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Let F be a finite extension of Q_p and K a quadratic extension of F. If (Π, V) is a representation of $GL_2(K)$, H a subgroup of $GL_2(K)$ and μ a character of the image subgroup $\det(H)$ of K^*, then Π is said to be μ-distinguished with respect to H if there exists a nonzero linear form l on V such that $l(\Pi(g)v) = \mu(\det g)l(v)$ for $g \in H$ and $v \in V$. We provide new proofs, using entirely local methods, of some well-known results in the theory of non-archimedean distinguished representations for $GL(2)$.

1. Introduction.

Let K/F be a quadratic extension of non-archimedean local fields of characteristic zero. For a local field F, $\mathcal{O}_F$ will be the ring of integers of F and $\mathcal{P}_F$ the maximal ideal of $\mathcal{O}_F$. Let π_F be a generator of $\mathcal{P}_F$. Let v_F be the valuation of F such that $v_F(\pi_F) = 1$. The cardinality of $\mathcal{O}_F/\mathcal{P}_F$ is denoted by q_F. Let σ be the nontrivial element of the Galois group of K over F. By $\omega_{K/F}$ we denote the nontrivial character of $\frac{F^*}{N_{K/F}(K^*)}$, where $N_{K/F}$ is the norm from K to F. Fix a nontrivial additive character ψ of F and set $\psi_K = \psi \circ tr_{K/F}$, where $tr_{K/F}$ is the trace from K to F. Let (Π, V) be an irreducible, admissible representation of $GL_2(K)$ and let ω_Π denote the central character of Π. Then $(\widetilde{\Pi}, \widetilde{V})$ denotes the representation contragredient to Π. The representation Π^σ is defined by $\Pi^\sigma(g) = \Pi(g^\sigma)$ for $g \in GL_2(K)$, where σ acts on g elementwise. For characters λ of K^*, $\gamma(\Pi \otimes \lambda, \psi_K)$ denotes the gamma factor involved in the functional equation for functions in the Kirillov model $K(\Pi, \psi_K)$ of Π with respect to ψ_K. We use the notation $\Pi(\chi_1, \chi_2)$ and $\sum(\chi_1, \chi_2)$ to denote principal series and special representations, respectively, of $GL_2(K)$ (where χ_1,χ_2 are characters of K^*) (see [**10**]) and $\pi(\mu_1, \mu_2)$ and $\sigma(\mu_1, \mu_2)$ for the corresponding representations of $GL_2(F)$. In general, Π would be an irreducible, admissible representation of $GL_2(K)$ and π a similar representation of $GL_2(F)$.

Let μ be a character of F^*. We say that Π is μ-distinguished with respect to $GL_2(F)$ if there exists a nonzero linear form l on the space of Π such that $l(\Pi(g)v) = \mu(\det g)l(v)$ for all $v \in V$ and $g \in GL_2(F)$. By a distinguished representation we mean a 1-distinguished representation. We will also consider distinguishedness with respect to another subgroup of

$GL_2(K)$, namely, the unitary group in two variables $U(2, K/F)$. Recall that $U(2, K/F)$ is the subgroup of fixed points of the involution τ on $GL_2(K)$ given by $g^\tau = w\ {}^t g^{-\sigma}\ w^{-1}$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus for a character η of the group of norm one elements of K^*, η-distinguished representations of $GL_2(K)$ with respect to $U(2, K/F)$ can be defined as above. Sometimes we do not specify the subgroup with respect to which the representation is distinguished if the subgroup is $GL_2(F)$.

We would like to stress that none of the theorems stated in this paper are new. All these are known results whose existing proofs use a mixture of local and global methods. We provide local proofs in those instances where only global proofs exist.

Throughout this paper we consider only infinite dimensional representations. We prove:

Theorem 1.1. *Let μ be a character of F^*. Let Π be an irreducible, admissible representation of $GL_2(K)$ with $\omega_\Pi = \mu \circ N_{K/F}$. Then the following statements are equivalent:*

(1) Π *is a base change lift of a representation of $GL_2(F)$ with central character $\mu\omega_{K/F}$.*
(2) $\gamma(\Pi \otimes \lambda^{-1}, \psi_K)\lambda(-1) = 1$ *for all characters λ of K^* which satisfy $\lambda|_{F^*} = \mu$.*
(3) Π *is μ-distinguished with respect to $GL_2(F)$.*

The equivalence of (2) and (3) follows from Theorem 4.1 of Hakim [**8**]. As has been pointed out by the referee, the global analogue of the equivalence of (1) and (3) is contained in the paper of Harder, Langlands and Rapoport [**9**]. Alternative proofs of the global result appeared in Ye's thesis [**18**], [**19**] and in Flicker [**4**]. The first purely local proof of (1) implies (2) appears in Saito's Corollary 2.4 [**15**]. Modulo the assertion on the central character, (3) implies (1) by a result of Hakim (Theorem 2.1 in [**8**]). To prove the assertion on the central character, we give a new argument which combines Tunnell's formula [**17**] as well as Saito's proof of it. The next theorem brings distinguishedness with respect to the unitary group into the picture.

Theorem 1.2. *Let μ be a character of F^* and Π an irreducible, admissible representation of $GL_2(K)$ with central character $\omega_\Pi = \mu \circ N_{K/F}$. Then the following are equivalent:*

(1) $\Pi \sim \Pi^\sigma$, *i.e.,* Π *is a base change lift from $GL_2(F)$.*

(2) Π *is distinguished with respect to $U(2, K/F)$.*

(3) Π *is μ-distinguished or $\mu\omega_{K/F}$-distinguished.*

Note that the equivalence of (1) and (2) above is vacuously true if ω_Π does not factor through the norm map $N_{K/F}$. This equivalence and the next theorem, which can easily be deduced from the equivalence of (1) and (3) of Theorem 1.2, are, in fact, conjectured to be true in the context of $GL(n)$ and this conjecture has been proved in many cases (cf. [**11**], [**12**] and [**14**]).

Theorem 1.3. *Let* Π *be an irreducible, admissible representation of* $GL_2(K)$ *with* $\omega_\Pi|_{F^*} = 1$. *Then* $\widetilde{\Pi} \sim \Pi^\sigma$ *if and only if* Π *is distinguished or* $\omega_{K/F}$*-distinguished with respect to* $GL_2(F)$.

Formulated in the language of the base change theory for $U(2, K/F)$ [**2**], Theorem 1.3 says that an irreducible, admissible representation of $GL_2(K)$ is distinguished or $\omega_{K/F}$-distinguished with respect to $GL_2(F)$ if and only if it is a base change of a representation of $U(2, K/F)$. There are precisely two base change maps from the class of admissible representations of $U(2, K/F)$ to the class of admissible representations of $GL_2(K)$, namely the stable and unstable base change maps. The following theorem, due to Flicker [**3**, Theorem 7], is thus stronger than Theorem 1.3.

Theorem 1.4. *An irreducible, admissible representation* Π *of* $GL_2(K)$ *is distinguished with respect to* $GL_2(F)$ *if and only if it is an unstable base change lift of a representation of* $U(2, K/F)$.

In the context of $GL(n)$ it is believed that unstable (resp. stable) base change lifts from $U(n, K/F)$ are precisely the distinguished representations with respect to $GL_n(F)$ when n is even (resp. when n is odd) (cf. [**3**]). In [**3**] Flicker deduces Theorem 1.4 from a similar global theorem. In this paper we will give a different proof which is purely local. For principal series representations and special representations of $GL_2(K)$ this will follow from two results due to Flicker and Hakim [**2**, **5**], whereas for supercuspidals, we adopt a method due to Saito [**15**] to get the desired result. Since the most substantial part of this paper deals with the proof of this theorem, we give the main ideas of the proof here.

