The epicyclic energy is the energy available to an observer on a circular orbit at the guiding center radius. The encounter is caused by differential rotation in the disk. We ignore completely the motion of the clouds perpendicular to the disk. This approximation is valid because the radius of GMCs is within a factor of 2 of their scale height. Our notation is similar to that of Petit & Hénon (1986).

Encounters occur in a disklike potential that can be characterized locally by the logarithmic derivative of its rotation curve

$$\beta \equiv \frac{d\ln v_c}{d\ln r} = 1 - 2\left(\frac{A}{\Omega}\right) = \frac{1}{2}\frac{\kappa^2}{\Omega^2} - 1 . \tag{5}$$

We choose units such that $G = \kappa = M_1 + M_2 = 1$, where κ is the epicyclic frequency. Table 1 describes a "fiducial" molecular cloud at galactocentric radius $R_0/2$, and is based on Scoville et al. (1987). Note that the length unit, effectively a tidal radius in the Galactic disk, is $[G(M_1 + M_2)/\kappa^2]^{1/3} \sim 94$ pc. The time unit is $\kappa^{-1} \sim 1.4 \times 10^7$ yr, and the mass unit for the typical encounter between equally massive clouds is $10^6\ M_\odot$. In these units, the rotation frequency, Ω, and Oort's constants A and B, are given by

$$\Omega = \frac{1}{\sqrt{2(1+\beta)}} , \quad A = \frac{1-\beta}{\sqrt{8(1+\beta)}} , \quad B = -\sqrt{\frac{1+\beta}{8}} , \tag{6}$$

TABLE 1

FIDUCIAL CLOUD PARAMETERS

Parameter	Unit	Typical Value
Mass unit	$2M_{\mathrm{cl}}$	$10^6\ M_\odot$
Length unit	$(2GM_{\mathrm{cl}}/\kappa^2)^{1/3}$	94 pc
Time unit	κ^{-1}	1.4×10^7 yr
Velocity unit	$(2GM_{\mathrm{cl}}\kappa)^{1/3}$	6.8 km s^{-1}
Mass	M_{cl}	$5 \times 10^5\ M_\odot$
Effective radius	$r_{\mathrm{min}}/2$	$0.2 \simeq 19$ pc
Galactocentric distance	r	$R_0/2$
β	$d\ln V_c/d\ln r$	0.0
κ	$\partial^2\phi_{\mathrm{eff}}/\partial r^2$	36 R_0/r km s^{-1} kpc^{-1} $1.17 \times 10^{-15}\ R_0/r\ \mathrm{s}^{-1}$

where Oort's constant A is

$$A \equiv -\frac{r}{2}\frac{d\Omega}{dr} , \tag{7}$$

so that both A and Ω are positive. We define $\mu \equiv M_1/(M_1 + M_2)$. Recall that $\kappa^2 = -4\Omega B = 1$.

We use a local Cartesian coordinate system that rotates with the guiding center of the center of mass of the two clouds, with the x-axis pointing in the direction of increasing radius and the y-axis pointing in the direction of increasing azimuth. In dimensionless form the equations of motion (Spitzer & Schwarzschild 1953) can be written

$$\begin{aligned} \ddot{x}_1 &= 4A\Omega x_1 + 2\Omega\dot{y}_1 - (1-\mu)(x_1 - x_2)/r^3 , \\ \ddot{y}_1 &= -2\Omega\dot{x}_1 - (1-\mu)(y_1 - y_2)/r^3 , \end{aligned} \tag{8}$$

for cloud 1 and conversely for cloud 2, with μ replaced by $1 - \mu$. Here r is the distance between clouds. We have neglected terms of second order and higher in S/R, where S is the impact parameter and R is the galactocentric radius of the center of mass. For molecular clouds in our Galaxy, the requirement $R \gg S$ is readily satisfied for the encounters of interest, since most of the molecular gas is at $R \gtrsim 4$ kpc (Scoville & Sanders 1987), and (see § 3) the most important encounters occur at an impact parameter of a few $\times\ (GM/\kappa^2)^{1/3} \sim 200$ pc for our fiducial cloud.

Transforming to relative and center of mass coordinates, we have

$$x_r \equiv x_1 - x_2 , \quad y_r \equiv y_1 - y_2 , \tag{9a}$$

$$x_{cm} \equiv \mu x_1 + (1-\mu)x_2 , \quad y_{cm} \equiv \mu y_1 + (1-\mu)y_2 . \tag{9b}$$

The subscripts cm and r are used to denote quantities in the center of mass and relative coordinates, respectively. The relative coordinate equations of motion are

$$\ddot{x}_r = 4A\Omega x_r + 2\Omega\dot{y}_r + \epsilon A_x , \tag{10a}$$

$$\ddot{y}_r = -2\Omega\dot{x}_r + \epsilon A_y , \tag{10b}$$

where

$$\epsilon \equiv S^{-2} , \quad A_x \equiv -\frac{(x_r/S)}{(r/S)^3} , \quad A_y \equiv -\frac{(y_r/S)}{(r/S)^3} . \tag{11}$$

Similar equations apply in the center of mass coordinates except that there is no interaction term (the total angular momentum of the system is conserved).

If we neglect the gravitational interaction (i.e., to zeroth order in ϵ) the solutions to the center of mass equations of motion are free epicycles, as are the solutions to the relative coordinate equations of motion:

$$\begin{aligned} x_r &= D_{1,r}\cos t + D_{2,r}\sin t + S_r , \\ y_r &= -2\Omega D_{1,r}\sin t + 2\Omega D_{2,r}\cos t - 2AS_r t + D_{3,r} , \\ x_{cm} &= D_{1,cm}\cos t + D_{2,cm}\sin t + S_{cm} , \\ y_{cm} &= -2\Omega D_{1,cm}\sin t + 2\Omega D_{2,cm}\cos t - 2AS_{cm} t + D_{3,cm} . \end{aligned} \tag{12}$$

Similar solutions hold for clouds 1 and 2, with the coefficients written as $D_{1,1}$, $D_{2,1}$, etc. The constants $D_{3,r}$, $D_{3,cm}$ and S_{cm} can be made to vanish in a suitably chosen coordinate system, where the center of mass oscillates about the origin and the encounter occurs at $t \sim 0$. We assume this choice from now on, and denote S_r as simply S.

The relative coordinate equations of motion (eqs. [10a, b]) admit a conserved quantity:

$$\Gamma \equiv \frac{1}{2}(\dot{x}_r^2 + \dot{y}_r^2) - 2A\Omega x_r^2 - \frac{1}{r}, \tag{13}$$

where r is the distance between clouds (e.g., Brouwer & Clemence 1961, § XII.18). This is the analog of the Jacobi integral for the linearized equations of motion. At large distances (early and late times) the last term in equation (13) can be neglected and Γ rewritten in terms of D_1, D_2, and S:

$$\Gamma_{\pm\infty} = D_{1,r}^2 + D_{2,r}^2 - \frac{A}{\Omega}S^2 = a_r^2 - \frac{A}{\Omega}S^2, \tag{14}$$

where a_r is the epicyclic amplitude in the relative coordinate.