Corresponding to the quadratic extension K of F we fix an embedding i of K^*/F^* in $U(2, K/F)$ given by

$$i(aF^*) = \begin{pmatrix} x & y \\ \Delta^2 y & x \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ where } a = x + \Delta y \in K^*.$$

If $g = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} -a^{-1} & 0 \\ 0 & 1 \end{pmatrix} w$, then observe that gg^τ (where τ is the involution $g \to w^t g^{-\sigma} w^{-1}$) and $i(aF^*)$ are conjugate in $GL_2(K)$. By means of the base change theory of $U(2, K/F)$, we get two formulae — one in the stable base change case and the other in the unstable base change case — for $\chi_{\{\pi\}}(i(aF^*))$, where $\{\pi\}$ is the packet of representations of $U(2, K/F)$ that base changes to Π and $\chi_{\{\pi\}}$ is the sum of characters of the representations in the packet of π. These formulae must be seen as the analogues of Tunnell's formula for characters of $GL(2)$. Flicker's theorem can be derived as a corollary to the proof of these formulae, just as Saito deduces Proposition 2.1 (Corollary 2.4 in [**15**]) from his proof of Tunnell's formula.

2. Proof of Theorem 1.1.

We will prove Theorem 1.1 through a series of propositions. We start with a result of Saito [**15**, Corollary 2.4].

Proposition 2.1. *Let π be a supercuspidal representation of $GL_2(F)$ with the central character ω_π, and let Π be the base change lift of π to $GL_2(K)$. Then for characters λ of K^* which satisfy $\lambda|_{F^*} = \omega_\pi \omega_{K/F}$, $\epsilon(\Pi \otimes \lambda^{-1}, \psi_K)\lambda(-1)$ is independent of λ.*

The epsilon factor is related to the gamma factor by [**10**]

$$\gamma(\Pi \otimes \lambda^{-1}, \psi_K) = \epsilon(\Pi \otimes \lambda^{-1}, \psi_K) \frac{L(\frac{1}{2}, \widetilde{\Pi} \otimes \lambda)}{L(\frac{1}{2}, \Pi \otimes \lambda^{-1})}.$$

The one dimensional epsilon factors and gamma factors are related by

$$\gamma(\chi, \psi_K) = \epsilon(\chi, \psi_K) \frac{L(\frac{1}{2}, \chi^{-1})}{L(\frac{1}{2}, \chi)}.$$

Our convention for the 1-dim ϵ-factor is the one used by Langlands. Thus for a character χ of F^* and an additive character ψ of F we have $\epsilon(\chi, \psi) = \chi(c)\frac{t}{|t|}$, where $t = \int_{U_F} \chi^{-1}(u)\psi(u/c)\, du$, U_F being the group of units in the ring of integers of F, du a Haar measure on U_F and c an element of F of valuation $a(\chi) + n(\psi)$. Here $a(\chi)$ is the conductoral exponent of χ and $n(\psi)$ the conductoral exponent of ψ. We refer to [**16**] for the basic properties of these ϵ-factors. For instance, we have

(i) $\epsilon(\chi, \psi_K) = \epsilon(\chi^\sigma, \psi_K)$,
(ii) $\epsilon(\chi, \psi_K)\epsilon(\chi^{-1}, \psi_K) = \chi(-1)$.

For the following two properties of ϵ-factors associated to representations of $GL(2)$ refer to [**10**].

(iii) $\epsilon(\Pi, \psi_K)\epsilon(\widetilde{\Pi}, \psi_K) = \omega_\Pi(-1)$.
(iv) $\epsilon(\Pi, (\psi_K)_a) = \omega_\Pi(a)\epsilon(\Pi, \psi_K)$ where $(\psi_K)_a(x) = \psi_K(ax)$.

Also we have

(v) (Frohlich-Queyrut [**6**, Theorem 3].) For any character χ of K^* which is trivial on F^* and any $\Delta \in K^*$ with $tr_{K/F}(\Delta) = 0$, $\epsilon(\chi, \psi_K) = \chi(\Delta)$.

Now we claim that the value of $\epsilon(\Pi \otimes \lambda^{-1}, \psi_K)\lambda(-1)$ in Proposition 2.1 is precisely 1. To prove this, write $\omega_\Pi = \mu_1\mu_2$, where μ_1 and μ_2 are characters of K^* such that $\mu_i|_{F^*} = \omega_\pi\omega_{K/F}, (i = 1, 2)$. Then for characters λ of K^* with very small conductors which satisfy $\lambda|_{F^*} = \omega_\pi\omega_{K/F}$, we have

$$\epsilon(\Pi \otimes \lambda^{-1}, \psi_K)\lambda(-1) = \epsilon(\mu_1\lambda^{-1}, \psi_K)\epsilon(\mu_2\lambda^{-1}, \psi_K)\lambda(-1)$$

by a result of Jacquet and Langlands [**10**, Proposition 3.8, p. 116]. Note that $\mu_i\lambda^{-1}|_{F^*} = 1$ for $i = 1, 2$. Thus, by Property (v) above, $\epsilon(\mu_i\lambda^{-1}, \psi_K) = \mu_i\lambda^{-1}(\Delta)$, where Δ is any trace zero element of K. Thus

$$\begin{aligned}\epsilon(\Pi \otimes \lambda^{-1}, \psi_K)\lambda(-1) &= \mu_1\lambda^{-1}(\Delta)\mu_2\lambda^{-1}(\Delta)\lambda(-1)\\ &= \mu_1\mu_2(\Delta)\lambda^{-1}(-\Delta^2)\\ &= \omega_{\Pi}(\Delta)\lambda^{-1}(N_{K/F}(\Delta))\\ &= \omega_{\Pi}(\Delta){\omega_\pi}^{-1}(N_{K/F}(\Delta))\\ &= 1\end{aligned}$$

whenever $\lambda|_{F^*} = \omega_\pi\omega_{K/F}$ and the conductor of λ is sufficiently small. This proves our claim. If Π is principal or special, then $\epsilon(\Pi \otimes \lambda^{-1}, \psi_K)\lambda(-1) = 1$ whenever $\lambda|_{F^*} = \omega_\pi\omega_{K/F}$ can be proved by a direct epsilon factor computation, and the condition that it be independent of λ is not needed. For then either $\Pi = \Pi(\chi_1, \chi_2)$, where χ_1 and χ_2 are characters of K^* such that $\chi_i = {\chi_i}^\sigma (i = 1, 2)$, or $\Pi = \sum(\chi_1, \chi_2)$, where $\chi_1{\chi_2}^{-1} = |\ |_K$ and if $\chi = \chi_1|\ |_K^{-1/2} = \chi_2|\ |_K^{1/2}$, then $\chi = \mu \circ N_{K/F}$ for a character μ of F^*. Now consider $\epsilon(\Pi \otimes \lambda^{-1}, \psi_K)\lambda(-1)$ for characters λ of K^* with $\lambda|_{F^*} = \omega_\pi\omega_{K/F}$. The condition on λ means that $\lambda\lambda^\sigma = \omega_{\Pi}$. If Π is the principal series representation considered here or the special representation with $\chi\lambda^{-1}$ ramified, then the $GL(2)$ ϵ-factor factorises into the $GL(1)$ factors as follows:

$$\epsilon(\Pi \otimes \lambda^{-1}, \psi_K) = \epsilon(\chi_1\lambda^{-1}, \psi_K)\epsilon(\chi_2\lambda^{-1}, \psi_K).$$