The change in epicyclic energy, $\Delta(E_{\mathrm{epi},i})$ of cloud i ($i = 1, 2$) over the course of the encounter can be written in terms of the $D_{i,j}$ (see eq. [12]) as

$$\Delta(E_{\mathrm{epi},i}) = \tfrac{1}{2}m_i\Delta(D_{1,i}^2 + D_{2,i}^2), \tag{15}$$

since

$$E_{\mathrm{epi},i} = \tfrac{1}{2}m_i\kappa^2 a_i^2 = \tfrac{1}{2}m_i\kappa^2(D_{1,i}^2 + D_{2,i}^2), \tag{16}$$

where $m_1 \equiv \mu$ and $m_2 \equiv 1 - \mu$. The following relations follow immediately from the definition of the relative and center of mass coordinates:

$$D_{i,1} = D_{i,cm} + (1-\mu)D_{i,r},$$
$$D_{i,2} = D_{i,cm} - \mu D_{i,r}. \tag{17}$$

Similar relations apply for the impact parameter S. We can now write

$$D_{i,1}^2 = D_{i,cm}^2 + 2(1-\mu)D_{i,cm}D_{i,r} + (1-\mu)^2 D_{i,r}^2. \tag{18}$$

We will calculate the change in epicyclic energy averaged over initial epicyclic phases ($\phi_i \equiv \tan^{-1}(D_{2,i}/D_{1,i})$), assuming that they are randomly distributed. The phase average in the physical coordinate system, denoted $\langle f \rangle$, is given by

$$\langle f \rangle = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} d\phi_1\, d\phi_2\, f, \tag{19}$$

where ϕ_i refers to the initial phase of cloud i. Thus we have

$$\langle \Delta(E_{\mathrm{epi},i}) \rangle = \tfrac{1}{2}m_i\Delta(\langle D_{1,i}^2 \rangle + \langle D_{2,i}^2 \rangle), \tag{20}$$

and

$$\langle \Delta D_{i,1}^2 \rangle = 2(1-\mu)\langle D_{i,cm}\Delta D_{i,r} \rangle + (1-\mu)^2\langle \Delta D_{i,r}^2 \rangle. \tag{21}$$

Now $\Delta D_{i,r}$ can be expanded in powers of $D_{1,r}$ and $D_{2,r}$, but does not depend on $D_{i,cm}$, so that the first term vanishes under phase averaging. A similar result obtains for $\langle \Delta D_{i,2}^2 \rangle$, so that

$$\langle \Delta D_{i,j}^2 \rangle = m_k^2 \langle \Delta D_{i,r}^2 \rangle, \tag{22}$$

where $k \neq j$. Hence

$$\langle \Delta(E_{\mathrm{epi},i}) \rangle = \tfrac{1}{2}m_i m_k^2 \langle \Delta(D_{1,r}^2 + D_{2,r}^2) \rangle. \tag{23}$$

It follows that $\langle \Delta(E_{\mathrm{epi},i}) \rangle / \langle \Delta(E_{\mathrm{epi},j}) \rangle = m_j/m_i$. Thus on average the less massive cloud obtains a larger share of the rotational energy tapped in gravitational encounters than the more massive cloud. This result was obtained earlier by JO (their Appendix C) for the special case of zero initial velocity of the clouds.

We now comment briefly on how the disk-scattering problem can be expressed in action-angle variables (e.g., Binney 1987). The radial action is $J_x \equiv \frac{1}{2}a_r^2$, while $J_y \equiv -2BS$ is a linear function of the z-component of the angular momentum. These actions and their associated angles are related to x, y, $\dot{x}$, and $\dot{y}$ by

$$J_x = \frac{1}{2}\left(\frac{A}{B}x + \frac{1}{2B}\dot{y}\right)^2 + \frac{1}{2}\dot{x}^2, \quad \Theta_x = \cos^{-1}\left(\frac{\dot{x}}{\sqrt{2J_x}}\right), \tag{24a}$$

$$J_y = 2\Omega x + \dot{y}, \quad \Theta_y = y - 2\Omega\dot{x}, \tag{24b}$$

where A and B are the Oort constants. The Hamiltonian is the "Jacobi constant" Γ expressed in action-angle variables:

$$H = J_x + \frac{A}{2B}J_y^2 + \psi_p, \tag{25}$$

where ψ_p is the perturbing potential. This coordinate system turns out to be convenient for the numerical investigation of distant encounters.

The disk scattering problem can be qualitatively understood by examining a few representative orbits. Figure 1 shows orbits in the relative coordinates (x_r, y_r). For simplicity we have set the initial epicyclic amplitude to 0. The orbits are labeled A through E, and enter the plot from the upper right. Orbit A is a distant encounter that results in a small increase in the epicyclic energy. This encounter is weak enough that it can be adequately treated with perturbation theory. Orbit B is a strong encounter with a large increase in epicyclic energy, but with no change in the sign of the impact parameter. Orbit C is a close encounter that approaches the origin more closely than $r_{\min} = 2 \times r_{\mathrm{cl}}$, where r_{cl} is the effective cloud radius, and is treated as a collision. Orbit D is a strong "horseshoe" encounter that results in a large increase in epicyclic amplitude and a change in the sign of the impact parameter. Orbit E is a weak horseshoe encounter. Orbits B through D lie in the range in impact parameter where there are strong encounters, chaotic orbits, physical collisions, and large variations in epicyclic energy. At large impact parameters the typical change in epicyclic energy per encounter declines rapidly (as $1/S^4$, where S is the impact parameter) while the encounter frequency increases as S, so it is encounters at intermediate values of the impact parameter, such as orbits B through D, that are most important for heating the cloud distribution. This may be contrasted with relaxation in a three-dimensional homogeneous system, where each logarithmic interval in impact parameter contributes equally. Finally, Figure 2 shows an orbit in action-angle variables, with the same impact parameter as Orbit A, above, but with $a_r^{(0)2} = 1.0$. The ordinate in each case is the *change* in the phase space coordinate from the value it would have had had the perturbing force been absent.

2.2. *First-Order Perturbation Theory*

The postencounter epicyclic energy can be expanded in powers of $\epsilon \equiv S^{-2}$, where S is the initial impact parameter. Here we calculate the lowest order (in ϵ) variation in the epicyclic amplitude under the assumption that the initial epicyclic amplitude is small, using the technique of variation of coordinates.