We have $\chi_2\lambda^{-1} = {\chi_1}^{-1}\lambda^\sigma$ and hence

$$\epsilon(\chi_2\lambda^{-1}, \psi_K) = \epsilon({\chi_1}^{-1}\lambda^\sigma, \psi_K) = \epsilon({\chi_1}^{-\sigma}\lambda, \psi_K) = \epsilon({\chi_1}^{-1}\lambda, \psi_K),$$

since $\chi_1 = {\chi_1}^\sigma$. Therefore,

$$\begin{aligned}\epsilon(\Pi \otimes \lambda^{-1}, \psi_K)\lambda(-1) &= \epsilon(\chi_1\lambda^{-1}, \psi_K)\epsilon({\chi_1}^{-1}\lambda, \psi_K)\lambda(-1)\\ &= \chi_1\lambda^{-1}(-1)\lambda(-1) = \chi_1(-1) = 1.\end{aligned}$$

If $\Pi = \sum(\chi_1, \chi_2)$ with $\chi\lambda^{-1}$ unramified, then K/F is necessarily unramified since $\chi\lambda^{-1}|_{F^*} = \omega_{K/F}$ in this situation. So we can take π_K to be π_F itself. Note that $\omega_{K/F}(\pi_F) = -1$. Now we have the factorisation (see [**10**, p. 109])

$$\epsilon(\Pi \otimes \lambda^{-1}, \psi_K) = \epsilon(\chi_1\lambda^{-1}, \psi_K)\epsilon(\chi_2\lambda^{-1}, \psi_K)\frac{L(\frac{1}{2}, {\chi_1}^{-1}\lambda)}{L(\frac{1}{2}, \chi_2\lambda^{-1})}.$$

Therefore

$$\begin{aligned}\epsilon(\Pi\otimes\lambda^{-1},\psi_K)\lambda(-1) &= \epsilon(\chi_1\lambda^{-1},\psi_K)\epsilon(\chi_2\lambda^{-1},\psi_K)\frac{1-\omega_{K/F}(\pi_F)}{1-\omega_{K/F}(\pi_F)}\lambda(-1)\\ &= \epsilon(\chi_1\lambda^{-1},\psi_K)\epsilon(\chi_2\lambda^{-1},\psi_K)\lambda(-1)\\ &= \chi\lambda^{-1}(-1)\lambda(-1) = \chi(-1)\\ &= 1.\end{aligned}$$

If Π is a base change lift of a representation π of $GL_2(F)$, then $\Pi \sim \Pi^\sigma$ and therefore, $\widetilde{\Pi}\otimes\lambda = (\Pi\otimes\lambda^{-1})^\sigma$ when $\lambda|_{F^*} = \omega_\pi\omega_{K/F}$. Hence $L(\frac{1}{2},\widetilde{\Pi}\otimes\lambda) = L(\frac{1}{2},\Pi\otimes\lambda^{-1})$. Thus it follows that in our situation $\epsilon(\Pi\otimes\lambda^{-1},\psi_K)\lambda(-1) = \gamma(\Pi\otimes\lambda^{-1},\psi_K)\lambda(-1)$. Now if Π is an irreducible, admissible representation of $GL_2(K)$ with $\omega_\Pi = \mu\circ N_{K/F}$, and if $\Pi\sim\Pi^\sigma$, then Π is a base change lift of a representation π of $GL_2(F)$ and ω_π can be either μ or $\mu\omega_{K/F}$. Thus from the preceding discussion we get:

Proposition 2.2. *Let μ be a character of F^* and let Π be an irreducible, admissible representation of $GL_2(K)$ with $\omega_\Pi = \mu\circ N_{K/F}$ such that $\Pi\sim\Pi^\sigma$. Then $\gamma(\Pi\otimes\lambda^{-1},\psi_K)\lambda(-1) = 1$ for all characters λ of K^* with $\lambda|_{F^*} = \mu$ or $\gamma(\Pi\otimes\lambda^{-1},\psi_K)\lambda(-1) = 1$ for all characters λ of K^* with $\lambda|_{F^*} = \mu\omega_{K/F}$.*

Next we prove:

Proposition 2.3. *Let μ be a character of F^* and let Π be an irreducible, admissible representation of $GL_2(K)$ with $\omega_\Pi = \mu\circ N_{K/F}$. Then Π is μ-distinguished with respect to $GL_2(F)$ if and only if $\gamma(\Pi\otimes\lambda^{-1},\psi_K)\lambda(-1) = 1$ for all characters λ of K^* with $\lambda|_{F^*} = \mu$.*

Proof. This is immediate from a result of Hakim [**8**, Theorem 4.1] which states that a representation Π of $GL_2(K)$ with trivial central character is distinguished if and only if $\gamma(\Pi\otimes\lambda^{-1},(\psi_K)_\Delta) = 1$ for all characters λ of K^* with $\lambda|_{F^*} = 1$. Though Hakim assumes that $\omega_\Pi = 1$, the same proof works under the milder condition $\omega_\Pi|_{F^*} = 1$. Here $(\psi_K)_\Delta$ is the additive character of K given by $(\psi_K)_\Delta(x) = \psi_K(\Delta x)$. Suppose Π is an irreducible, admissible representation of $GL_2(K)$ with $\omega_\Pi = \mu\circ N_{K/F}$. Let $\widetilde{\mu}$ be a character of K^* such that $\widetilde{\mu}|_{F^*} = \mu$. Now Π is μ-distinguished if and only if $\Pi\otimes\widetilde{\mu}^{-1}$ is distinguished. Note that $\omega_{\Pi\otimes\widetilde{\mu}^{-1}}|_{F^*} = 1$. Thus, by the result of Hakim, Π is distinguished if and only if $\gamma(\Pi\otimes\widetilde{\mu}^{-1}\lambda^{-1},(\psi_K)_\Delta) = 1$ for all characters λ of K^* which satisfy $\lambda|_{F^*} = 1$, i.e., Π is μ-distinguished if and only if

$\gamma(\Pi\otimes\lambda^{-1},(\psi_K)_\Delta)=1$ for all characters λ of K^* with $\lambda|_{F^*}=\mu$. Also

$$\begin{aligned}\gamma(\Pi\otimes\lambda^{-1},(\psi_K)_\Delta) &= \omega_{\Pi\otimes\lambda^{-1}}(\Delta)\gamma(\Pi\otimes\lambda^{-1},\psi_K)\\ &= (\omega_\Pi\lambda^{-2})(\Delta)\gamma(\Pi\otimes\lambda^{-1},\psi_K)\\ &= \mu(N_{K/F}(\Delta))\lambda(-1)\lambda^{-1}(N_{K/F}(\Delta))\gamma(\Pi\otimes\lambda^{-1},\psi_K)\\ &= \gamma(\Pi\otimes\lambda^{-1},\psi_K)\lambda(-1)\end{aligned}$$

and the proposition follows.

Proposition 2.4. *Let μ be a character of F^*, and let Π be an irreducible, admissible representation of $GL_2(K)$ with $\omega_\Pi=\mu\circ N_{K/F}$. Suppose Π is μ-distinguished. Then Π is a base change lift of a representation of $GL_2(F)$ with central character $\mu\omega_{K/F}$.*

Proof. This also follows from a result due to Hakim which says that $\widetilde{\Pi}\sim\Pi^\sigma$ for a distinguished representation Π [**8**, Theorem 2.1], [**3**, Prop. 12]. Now let Π be μ-distinguished. Then $\Pi\otimes\widetilde{\mu}^{-1}$is distinguished, where $\widetilde{\mu}$ is an extension of μ to K^* and hence $(\widetilde{\Pi\otimes\widetilde{\mu}^{-1}})\sim(\Pi\otimes\widetilde{\mu}^{-1})^\sigma$, i.e., $\widetilde{\Pi}\sim\Pi^\sigma\otimes(\widetilde{\mu}\widetilde{\mu}^\sigma)^{-1}$. But $\widetilde{\Pi}\sim\Pi\otimes{\omega_\Pi}^{-1}$ and $\omega_\Pi=\mu\circ N_{K/F}=\widetilde{\mu}\widetilde{\mu}^\sigma$. Thus it follows that $\Pi\sim\Pi^\sigma$.

What remains to be proved is the assertion on the central character. To this end we will make use of two results which we state now. The first one is due to Flicker and Hakim [**5**, Proposition B17] and the second is Tunnell's formula for characters of $GL(2)$ proved by Saito in full generality [**15**, **17**].

Theorem 2.5. *The principal series representation $\Pi(\chi,\chi^{-\sigma})$ of $GL_2(K)$ is distinguished (and $\omega_{K/F}$-distinguished). The principal series representation $\Pi(\chi_1,\chi_2)$, $\chi_1\neq\chi_2$ is distinguished precisely when $\chi_i|_{F^*}=1$ $(i=1,2)$. The special representation $\sigma(\chi|\ |_K^{1/2},\chi|\ |_K^{-1/2})$ is distinguished precisely when $\chi|_{F^*}=\omega_{K/F}$.*

Fix an embedding of K^* in $GL_2(F)$.

Theorem 2.6. *Let π be an irreducible, admissible representation of $GL_2(F)$ and χ_π its character. Let Π be the base change lift of π to K. Then*

$$\chi_\pi|_{(K^*-F^*)}=\sum_{\lambda|_{F^*}=\omega_\pi}\frac{1+\epsilon(\Pi\otimes\lambda^{-1},\psi_K)\lambda(-1)}{2}\lambda$$

where the summation on the right is by partial sums over all characters of K^* of conductoral exponent $\leq n$.

In order to prove our assertion on the central character for the principal series and special representations of $GL_2(K)$, we need to show the following:

(i) The principal series representation $\Pi(\chi,\chi^\sigma)$ $(\chi\neq\chi^\sigma)$of $GL_2(K)$ is $\chi|_{F^*}$-distinguished and not $\chi|_{F^*}\omega_{K/F}$- distinguished. (This is because

$\Pi(\chi, \chi^\sigma)$ is the base change lift of a supercuspidal representation with central character $\chi|_{F^*}\omega_{K/F}$.)

(ii) The principal series representation $\Pi(\chi_1, \chi_2)$ of $GL_2(K)$ with $\chi_1 = \mu_1 \circ N_{K/F}, \chi_2 = \mu_2 \circ N_{K/F}$ (where μ_1, μ_2 are characters of F^*) is $\mu_1\mu_2$-distinguished as well as $\mu_1\mu_2\omega_{K/F}$-distinguished.

(iii) The representation $\sum(\chi| \,|_K^{1/2}, \chi| \,|_K^{-1/2})$ with $\chi = \mu \circ N_{K/F}$ (where μ is a character of F^*) is $\mu^2\omega_{K/F}$-distinguished and not μ^2-distinguished.

$\Pi(\chi, \chi^\sigma) \otimes \chi^{-1} = \Pi(1, \chi^{-1}\chi^\sigma)$ is distinguished by Theorem 2.5. Now take an extension $\widetilde{\omega_{K/F}}$ of $\omega_{K/F}$ to K^* and consider $\Pi(\chi, \chi^\sigma) \otimes \chi^{-1}\widetilde{\omega_{K/F}}^{-1}$. This is $\Pi(\widetilde{\omega_{K/F}}^{-1}, \chi^{-1}\chi^\sigma\widetilde{\omega_{K/F}}^{-1})$ and since the restriction to F^* of these two characters are not trivial, it follows by Theorem 2.5 that $\Pi(\chi, \chi^\sigma) \otimes \chi^{-1}\widetilde{\omega_{K/F}}^{-1}$ is not distinguished, or equivalently, $\Pi(\chi, \chi^\sigma)$ is not $\chi|_{F^*}\omega_{K/F}$-distinguished.

If $\Pi = \Pi(\chi_1, \chi_2)$ with $\chi_1 = \mu_1 \circ N_{K/F}, \chi_2 = \mu_2 \circ N_{K/F}$ (μ_1, μ_2 are characters of F^*), then for characters λ of K^* with $\lambda|_{F^*} = \mu_1\mu_2$,

$$\begin{aligned}
\gamma(\Pi \otimes \lambda^{-1}, \psi_K)\lambda(-1) &= \gamma(\chi_1\lambda^{-1}, \psi_K)\gamma(\chi_2\lambda^{-1}, \psi_K)\lambda(-1) \\
&= \gamma(\chi_1\lambda^{-1}, \psi_K)\gamma({\chi_1}^{-1}\lambda, \psi_K)\lambda(-1) \\
&= \chi_1\lambda^{-1}(-1)\lambda(-1) \\
&= \chi_1(-1) = 1
\end{aligned}$$

since $\chi_2\lambda^{-1} = {\chi_1}^{-1}\lambda^\sigma$ and $\chi_1 = {\chi_1}^\sigma$. The same argument works if we take λ such that $\lambda|_{F^*} = \mu_1\mu_2\omega_{K/F}$. Thus Π is both $\mu_1\mu_2$- distinguished and $\mu_1\mu_2\omega_{K/F}$-distinguished by Proposition 2.3.

For a character $\widetilde{\mu}$ of K^*, the special representation $\sum(\chi| \,|_K^{1/2}, \chi| \,|_K^{-1/2}) \otimes \widetilde{\mu}^{-1}$ is distinguished precisely when $\chi\widetilde{\mu}^{-1}|_{F^*} = \omega_{K/F}$ by Theorem 2.5, i.e., $\sum(\chi| \,|_K^{1/2}, \chi| \,|_K^{-1/2}) \otimes \widetilde{\mu}^{-1}$ is distinguished precisely when $\widetilde{\mu}|_{F^*} = \chi|_{F^*}\omega_{K/F} = \mu^2\omega_{K/F}$. Thus $\sum(\chi| \,|_K^{1/2}, \chi| \,|_K^{-1/2})$ is $\mu^2\omega_{K/F}$-distinguished and not μ^2-distinguished.

Now suppose that Π is a supercuspidal representation of $GL_2(K)$ with $\omega_\Pi = \mu \circ N_{K/F}$ which is μ-distinguished. We must show that Π is a base change lift of a representation of $GL_2(F)$ with central character $\mu\omega_{K/F}$. By what has already been shown, Π is a base change lift of a representation of $GL_2(F)$ (say π). Since $\omega_\Pi = \mu \circ N_{K/F}$, ω_π can be either μ or $\mu\omega_{K/F}$. What we need to show is that $\omega_\pi = \mu\omega_{K/F}$ and not μ. This will follow from Saito's proof of Tunnell's formula [**15**]. Using the relation $\omega_\Pi = \mu \circ N_{K/F}$

in Saito's proof, we will finally get the identity

$$\chi_\pi(a) = \sum_{\lambda|_{F^*}=\mu} \frac{1+\epsilon(\Pi\otimes\lambda^{-1},\psi_K)\lambda(-1)}{2}\lambda(a) + \sum_{\lambda|_{F^*}=\mu\omega_{K/F}} \frac{1+\epsilon(\Pi\otimes\lambda^{-1},\psi_K)\lambda(-1)}{2}\lambda(a)$$

where χ_π is the character of π and $a \in K^* - F^*$.

Since γ-factor and ϵ- factor are the same for supercuspidals, and since Π is given to be μ- distinguished, $\epsilon(\Pi\otimes\lambda^{-1},\psi_K)\lambda(-1) = 1$ for all characters λ of K^* with $\lambda|_{F^*} = \mu$ by Proposition 2.3. Thus the first sum in the above identity vanishes and we get

$$\chi_\pi|_{(K^*-F^*)} = \sum_{\lambda|_{F^*}=\mu\omega_{K/F}} \frac{1+\epsilon(\Pi\otimes\lambda^{-1},\psi_K)\lambda(-1)}{2}\lambda.$$

Comparing with Tunnell's formula (Theorem 2.6), we have $\omega_\pi = \mu\omega_{K/F}$. This finishes the proof of Proposition 2.4.