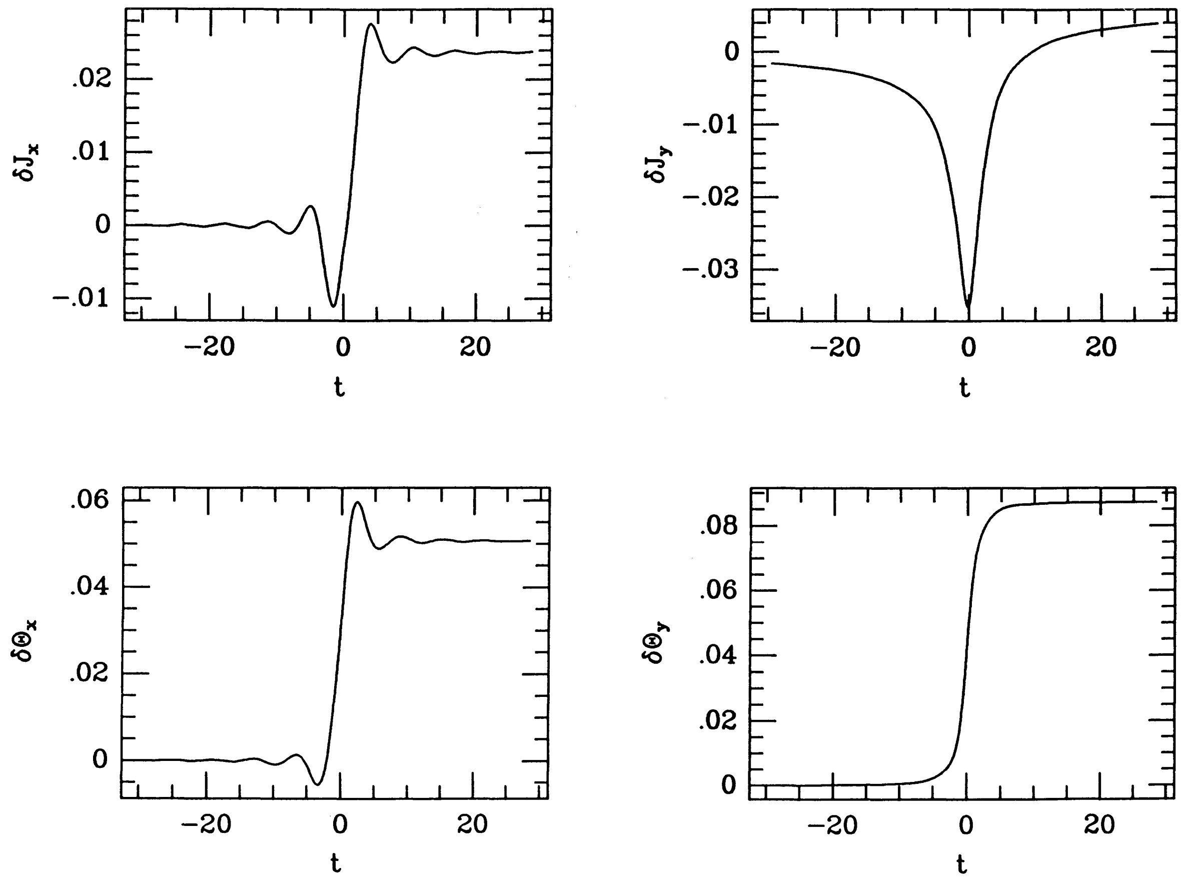

FIG. 2.—Typical orbit in the action-angle variables (see eq. [24]), with $S = 6$ and $a_r^{(0)2} = 1.0$

The second equation of motion (eq. [10b]) can be integrated and substituted into the first (eq. [10a]) to obtain

$$\ddot{x}_r + x_r = \epsilon\left(A_x + 2\Omega \int_{-\infty}^{t} dt'\, A_y\right) . \tag{26}$$

We then expand x_r in powers of ϵ and find that

$$\ddot{x}_r^{(1)} + x_r^{(1)} = A_x^{(0)} + 2\Omega \int_{-\infty}^{t} dt'\, A_y^{(0)} , \tag{27}$$

where the superscript (i) indicates that the expression is evaluated to order i in ϵ; $A_x^{(0)}$ and $A_y^{(0)}$ are then evaluated along the unperturbed orbit. The solution is

$$x_r^{(1)} = u_1^{(1)} \cos t + u_2^{(1)} \sin t , \tag{28}$$

where

$$u_1^{(1)} = -\int_{-\infty}^{t} dt' \sin t' \left(A_x^{(0)} + 2\Omega \int_{-\infty}^{t'} dt'' A_y^{(0)}\right) , \tag{29a}$$

$$u_2^{(1)} = \int_{-\infty}^{t} dt' \cos t' \left(A_x^{(0)} + 2\Omega \int_{-\infty}^{t'} dt''\, A_y^{(0)}\right) . \tag{29b}$$

If the initial epicyclic amplitude is zero, so that

$$x_r^{(0)} = S , \quad y_r^{(0)} = -2ASt , \tag{30}$$

then

$$A_x^{(0)} = -\frac{1}{(1 + 4A^2t^2)^{3/2}} , \quad A_y^{(0)} = \frac{2At}{(1 + 4A^2t^2)^{3/2}} . \tag{31}$$

It follows that

$$2\Omega \int_{-\infty}^{t} dt'\, A_y^{(0)} = -\frac{2\Omega}{(1 + 4A^2t^2)^{1/2}} . \tag{32}$$

The term in parentheses in equation (29a) is an even function of t, so that $u_1^{(1)}$ vanishes, and

$$u_2^{(1)} = -\int_{-\infty}^{t} dt' \cos t' \left(\frac{1}{(1 + 4A^2t'^2)^{3/2}} + \frac{2\Omega}{(1 + 4A^2t'^2)^{1/2}}\right) . \tag{33}$$

These integrals can be evaluated using the identity (Gradshteyn & Ryzhik 1980, p. 959)

$$\int_0^{\infty} \frac{\cos az\, dz}{(1 + z^2)^{\nu + 1/2}} = K_\nu(a) \frac{a^\nu \Gamma(\frac{1}{2})}{2^\nu \Gamma(\nu + \frac{1}{2})} , \tag{34}$$

where K_ν is a modified Bessel function of the second kind of order ν and Γ is Euler's Γ-function. Then

$$x_r^{(1)} = -\frac{1}{2A}\left[2\Omega K_0\left(\frac{1}{2A}\right) + K_1\left(\frac{1}{2A}\right)\right] \sin t , \tag{35}$$

which was first obtained by Julian & Toomre (1966).

In the limit of small initial epicyclic amplitude $[a_r^{(0)2} \ll (2GM_{\rm cl}/\kappa^2)^{1/3}]$, we then have

$$\langle \Delta a_r^2 \rangle = 2\epsilon^2 \langle x_r^{(1)2} \rangle . \tag{36}$$

If we do not require that the initial epicyclic amplitude be small, lower order terms arise that vanish under phase averaging. Variation of coordinates fails for higher order terms because of a logarithmic divergence in $y_r^{(1)}$. These higher order terms are straightforward, in principle, to calculate by perturbing the actions.

We can now comment on the perturbation theory result of JO, which is of the form

$$\langle \Delta E_{\rm epi} \rangle = \alpha_1 \frac{1}{S^4} + \alpha_2 \frac{a_r^{(0)2}}{S^6} . \tag{37}$$

Under the equations of motion adopted by JO variation of parameters works to all orders, and α_2 can be directly calculated. However, their Taylor series expansion of the perturbing force includes only terms of up to linear order (the "tidal" terms). It turns out (see § 3) that higher order terms in the expansion can make contributions to the change in epicyclic amplitude of order ϵ^3, so that JO neglect a term proportional to S^{-6} and their value of α_2 is of the wrong sign.

2.3. *Comparison to Previous Results*

The lowest order change in $\langle a_r^2 \rangle$ per encounter is

$$\langle \Delta a_r^2 \rangle = \epsilon^2 \left(\frac{1}{4A^2}\right)\left[2\Omega K_0\left(\frac{1}{2A}\right) + K_1\left(\frac{1}{2A}\right)\right]^2 \equiv 2\epsilon^2 F(\beta) . \tag{38}$$

Then

$$\langle \Delta(E_{{\rm epi},i}) \rangle = \frac{1}{2} \epsilon^2 m_i m_k^2 \langle a_r^{(1)2} \rangle$$

$$= \frac{m_i m_k^2}{S^4} F(\beta) . \tag{39}$$

The function F is shown in Figure 3; note that it peaks for $\beta \simeq 0$. If we assume that both clouds have the same mass ($\mu = 0.5$), and let $\beta = 0$ (flat rotation curve), then $\Omega = 2A = 1/\sqrt{2}$ and

$$\langle \Delta(E_{{\rm epi},i}) \rangle = \frac{\eta}{S^4} , \quad \eta \equiv 0.4256 . \tag{40}$$

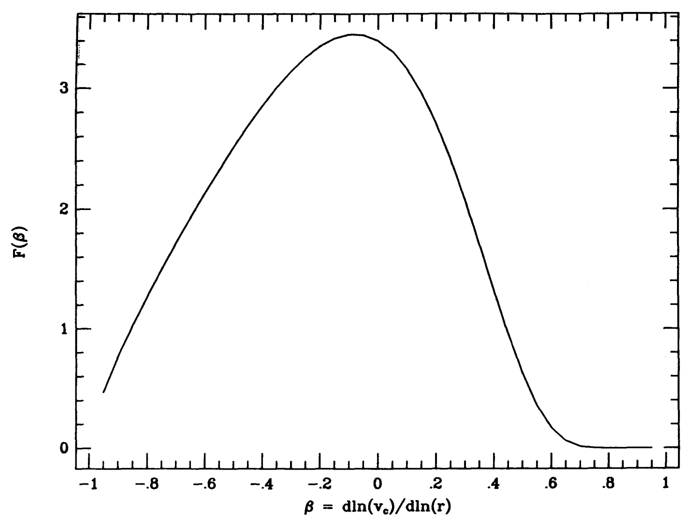

FIG. 3.—$F(\beta)$, which is proportional to the gain in epicyclic energy at large impact parameter (see eq. [38]).

Note that $\langle \Delta(E_{{\rm epi},i}) \rangle$ is a rapidly decreasing function of impact parameter, implying that the most important encounters for epicyclic excitation are the closest ones. We compare this result to that obtained by JO (see their eq. [105])

$$\langle \Delta(E_{{\rm epi},i}) \rangle_{\rm JO} = \frac{1}{4} m_t \left(\frac{G m_f}{A S^3}\right)^2 \left(S^2 g(\beta') + \left[\frac{V_{r,{\rm epi}}}{\kappa}\right]^2 f(\beta')\right) , \tag{41}$$

where β' is 2.828 for a flat rotation curve. If we cast this in dimensionless form, and assume that the masses of the two clouds are equal, we can write

$$\langle \Delta(E_{{\rm epi},i}) \rangle_{\rm JO} = \frac{g(\beta')}{32 A^2 S^4} \left(1 + \frac{f(\beta') a_r^2}{g(\beta') S^2}\right) . \tag{42}$$

For a flat rotation curve, $A = 1/\sqrt{8}$, $f(\beta') = 1.847$, $g(\beta') = 0.626$. Then

$$\langle \Delta(E_{{\rm epi},i}) \rangle_{\rm JO} = \frac{0.157}{S^4} \left(1 + 2.95 \frac{a_r^2}{S^2}\right) . \tag{43}$$

Comparing this to equation (40) above, we have

$$\frac{\langle \Delta(E_{{\rm epi},i}) \rangle_{\rm JO}}{\langle \Delta(E_{{\rm epi},i}) \rangle_{\rm GOJ}} = 0.369 \left(1 + 2.95 \frac{a_r^2}{S^2}\right) , \tag{44}$$

so that the ratio of the two results is of order unity.

3. NUMERICAL RESULTS

Since the most important encounters for heating the cloud distribution occur at intermediate impact parameters [$S \sim (GM_{\rm cl}/\kappa^2)^{1/3}$] where the orbits are analytically intractable, the true heating and cooling rates (in our model) can only be found by the direct numerical integration of orbits. First, however, it is of interest to numerically estimate higher order terms in the perturbation series expansion of Δa_r^2 (eq. [38]), and to see where and how seriously perturbation theory breaks down.

3.1. *Corrections to Perturbation Theory*

The expression for the average change in energy per encounter obtained from perturbation theory in the last section can be extended numerically by directly calculating the phase-averaged change in energy and then fitting the result to a series expansion. The equations of motion suggest a series expansion of the form

$$\langle \Delta a_r^2(S, a_r^{(0)}) \rangle = \epsilon^2 \sum_{i,j=0}^{\infty} \alpha_{i,j} \epsilon^i (a_r^{(0)2})^j \tag{45}$$

where we may set $\alpha_{0,0} = 6.8099$ from equation (38) in the case of a flat rotation curve. We then fit for the coefficients in equation (45) in the range $S = [7, 20]$, $a_r^2 = [1, 2]$, sampling on a grid with $\delta S = 1.0$ and $\delta a_r^2 = 0.1$. The range in impact parameter was chosen so that impact parameter was large enough that the perturbation expansion was approximately valid, but small enough so that $S/R \lesssim 1$ for the fiducial cloud. The range in initial epicyclic amplitude was chosen with hindsight from calculations described below so as to include the final equilibrium value.