Theorem 1.1 follows from the above propositions.

3. Proofs of Theorems 1.2 and 1.3.

We now prove that Statements (2) and (3) in Theorem 1.2 are equivalent. Recall that

$$U(2,K/F) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K) \mid w\ {}^tg^{-\sigma}\ w^{-1} = g \right\}$$

where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $g^\sigma = \begin{pmatrix} a^\sigma & b^\sigma \\ c^\sigma & d^\sigma \end{pmatrix}$. Thus

$$U(2,K/F) = \left\{ g \in GL_2(K) \mid \frac{1}{\sigma(\det g)} g^\sigma = g \right\}$$

and the centre of $U(2,K/F)$ is

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in GL_2(K) \mid N_{K/F}(a) = 1 \right\}.$$

Therefore, if a representation Π of $GL_2(K)$ is distinguished with respect to $U(2,K/F)$, ω_Π factors through the norm map $N_{K/F}$. Define $GL_2^+(F)$ to be the subgroup of $GL_2(F)$ consisting of matrices whose determinant lies in $N_{K/F}K^*$. We observe that $Z(GL_2(K))GL_2^+(F) = Z(GL_2(K))U(2,K/F)$, where $Z(GL_2(K))$ is the centre of $GL_2(K)$. Hence if (Π,V) is μ-distinguished, and l is a nonzero linear functional on V such that $l(\Pi(g)v) = \mu(\det g)l(v)$ for $g \in GL_2(F)$, then $l(\Pi(g)v) = l(v)$ for $g \in U(2,K/F)$. The case when Π is $\mu\omega_{K/F}$-distinguished is similar. Conversely, if Π is distinguished for $U(2,K/F)$, and l is a nonzero linear functional on the space of Π such that $l(\Pi(g)v) = l(v)$ for $g \in U(2,K/F)$, then $l(\Pi(g)v) = \mu(\det g)l(v)$ for $g \in GL_2^+(F)$. We define a linear functional l' on the space of Π by

$$l'(v) = l(v) + \mu^{-1}(a)l\left(\Pi\left(\left(\begin{array}{cc} a & 0 \\ 0 & 1 \end{array}\right)\right)v\right)$$

where $a \in F^* - N_{K/F}K^*$. Then it is easy to check that $l'(\Pi(g)v) = \mu(\det g)l'(v)$ for $g \in GL_2(F)$. Thus if $l' \neq 0$, then Π is μ-distinguished. If $l' = 0$ then

$$l\left(\Pi\left(\left(\begin{array}{cc} a & 0 \\ 0 & 1 \end{array}\right)\right)v\right) = \mu\omega_{K/F}(a)l(v)$$

and so Π is $\mu\omega_{K/F}$-distinguished.

The above together with the proof of Theorem 1.1 completes the proof of Theorem 1.2.

We now prove Theorem 1.3. Let Π be an irreducible, admissible representation of $GL_2(K)$ with $\omega_\Pi|_{F^*} = 1$. Since $\omega_\Pi|_{F^*} = 1$, we have $\omega_\Pi = \eta^{-1}\eta^\sigma$ for a character η of K^* (by Hilbert 90). Note that

$$\begin{aligned} \omega_{\Pi\otimes\eta} &= \omega_\Pi \eta^2 \\ &= \eta\eta^\sigma \\ &= \eta|_{F^*} \circ N_{K/F}. \end{aligned}$$

Hence

$$\begin{aligned} \Pi \text{ is dist. or } \omega_{K/F}\text{-dist.} &\Leftrightarrow \Pi\otimes\eta \text{ is } (\eta|_{F^*})\text{-dist. or } (\eta|_{F^*})\omega_{K/F}\text{-dist.} \\ &\Leftrightarrow \Pi\otimes\eta \text{ is a base change lift for } GL_2(F) \\ &\Leftrightarrow (\Pi\otimes\eta)^\sigma \sim \Pi\otimes\eta \\ &\Leftrightarrow \Pi^\sigma \sim \Pi\otimes\omega_\Pi^{-1} \\ &\Leftrightarrow \Pi^\sigma \sim \widetilde{\Pi}. \end{aligned}$$

If a supercuspidal Π is a base change of π and π', then $\pi' = \pi \otimes \omega_{K/F}$. In particular, $\omega_{\pi'} = \omega_\pi$. Therefore, from the above argument we conclude that Π is either distinguished (when $\Pi\otimes\eta$ is a base change of a representation of central character $\eta|_{F^*}\omega_{K/F}$) or $\omega_{K/F}$-distinguished (when $\Pi\otimes\eta$ is a base change of a representation of central character $\eta|_{F^*}$), but not both. This discussion together with Theorem 2.5 proves the following proposition.

Proposition 3.1. *Let Π be an irreducible, admissible representation of $GL_2(K)$. Then Π is both μ-distinguished and $\mu\omega_{K/F}$-distinguished with respect to $GL_2(F)$ exactly when $\Pi = \Pi(\mu_1 \circ N_{K/F}, \mu_2 \circ N_{K/F})$ for some characters μ_1 and μ_2 of F^* with $\mu = \mu_1\mu_2$.*

4. A local proof of Flicker's Theorem.

First let us view Theorem 1.3 in the language of base change theory for $U(2, K/F)$. The local base change lift is defined in [**2**] in terms of character identities and the existence of this lifting is proved there. The image of the base change map from the class of admissible representations of $U(2, K/F)$ to the class of admissible representations of $GL_2(K)$ consists of τ-invariant Π, where we recall that τ is the involution $g \to w^t g^{-\sigma} w^{-1}$ of $GL_2(K)$. The central character of any irreducible, admissible representation Π of $GL_2(K)$, which is in the image of the base change map, is trivial on F^*. If Π is τ-invariant and $\omega_\Pi|_{F^*} = 1$, then Π is obtained as the base change of a unique L-packet of $U(2, K/F)$. This L-packet consists of one or two irreducible, admissible representations of $U(2, K/F)$. If Π is an admissible representation of $GL_2(K)$ such that $\Pi \sim \Pi^\tau$, then take an intertwining operator between the spaces of Π and Π^τ, and use this operator to extend Π to the semi direct product $GL_2(K) \times \mathrm{Gal}(K/F)$, where $\mathrm{Gal}(K/F)$ acts on $GL_2(K)$ by $\sigma.g = g^\tau$. Let $\chi_{\Pi,\sigma}$ denote the character of this extended representation. There are precisely two base change maps - stable and unstable - from the class of admissible representations of $U(2, K/F)$ to the class of admissible representations of $GL_2(K)$. Let $\widetilde{\omega_{K/F}}$ be an extension of $\omega_{K/F}$ to K^*. We say that Π is a stable base change of a representation π of $U(2, K/F)$ if

$$\chi_{\Pi,\sigma}(g) = \chi_{\{\pi\}}(gg^\tau)$$

whenever g is such that gg^τ is regular in $U(2, K/F)$. Here $\{\pi\}$ is the L-packet of π and $\chi_{\{\pi_1,\pi_2\}} = \chi_{\pi_1} + \chi_{\pi_2}$. The character $\chi_{\{\pi\}}$ depends only on the conjugacy class of gg^τ in $GL_2(K)$. Further, Π is said to be an unstable base change lift of π if

$$\chi_{\Pi,\sigma}(g) = \widetilde{\omega_{K/F}}(\det g)\chi_{\{\pi\}}(gg^\tau)$$

for all $g \in GL_2(K)$ with gg^τ regular in $U(2, K/F)$.