The orbits were integrated in action-angle space using a combination of the Bulirsch-Stoer and Runge-Kutta methods as implemented by Press et al. (1986, Ch. 15). Using action-angle variables permitted more reasonable control over the accuracy of the integration. The Runge-Kutta method was used for a 30 time unit interval near the encounter (at $t \sim 0$), and the Bulirsch-Stoer for the rest. The phase averaging was carried out using Gauss-Legendre integration with 15 weights, which provided adequate accuracy since at large impact parameters the final epicyclic amplitude is a smooth function of epicyclic phase (see Fig. 5*a*, which shows the variation of the change in epicyclic amplitude over phase at large impact parameter). There is a slight error incurred by integrating over a finite interval around $t = 0$. This was minimized by averaging over an epicyclic period at the end of the integration (to accelerate convergence of the actions, which oscillate slightly) and then beginning at early enough times so that this contribution to the error was insignificant.

After fitting for the leading terms in equation (45) by minimizing the squared relative error at each point on the numerical grid, we find

$$\langle \Delta a_r^2 \rangle \simeq \frac{6.8099}{S^4}\left(1 + 1.6\,\frac{1}{S^2} - 1.0\,\frac{a_r^{(0)2}}{S^2}\right). \tag{46}$$

The error is everywhere less than 0.5%. The coefficients are a fitting formula and not what would be obtained in a higher order perturbation calculation, but we do not expect the perturbation theory coefficients to differ in sign or order of magnitude. The accuracy of this extended perturbation expansion is evaluated in Figure 4. Here $\langle \Delta a_r^2(S,\, a_r^{(0)2} = 1.8,\, \beta = 0)\rangle$, as calculated by techniques to be described below, is shown as a solid line, while the extended perturbation result is shown as a dashed line. The extended perturbation expansion is in serious error for $S \lesssim 3.4$.

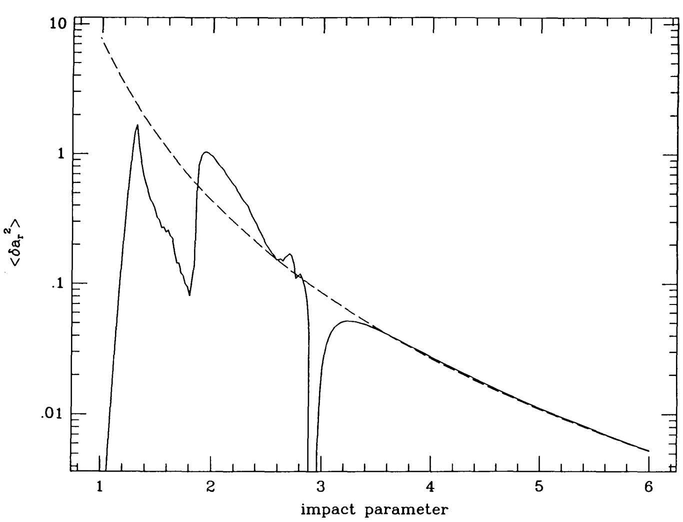

FIG. 4.—$\langle \Delta a_r^2(S,\, a_r^{(0)2} = 1.8)\rangle$, the variation in the squared epicyclic amplitude averaged over phase, from direct integration of the orbits (*solid line*) and from perturbation theory (*broken line*).

3.2. *Direct Calculation of Gains and Losses*

Now we evaluate the full epicyclic excitation and damping rates. Following JO, we use a simple model for collisions: if the cloud separation becomes less than $r_{\min}$, the *relative* epicyclic amplitude and impact parameter are set to zero, and it is imagined that the clouds drift to some large separation. Each cloud retains some residual epicyclic energy from the motion of the center of mass. All energy loss is then the result of physical collisions between clouds. In other words, we can think of the collisions as occurring with a coefficient of restitution that is large enough that the clouds do not stick.

We calculate the gains and losses for an individual cloud with a given value of the epicyclic energy interacting with an ensemble of clouds with the same energy. Equating gains and losses for this cloud will give the correct result for the equilibrium epicyclic energy if the distribution of clouds over epicyclic energy is sharply peaked at the equilibrium value. The ensemble averaged energy gain per unit time per cloud is

$$\left(\frac{dE_{\rm epi}}{dt}\right)_{\rm gain} = \frac{1}{16}\int_{-\infty}^{\infty} \langle \Delta a_r^2 \rangle \omega_{\rm enc}\, dS \tag{47}$$

where $\omega_{\rm enc} \equiv \Sigma 2AS$ is the encounter frequency per unit impact parameter, Σ is the surface number density, and we have assumed that $m_i = m_k = 1/2$. The brackets indicate as before that Δa_r^2 has been averaged over phase. The energy gain per unit time is then proportional to the integral

$$P(a_r^{(0)},\, \beta,\, r_{\min}) \equiv \int_0^{\infty} \langle \Delta a_r^2 \rangle S\, dS\,, \tag{48}$$

where $r_{\min}$ is such that if the orbit passes within $r_{\min}$ of the origin, then Δa_r^2 is set to 0. The ensemble average energy loss per unit time cloud is given by

$$\left(\frac{dE_{\rm epi}}{dt}\right)_{\rm loss} = \frac{1}{16}\int_{-\infty}^{\infty} \langle \Theta \rangle a_r^{(0)2} \omega_{\rm enc}\, dS\,, \tag{49}$$

where Θ is 1 if there is a collision ($r < r_{\min}$) and zero otherwise. This is proportional to the integral

$$W(a_r^{(0)},\, \beta,\, r_{\min}) \equiv \int_0^{\infty} \langle \Theta \rangle S\, dS\,. \tag{50}$$

Energy balance now requires that

$$P = a_r^{(0)2} W\,. \tag{51}$$

Here, as in planetary rings in the limit $\Omega/\omega_{\rm coll} \gg 1$ the collision frequency (surface density, or optical depth for rings) does not appear in the equilibrium equation. However, since we are considering long-range force and nearly inelastic collisions, both our gain and loss terms have different dependences on the velocity dispersion than is the case for planetary rings. In particular, the effective viscosity induced by gravitational interactions in the Galactic disk is at a local maximum when all the GMCs are on perfectly circular orbits.