Thus Theorem 1.3 can be reformulated as follows.

Theorem 4.1. *Let* Π *be an irreducible, admissible representation of* $GL_2(K)$*. Then* Π *is a base change lift of a representation of* $U(2, K/F)$ *if and only if* Π *is distinguished or* $\omega_{K/F}$*-distinguished with respect to* $GL_2(F)$*.*

But something more is true. In the above theorem, distinguished representations with respect to $GL_2(F)$ correspond to the unstable base change lift from $U(2, K/F)$, and the $\omega_{K/F}$-distinguished representations with respect to $GL_2(F)$ correspond to the stable base change lift from $U(2, K/F)$. This is proved by Flicker [**3**, Theorem 7] by global means. We produce a purely local proof of this theorem. Our proof essentially imitates Saito's proof of Tunnell's formula [**15**].

We start with a proposition (cf. [**3**, p. 161], [**2**, p. 717]).

Proposition 4.2.

(i) *The principal series representation* $\Pi(\chi, \chi^{-\sigma})$ *of* $GL_2(K)$ *is in the image of both the stable and the unstable base change maps.*
(ii) *The principal series representation* $\Pi(\chi_1, \chi_2)$ *with* $\chi_1 \neq \chi_2$ *and* $\chi_i|_{F^*} = 1$ $(i = 1, 2)$ *is in the image of the unstable base change map and it is not obtained by the stable lifting.*
(iii) *The special representation* $\sigma(\chi| \,|_K^{1/2}, \chi| \,|_K^{-1/2})$ *is obtained through the unstable lifting precisely when* $\chi|_{F^*} = \omega_{K/F}$.

Comparing this proposition with Theorem 2.1, we see that Theorem 1.4 is verified for principal series and special representations. We need to verify Theorem 1.4 for supercuspidal representations of $GL_2(K)$.

Let Π be a supercuspidal representation of $GL_2(K)$ which is a base change lift of a representation of $U(2, K/F)$. Then $\Pi \sim \Pi^\tau$ and $\omega_\Pi|_{F^*} = 1$, i.e., $\Pi \sim \omega_\Pi \otimes \Pi^\sigma$ and $\omega_\Pi|_{F^*} = 1$. So by the uniqueness of the Kirillov model, $K(\Pi, \psi_K) = K(\omega_\Pi \otimes \Pi^\sigma, \psi_K)$. Note that I_σ, defined on the Kirillov model $K(\Pi, \psi_K)$ of Π by $I_\sigma f(x) = \omega_\Pi(x) f(x^\sigma)$, gives an intertwining operator from $(\Pi, K(\Pi, \psi_K))$ to $(\omega_\Pi \otimes \Pi^\sigma, K(\Pi, \psi_K))$. Also $I_\sigma^2 =$ identity since $\omega_\Pi|_{F^*} = 1$ and $\Pi(\sigma.h) = I_\sigma \Pi(h) I_\sigma$. We extend Π to $GL_2(K) \times \mathrm{Gal}(K/F)$ by $\Pi(g, \sigma) = \Pi(g) I_\sigma$. Now we compute the value of the twisted character $\chi_{\Pi,\sigma}$ at

$$g = \begin{pmatrix} 0 & -a^{-1}\Delta \\ -\Delta & 0 \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} -a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $a \in K^*, \Delta \in K^*$ such that $tr_{K/F}(\Delta) = 0$.

Since Π is a supercuspidal representation, $K(\Pi, \psi_K)$ coincides with the space of Schwartz-Bruhat functions $\mathcal{S}(K^*)$ on K^*, and a basis of this space is given by the set of following functions

$$\xi_\lambda^{(n)}(x) = \begin{cases} \lambda(x) & \text{if } v_K(x) = -n \\ 0 & \text{otherwise.} \end{cases}$$

Here n varies over all integers and λ varies over a complete set of representatives of all characters of K^* modulo $\sim$, where $\lambda_1 \sim \lambda_2$ if and only if $\lambda_1 \lambda_2^{-1}$ is unramified. We have the lemma [**15**, Lemma 2.1]:

Lemma 4.3. $\Pi(w)\xi_\lambda^{(n)} = \epsilon(\Pi \otimes \lambda^{-1}, \psi_K)\xi^{(m)}_{\omega_\Pi \lambda^{-1}}$ *where* $m = f(\Pi \otimes \lambda^{-1}) + 2n(\psi_K) - n$.

Here $f(\Pi \otimes \lambda^{-1})$ and $n(\psi_K)$ denote the conductoral exponents of $\Pi \otimes \lambda^{-1}$ and ψ_K respectively.

Using this lemma we compute $\Pi(g)I_\sigma\xi_\lambda^{(n)}$.

$$\begin{aligned}\Pi(g)I_\sigma\xi_\lambda^{(n)} &= \Pi\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\Pi\begin{pmatrix}-a^{-1} & 0\\ 0 & 1\end{pmatrix}\Pi(w)I_\sigma\xi_\lambda^{(n)}\\ &= \Pi\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\Pi\begin{pmatrix}-a^{-1} & 0\\ 0 & 1\end{pmatrix}\Pi(w)\xi_{\omega_\Pi\lambda^\sigma}^{(n)}\\ &= \epsilon(\Pi\otimes{\omega_\Pi}^{-1}\lambda^{-\sigma},\psi_K)\Pi\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\Pi\begin{pmatrix}-a^{-1} & 0\\ 0 & 1\end{pmatrix}\xi_{\lambda^{-\sigma}}^{(m)}\\ &\qquad(\text{where } m = f(\Pi\otimes{\omega_\Pi}^{-1}\lambda^{-\sigma}) + 2n(\psi_K) - n\ )\\ &= \omega_\Pi(\Delta)\epsilon(\Pi\otimes{\omega_\Pi}^{-1}\lambda^{-\sigma},\psi_K)\lambda^{-\sigma}(-a^{-1})\xi_{\lambda^{-\sigma}}^{(m-v_K(a))}.\end{aligned}$$

But $\Pi\otimes{\omega_\Pi}^{-1}\lambda^{-\sigma} \sim \widetilde{\Pi}\otimes\lambda^{-\sigma} \sim \Pi^\sigma\otimes\lambda^{-\sigma} = (\Pi\otimes\lambda^{-1})^\sigma$. Therefore, $\epsilon(\Pi\otimes{\omega_\Pi}^{-1}\lambda^{-\sigma},\psi_K) = \epsilon(\Pi\otimes\lambda^{-1},\psi_K)$ and $f(\Pi\otimes{\omega_\Pi}^{-1}\lambda^{-\sigma}) = f(\Pi\otimes\lambda^{-1})$. Thus, $\Pi(g)I_\sigma\xi_\lambda^{(n)} = \omega_\Pi(\Delta)\epsilon(\Pi\otimes\lambda^{-1},\psi_K)\lambda^\sigma(-a)\xi_{\lambda^{-\sigma}}^{(m-v_K(a))}$ where $m = f(\Pi\otimes\lambda^{-1}) + 2n(\psi_K) - n$.

We have thus proved:

Lemma 4.4. *For $a \in K^* - F^*$,*

$$\Pi\left(\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}-a^{-1} & 0\\ 0 & 1\end{pmatrix}w\right)I_\sigma\xi_\lambda^{(n)} = \omega_\Pi(\Delta)\epsilon(\Pi\otimes\lambda^{-1},\psi_K)\lambda^\sigma(-a)\xi_{\lambda^{-\sigma}}^{(m-v_K(a))}$$

where $m = f(\Pi\otimes\lambda^{-1}) + 2n(\psi_K) - n$.

We want to compute $\chi_{\Pi,\sigma}(g)$, where $g = \begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}-a^{-1} & 0\\ 0 & 1\end{pmatrix}w$. There is a standard method to do this and we refer to [**15**, pp. 102-103] for the details. Set $\Gamma_n = \begin{pmatrix}1+\mathcal{P}_K^n & \mathcal{P}_K^n\\ \mathcal{P}_K^n & 1+\mathcal{P}_K^n\end{pmatrix}\cap GL_2(\mathcal{O}_K)$. Let $K(\Pi,\psi_K)^n$ be the subspace of $K(\Pi,\psi_K)$ consisting of elements invariant under Γ_n.

Let

$$B_n = \left\{\xi_\lambda^{(m)}\ \middle|\ \begin{matrix}\text{conductor of } \lambda \le n\\ f(\Pi\otimes\lambda^{-1}) + n(\psi_K) - n \le m \le n(\psi_K) + n\end{matrix}\right\}.$$

Then B_n gives a basis of $K(\Pi,\psi_K)^n$ for n sufficiently large, and $\bigcup_n B_n$ gives a basis of $K(\Pi,\psi_K)$.

Let P_n be the projection of $K(\Pi,\psi_K)$ onto $K(\Pi,\psi_K)^n$ defined by $\frac{\int_{\Gamma_n}\Pi(g)dg}{\int_{\Gamma_n}dg}$, where dg is a Haar measure on $GL_2(K)$. Then the value of $\chi_{\Pi,\sigma}(g)$ can be calculated as trace$(\Pi(g)I_\sigma P_n)$ with respect to this basis for a sufficiently large n.

Suppose $\xi_\lambda^{(n)}$ contribute to $\chi_{\Pi,\sigma}\left(\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}-a^{-1} & 0\\ 0 & 1\end{pmatrix}w\right)$. Then we have:

(i) $n = n(\psi_K) + \frac{1}{2}(f(\Pi \otimes \lambda^{-1}) - v_K(a))$;
(ii) $\lambda|_{\mathcal{O}_K^*} = \lambda^{-\sigma}|_{\mathcal{O}_K^*}$.

First assume that K/F is unramified. Then from (ii) we have $\lambda|_{\mathcal{O}_F^*} = 1$. As a representative of the class of λ, take λ such that $\lambda(\pi_F) = 1$. Then we have $\lambda^{-\sigma} = \lambda$. Thus the contribution to

$$\chi_{\Pi,\sigma}\left(\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} -a^{-1} & 0 \\ 0 & 1 \end{pmatrix} w\right)$$

of $\xi_\lambda^{(n)}$ for the above λ is equal to

$$\begin{cases} \omega_\Pi(\Delta)\epsilon(\Pi \otimes \lambda^{-1}, \psi_K)\lambda^\sigma(-a) & \text{if } v_K(a) \equiv f(\Pi \otimes \lambda^{-1}) (\operatorname{mod} 2) \\ 0 & \text{otherwise.} \end{cases}$$

Since K/F is unramified, we have an extension $\widetilde{\omega_{K/F}}$ of $\omega_{K/F}$ to K^* which is unramified. Then

$$\epsilon(\Pi \otimes \lambda^{-1}\widetilde{\omega_{K/F}}^{-1}, \psi_K) = (-1)^{f(\Pi\otimes\lambda^{-1})}\epsilon(\Pi \otimes \lambda^{-1}, \psi_K).$$

Therefore the contribution of $\xi_\lambda^{(n)}$ to $\chi_{\Pi,\sigma}\left(\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} -a^{-1} & 0 \\ 0 & 1 \end{pmatrix} w\right)$ is equal to

$$\frac{1}{2}\omega_\Pi(\Delta)(\epsilon(\Pi \otimes \lambda^{-1}, \psi_K)\lambda^\sigma(-a) + \epsilon(\Pi \otimes \lambda^{-1}\widetilde{\omega_{K/F}}^{-1}, \psi_K)(\lambda\widetilde{\omega_{K/F}})^\sigma(-a)).$$

Thus

$$\begin{aligned}
&\chi_{\Pi,\sigma}\left(\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} -a^{-1} & 0 \\ 0 & 1 \end{pmatrix} w\right) \\
&\qquad = \omega_\Pi(\Delta)\left(\sum_{\lambda|_{F^*}=1} \frac{\epsilon(\Pi \otimes \lambda^{-1}, \psi_K)}{2}\lambda^\sigma(-a)\right. \\
&\qquad\qquad \left. + \sum_{\lambda|_{F^*}=\omega_{K/F}} \frac{\epsilon(\Pi \otimes \lambda^{-1}, \psi_K)}{2}\lambda^\sigma(-a)\right) \\
&\qquad = \sum_{\lambda|_{F^*}=1} \frac{\epsilon(\Pi \otimes \lambda^{-1}, (\psi_K)_\Delta)}{2}\lambda^\sigma(a) \\
&\qquad\qquad + \sum_{\lambda|_{F^*}=\omega_{K/F}} \frac{\epsilon(\Pi \otimes \lambda^{-1}, (\psi_K)_\Delta)}{2}\lambda^\sigma(a) \\
&\qquad = \sum_{\lambda|_{F^*}=1} \frac{1 + \epsilon(\Pi \otimes \lambda^{-1}, (\psi_K)_\Delta)}{2}\lambda^\sigma(a) \\
&\qquad\qquad + \sum_{\lambda|_{F^*}=\omega_{K/F}} \frac{1 + \epsilon(\Pi \otimes \lambda^{-1}, (\psi_K)_\Delta)}{2}\lambda^\sigma(a)
\end{aligned}$$

since $\sum_{\lambda|_{F^*}=1}\lambda^\sigma = 0$ and $\sum_{\lambda|_{F^*}=\omega_{K/F}}\lambda^\sigma = 0$.

Now suppose K/F is a ramified extension. Let π_F be a uniformizing element of F that is contained in the norm of K. Condition (ii) implies $\lambda|_{N_{K/F}(\mathcal{O}_K^*)} = 1$. Therefore, $\lambda|_{\mathcal{O}_F^*} = 1$ or $\omega_{K/F}|_{\mathcal{O}_F^*}$. In the class of λ satisfying (ii) there are exactly two characters satisfying $\lambda_i(\pi_F) = 1, (i = 1, 2)$ and they satisfy $\lambda_i = \lambda_i^{-\sigma}$. Since λ_1 and λ_2 belong to the same class, $\lambda_1\lambda_2^{-1}$ is unramified. Hence, $\lambda_1\lambda_2^{-1}(\pi_K^2) = \lambda_1\lambda_2^{-1}(\pi_F) = 1$. Thus $\lambda_1\lambda_2^{-1}(\pi_K) = \pm 1$. But $\lambda_1\lambda_2^{-1}(\pi_K) = 1$ implies that $\lambda_1 = \lambda_2$ which is not true and so $\lambda_1(\pi_K) = -\lambda_2(\pi_K)$. Thus $\lambda_2 = \lambda_1\eta$ where $\eta(x) = (-1)^{v_K(x)}$. Now the contribution of $\xi_\lambda^{(n)}$ to $\chi_{\Pi,\sigma}\left(\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} -a^{-1} & 0 \\ 0 & 1 \end{pmatrix} w\right)$ is

$$\frac{1}{2}\omega_\Pi(\Delta)(\epsilon(\Pi\otimes\lambda_1^{-1}, \psi_K)\lambda_1^\sigma(-a) + \epsilon(\Pi\otimes\lambda_2^{-1}, \psi_K)\lambda_2^\sigma(-a))$$

and $\lambda_1|_{F^*} = \lambda_2|_{F^*}$. But $\lambda_1|_{F^*}$ can be 1 or $\omega_{K/F}$. So the total contribution is once again

$$\sum_{\lambda|_{F^*}=1}\frac{\epsilon(\Pi\otimes\lambda^{-1}, (\psi_K)_\Delta)}{2}\lambda^\sigma(a) + \sum_{\lambda|_{F^*}=\omega_{K/F}}\frac{\epsilon(\Pi\otimes\lambda^{-1}, (\psi_K)_\Delta)}{2}\lambda^\sigma(a).$$