We have evaluated P and W by directly integrating orbits in the relative coordinate, phase averaging, and integrating over impact parameter. The integrals over phase and impact parameter were evaluated using the trapezoidal rule. Since there are four parameters that characterize the accuracy of the integration, it is diifficult to obtain a simple error estimate. After changing each of these parameters in turn by a factor of 2 toward higher accuracy and obtaining changes that were always substantially less than 1%, we informally estimate that the errors are less than 1%, and this is the origin of the error bars in Figures 9 and 10 below. For all the encounters, we set $\beta = 0$, appropriate for a flat rotation curve, and $r_{\min} = 0.4$, corresponding to twice our estimate for the effective radius of the fiducial cloud. Although we have prevented the closest encounters from contributing to the energy gain per unit time, P is *not* divergent in the limit $r_{\min} \to 0$; rather it tends to a finite limit that is of the same order as the results for $r_{\min} = 0.4$. Indeed, our numerical results and those of Petit & Hénon

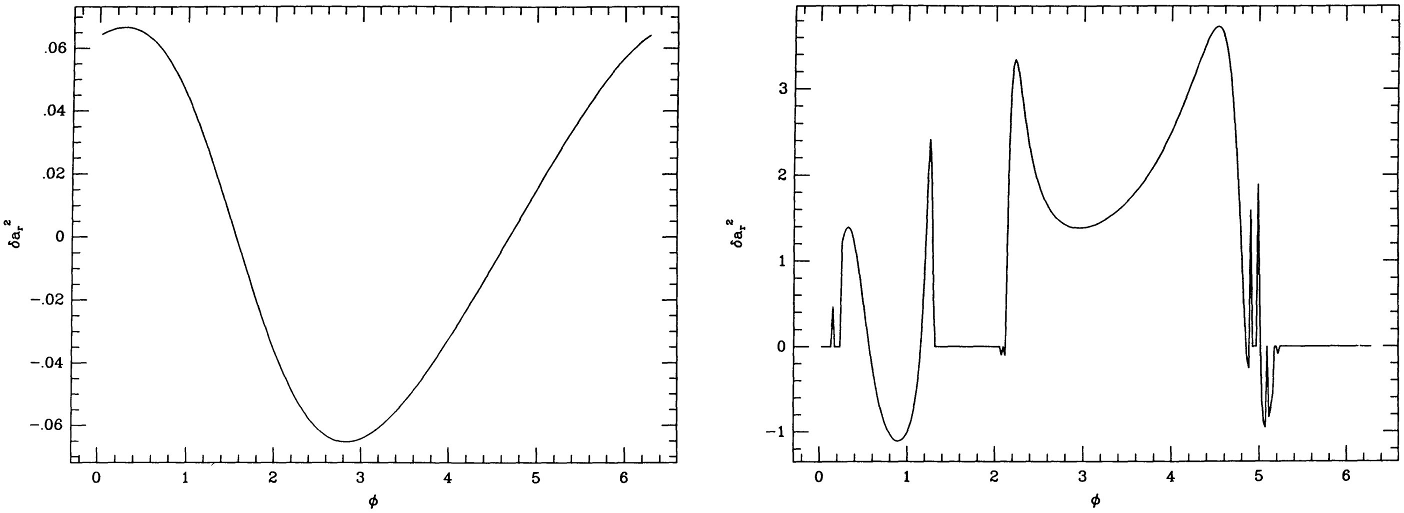

FIG. 5a

FIG. 5b

FIG. 5.—$\Delta a_r^2(\phi)$, the variation in squared epicyclic amplitude over relative epicyclic phase at (*a*) $S = 10.0$ and (*b*) $S = 2.0$ and with $a_r^{(0)2} = 1.5$

(1986) suggest, but of course do not prove, that $\Delta a_r^2(S, \phi)$ is bounded above.

The results of our numerical integrations are presented graphically in Figures 5 through 8. Figures 5*a* and 5*b* show the function $\Delta a^2(\phi_r)$ for a single initial value of the epicyclic amplitude and for two values of the impact parameter. At large impact parameter the function is smooth and sinusoidal over ϕ_r (Fig. 5*a*), while at impact parameters where collisions occur, the function is ill behaved (Fig. 5*b*) due to complex interactions, including trapping and temporary capture, as was noted earlier by Icke (1982, Fig. 4), and Petit & Hénon (1986, Fig. 1). The phases where $\Delta a_r^2 = 0$ in Figure 5*b* correspond to orbits with collisions. Figure 6 shows the variation in $\langle \Delta a^2(S) \rangle$ over S for $a_r^{(0)2} = 1.5$. The large dip in the middle of the curve is due to physical collisions, which give zero increase in the epicyclic energy, under our accounting. Figure 7 shows a scatter of points about the phase-averaged function $\langle \Delta a^2(1.5, S) \rangle$. The points are the change in the squared epicyclic amplitude at various values of the initial epicyclic phase. The variation over phase is an order larger (in ϵ) than the phase average at large impact parameter, so phase randomness must be well satisfied for our calculation to be valid. Figure 8 shows the loss function $\langle \Theta(1.5, S) \rangle$. There is a sharp peak in the collision frequency near $S = 1.8$; for impact parameters less than 1.4, no collisions occur and the clouds repel one another on horseshoe orbits.

The functions $P(a_r^{(0)2})$ and $W(a_r^{(0)2})$ defined by equations (48) and (50) are shown in Figures 9 and 10. A least-squares fit to our numerical integrations in the range $a_r^{(0)2} = [0.9, 1.8]$ yields

$$P(a_r^{(0)}, 0, 0.4) = 2.12 - 0.37 a_r^{(0)2} \tag{52}$$

and

$$W(a_r^{(0)}, 0, 0.4) = 0.87 + 0.13 a_r^{(0)2} \ . \tag{53}$$

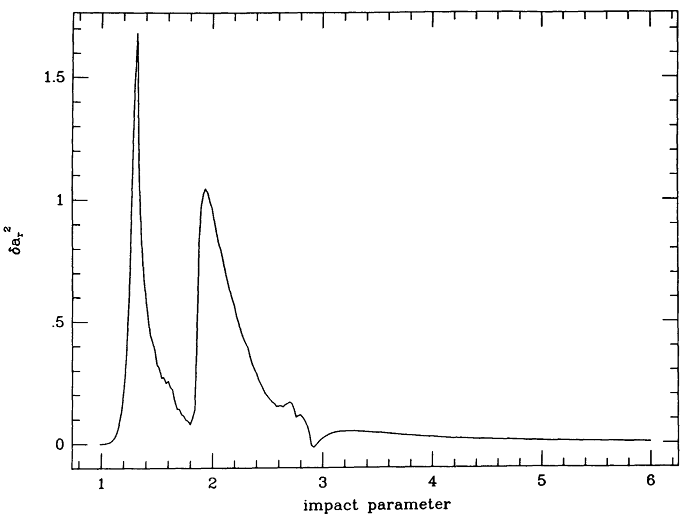

FIG. 6.—$\Delta a_r^2(S)$, the variation in squared epicyclic amplitude over the range in impact parameter $S = [1.0, 6.0]$ for $a_r^{(0)2} = 1.5$.

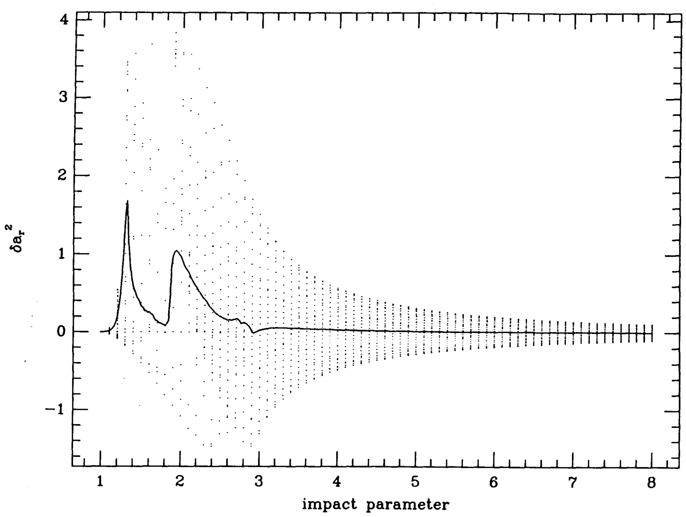

FIG. 7.—$\Delta a_r^2(S)$, the variation in squared epicyclic amplitude over impact parameter with $a_r^{(0)2} = 0.4$. The points plotted in the background show $\Delta a_r^2(S, \phi_r)$ sampled evenly in ϕ_r.

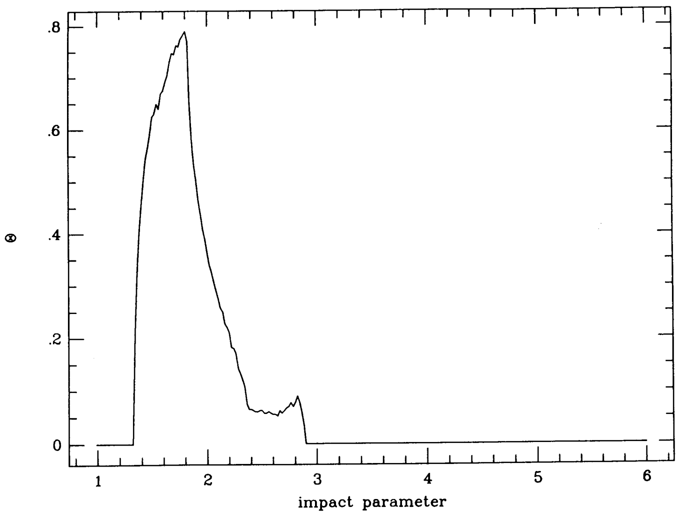

FIG. 8.—$\Theta(S)$ (see eq. [49]), which is proportional to the loss rate and collision frequency per unit impact parameter, at $a_r^{(0)2} = 1.5$.

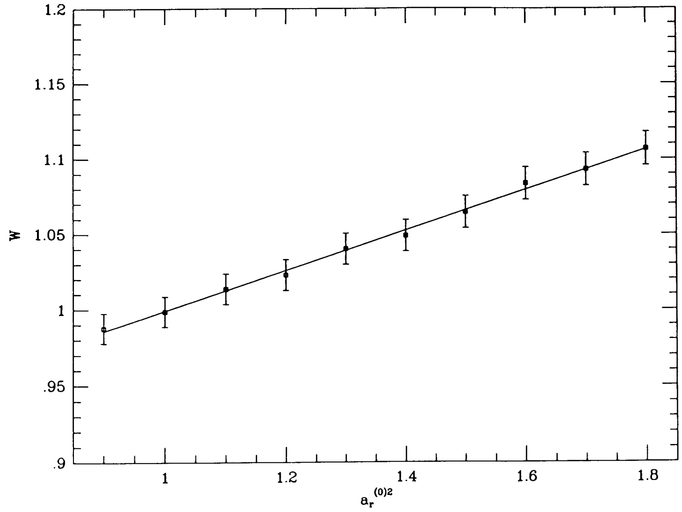

FIG. 10.—$W(a_r^{(0)2})$, defined by eq. (50), proportional to the loss of epicyclic energy in collisions per unit time, according to the simple model for collisions used. The line is the least-squares fit given by eq. (55). The error bars are informal estimates.

Qualitatively, the dependence shown here is what one expects in dynamical problems: the larger the velocity dispersion, the smaller the energy exchange in individual gravitational encounters (eq. [52]) but the larger the loss in physical encounters (eq. [53]).

The solution to the energy balance equation (eq. [51]) is $a_{r,\mathrm{eq}}^{(0)2} = 1.5$. Dependences on cloud size $r_{\mathrm{cl}} = \frac{1}{2} r_{\min}$ and the shape of the rotation curve β, have been suppressed to obtain this result. We can now evaluate the dependence on β and $r_{\min}$ near this equilibrium point. A least-squares fit yields

$$P(1.5, \beta, r_{\min}) \simeq 1.57 - 4.43\beta - 0.65(r_{\min} - 0.4) \quad (54)$$

and

$$W(1.5, \beta, r_{\min}) \simeq 1.07 + 0.65\beta + 0.86(r_{\min} - 0.4) . \quad (55)$$

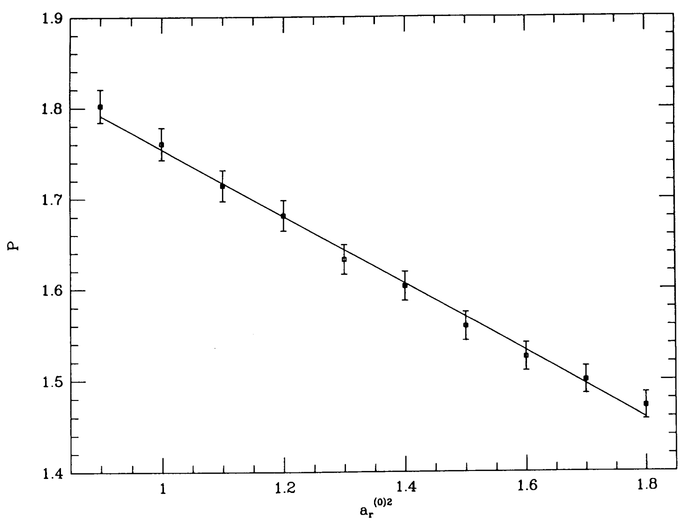

FIG. 9.—$P(a_r^{(0)2})$, defined by eq. (48), proportional to the gain in epicyclic energy of a point mass per unit time interacting gravitationally with a population of clouds such that the relative epicyclic amplitude is typically a_r. The line is the least-squares fit given by eq. (54). The error bars are informal estimates.

Note the steep dependence of the heating rate P on the logarithmic derivative of the rotation curve β. This implies that for a slightly declining rotation curve the heating rate should increase dramatically. As expected, energy losses are greater and gains are less for larger clouds at a fixed value of the epicyclic energy. This requires a compensating decrease in a_r^2 to achieve equilibrium. Quantitatively, we find that on combining equation (51), (54), and (55), and assuming $|\beta| \ll 1$ and $|r_{\min} - 0.4| \ll 1$,

$$a_{r,\mathrm{eq}}^{(0)2} = 1.5 - 5.0\beta - 1.8(r_{\min} - 0.4) . \quad (56)$$

The relative coordinate epicyclic amplitude can be related to the amplitude for individual clouds as follows. From the definition of the relative coordinate and an average over the epicyclic phases of each of the two clouds involved in an encounter,

$$a_{i,\mathrm{eq}}^{(0)2} = \tfrac{1}{2} a_{r,\mathrm{eq}}^{(0)2} . \quad (57)$$

The cloud epicyclic amplitudes are then related to the local, two-dimensional velocity dispersion by

$$\sigma^2 = \sigma_r^2(1 + 1/4\Omega^2) = \tfrac{1}{2} a_{i,\mathrm{eq}}^{(0)2}(1 + 1/4\Omega^2) \quad (58)$$

(e.g., Binney & Tremaine 1987). If $\beta = 0$,

$$\sigma^2 = \tfrac{3}{8} a_{r,\mathrm{eq}}^{(0)2} , \quad (59)$$

so that the equilibrium velocity dispersion $\sigma = 0.75$. In physical units our basic result is then

$$\sigma = 0.94(GM_{\mathrm{cl}}\,\kappa)^{1/3}(1.0 - 1.7\beta - 0.61(r_{\min} - 0.4r_t)) = 5.1 \text{ km s}^{-1} \quad (60)$$

for the planar velocity dispersion of our fiducial cloud of $5 \times 10^5\ M_\odot$ at $R_0/2$, where r_t is the length unit and $0.4r_t = 19$ pc. Note that for a declining rotation curve the shear rate increases, and hence the velocity dispersion, while increasing $r_{\min}$ serves to increase the damping and decrease the velocity dispersion.

4. CONCLUSION

To summarize, we have calculated the phase-averaged change in epicyclic energy in the limit of large impact parameter. The resulting expression is

$$\langle \Delta(E_{\mathrm{epi},i}) \rangle = \frac{m_i m_k^2}{S^4} F(\beta) \tag{61}$$

(see Fig. 3 and eq. [39]). We have also calculated the total power transfer from orbital energy to random, epicyclic energy as a function of initial epicyclic amplitude using direct numerical integration of the orbits. We find that distant encounters do not contribute significantly to the increase in epicyclic energy, nor do encounters at small impact parameter; the most important encounters all occur at impact parameters of a few $\times (GM_{\mathrm{cl}}/\kappa^2)^{1/3}$. Balancing gains from gravitational scattering against losses from direct collisions, we find the simple result that the equilibrium two-dimensional velocity dispersion in our model is

$$\sigma = 0.94(GM_{\mathrm{cl}}\,\kappa)^{1/3}(1.0 - 1.7\beta - 0.61(r_{\min} - 0.4r_t)) = 5.1 \text{ km s}^{-1} \tag{62}$$

in our Galactic disk at $R_0/2$, slightly smaller than the equilibrium value found in JO. The most significant part of this result is not the exact value of the velocity dispersion, but rather the scaling with M_{cl} and κ, and the strong dependence on the slope of the rotation curve.

To the precision of our calculation our results agree with observations. Stark & Brand (1989) measure the rms difference between the cloud's radial velocity and the radial velocity of the LSR at the location of the cloud. They find this to be between 7 and 8 km s^{-1}, corresponding to a two-dimensional velocity dispersion of between 10 and 11 km s^{-1}. This includes local streaming motions. Knapp, Stark, & Wilson (1985) and Clemens (1985) measure the radial velocity dispersion of gas at the tangent point, or the azimuthal velocity dispersion, obtaining 5.6 km s^{-1} and 3 km s^{-1}, respectively. Clemens (1985) has subtracted local streaming motions, and thus obtains a lower velocity dispersion. The azimuthal and planar velocity dispersions are related by

$$\sigma^2 = \sigma_\phi^2(1 + 4\Omega^2) , \tag{63}$$

or

$$\sigma = \sigma_\phi \sqrt{1 + 2/(1 + \beta)} . \tag{64}$$

If the rotation curve is flat at the tangent points, then $\sigma = \sqrt{3}\sigma_\phi$, implying two dimensional velocity dispersions of 9.7 and 5.2 km s^{-1}, respectively.

Some caution is required in applying these results to the Galaxy. Throughout we have assumed that only two-body scattering is important. This is a valid approximation if $\Sigma \ll 1$, where Σ is the dimensionless surface number density expressed in units of $(2GM_{\mathrm{cl}}/\kappa^2)^{-2/3}$. Since this condition is only marginally satisfied at the molecular ring, it is likely that collective effects will enhance the gravitational scattering. A more realistic calculation would allow for a distribution of clouds over mass and velocity, and the finite lifetimes of molecular clouds, but must also include collective effects, possibly magnetic fields, and a detailed model of cloud-cloud collisions.

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APPENDIX

ACCRETION RATES

Accretion in a differentially rotating disk differs from accretion in a homogeneous background in that the accretion rate is controlled not only by the random velocities of particles, but also by the shear rate.

Given the function $W(a_r^{(0)}, \beta, r_{\min})$, it is trivial to calculate the accretion rate for a pressureless fluid. First we set the mass of one of the objects in the encounter to ≈ 0. This implies that we are using a larger effective capture radius (36 pc for our fiducial cloud) for GMC-small cloud encounters than for GMC-GMC encounters. Then the accretion rate is given by

$$\dot{M} = \int_{-\infty}^{\infty} \langle \Theta \rangle 2AS\Sigma_g\, dS = 4A\Sigma_g W , \tag{65}$$

where Σ_g is the surface *mass* density of the interstellar medium. In dimensional form

$$\dot{M} = 4A\Sigma_g r_t^2 W[\mathrm{g\ s^{-1}}] , \tag{66}$$

where $r_t = (GM_{\mathrm{cl}}/\kappa^2)^{1/3}$. This implies a growth time of

$$\frac{M}{\dot{M}} = \frac{M^{1/3}\kappa^{4/3}}{4A\Sigma_g G^{2/3}} (0.93 - 0.57\beta - 0.75(r_{\min}/r_t - 0.4))\ [\mathrm{s}] , \tag{67}$$

where we have included the dependence on $r_{\min}$ and β near the equilibrium point (eq. [55]). For the fiducial cloud, assuming a surface density of $\sim 10\ M_\odot\ \mathrm{pc}^{-2}$, the accretion rate is

$$\dot{M} = 3.3 \times 10^{-3}\ M_\odot\ \mathrm{yr}^{-1}$$

corresponding to a growth time of 1.6×10^8 yr. This estimate agrees with that of Larson (1988). This growth time is an upper limit since the pressure and self-gravity of the accreting medium could substantially enhance the accretion rate.

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