Thus, as in the unramified case, we get

$$\begin{aligned}
&\chi_{\Pi,\sigma}\left(\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} -a^{-1} & 0 \\ 0 & 1 \end{pmatrix} w\right) \\
&= \sum_{\lambda|_{F^*}=1}\frac{1+\epsilon(\Pi\otimes\lambda^{-1}, (\psi_K)_\Delta)}{2}\lambda^\sigma(a) \\
&\quad + \sum_{\lambda|_{F^*}=\omega_{K/F}}\frac{1+\epsilon(\Pi\otimes\lambda^{-1}, (\psi_K)_\Delta)}{2}\lambda^\sigma(a).
\end{aligned}$$

Fix an embedding i of K^*/F^* into $U(2, K/F)$ given by $i(aF^*) = \begin{pmatrix} x & y \\ \Delta^2 y & x \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix}$ where $a = x + \Delta y \in K^*$. If $g = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} -a^{-1} & 0 \\ 0 & 1 \end{pmatrix} w$, then observe that $gg^\tau = \begin{pmatrix} a^{-1}a^\sigma & 0 \\ 0 & 1 \end{pmatrix}$ is conjugate to the image of aF^* under i.

If Π is a stable base change of a representation π of $U(2, K/F)$, then

$$\chi_{\Pi,\sigma}\left(\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} -a^{-1} & 0 \\ 0 & 1 \end{pmatrix} w\right) = \chi_{\{\pi\}}(i(aF^*)),$$

and if Π is an unstable base change of π, then

$$\chi_{\Pi,\sigma}\left(\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} -a^{-1} & 0 \\ 0 & 1 \end{pmatrix} w\right) = \widetilde{\omega_{K/F}}(a^{-1})\chi_{\{\pi\}}(i(aF^*)).$$

Thus we get the following two identities, the first obtained when Π is a stable base change, and the second, when Π is an unstable base change.

$$\begin{aligned}\chi_{\{\pi\}}(i(aF^*)) = &\sum_{\lambda|_{F^*}=1} \frac{1+\epsilon(\Pi\otimes\lambda^{-1},(\psi_K)_\Delta)}{2}\lambda^\sigma(a)\\ &+ \sum_{\lambda|_{F^*}=\omega_{K/F}} \frac{1+\epsilon(\Pi\otimes\lambda^{-1},(\psi_K)_\Delta)}{2}\lambda^\sigma(a)\end{aligned}$$

$$\begin{aligned}\chi_{\{\pi\}}(i(aF^*)) = &\widetilde{\omega_{K/F}}(a)\sum_{\lambda|_{F^*}=1} \frac{1+\epsilon(\Pi\otimes\lambda^{-1},(\psi_K)_\Delta)}{2}\lambda^\sigma(a)\\ &+ \widetilde{\omega_{K/F}}(a)\sum_{\lambda|_{F^*}=\omega_{K/F}} \frac{1+\epsilon(\Pi\otimes\lambda^{-1},(\psi_K)_\Delta)}{2}\lambda^\sigma(a).\end{aligned}$$

Let $r \in F^* - N_{K/F}(K^*)$ and change a to ar. In both these identities the left side remain unchanged, whereas a change of sign occurs in the second sum of the first identity and in the first sum of the second identity. Thus it follows that the second sum vanishes in the first identity and the first sum vanishes in the second identity. Thus we get $\gamma(\Pi\otimes\lambda^{-1},(\psi_K)_\Delta) = 1$ for all characters λ of K^* with $\lambda|_{F^*} = \omega_{K/F}$ (respectively $\lambda|_{F^*} = 1$) if Π is a stable (resp. unstable) base change lift of a representation of $U(2, K/F)$. (Note that the γ- factor is the same as the ϵ-factor since Π is supercuspidal.) Hence, if Π is a stable base change, then it is $\omega_{K/F}$-distinguished, and if Π is an unstable base change, then it is distinguished (by the result of Hakim cited in the proof of Proposition 2.3).

Remark. Here $\epsilon(\Pi\otimes\lambda^{-1},(\psi_K)_\Delta) = \pm 1$ when $\lambda|_{F^*} = 1$ or $\omega_{K/F}$.

Proof. We have $\epsilon(\Pi\otimes\lambda^{-1},\psi_K)\epsilon((\widetilde{\Pi\otimes\lambda^{-1}}),\psi_K) = \omega_{\Pi\otimes\lambda^{-1}}(-1)$.

But $(\widetilde{\Pi\otimes\lambda^{-1}}) \sim \widetilde{\Pi}\otimes\lambda \sim \Pi^\sigma\otimes\lambda$.

Therefore $\epsilon((\widetilde{\Pi\otimes\lambda^{-1}}),\psi_K) = \epsilon(\Pi\otimes\lambda^\sigma,\psi_K) = \epsilon(\Pi\otimes\lambda^{-1},\psi_K)$.

Therefore $\epsilon(\Pi\otimes\lambda^{-1},\psi_K)^2 = 1$.

Now $\epsilon(\Pi\otimes\lambda^{-1},(\psi_K)_\Delta) = \omega_\Pi(\Delta)\lambda(-1)\epsilon(\Pi\otimes\lambda^{-1},\psi_K) = \pm 1$.

What remains to show in order to prove Theorem 1.4 is that a representation Π of $GL_2(K)$, distinguished with respect to $GL_2(F)$, is obtained by the unstable base change map. Now by Theorem 4.1, we know that Π is a base change lift of a representation of $U(2, K/F)$. We must show that Π is in the image of the unstable base change map and not in the image of the stable base change map. Suppose Π is a stable base change lift of a representation of $U(2, K/F)$. Then by what has been proved already,

Π is $\omega_{K/F}$-distinguished with respect to $GL_2(F)$. Thus Π is both distinguished and $\omega_{K/F}$-distinguished, which contradicts Proposition 3.1. This proves Theorem 1.4.

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References

[1] P. Anuradha Kameswari and R. Tandon, *A converse theorem for epsilon factors*, J. of Number Theory, **89** (2001), 308-323, MR 2002h:11122.

[2] Y. Flicker, *Stable and labile base change for $U(2)$*, Duke Math. J., **49** (1982), 691-729, MR 84i:22016, Zbl 0502.12013.

[3] ______, *On distinguished representations*, J. Reine Angew. Math., **418** (1991), 139-172, MR 92i:22019, Zbl 0725.11026.

[4] ______, *Twisted tensors and Euler products*, Bull. Soc. Math. France, **116** (1988), 295-313, MR 89m:11049, Zbl 0674.10026.

[5] Y. Flicker and J. Hakim *Quarternionic distinguished representations*, Amer. J. Math., **116** (1994), 683-736, MR 95i:22028, Zbl 0837.11030.

[6] A. Frohlich and J. Queyrut, *On the functional equation of the Artin L-function for characters of real representations*, Inv. Math., **20** (1973), 125-138, MR 48 #253, Zbl 0256.12010.

[7] J. Hakim, *Distinguished Nonarchimedean Representations for $GL(2)$*, Ph.D. thesis, Columbia University, 1989.

[8] ______, *Distinguished p-adic representations*, Duke Math J., **62** (1991), 1-22, MR 92c:22037, Zbl 0724.22016.

[9] G. Harder, R.P. Langlands and M. Rapoport, *Algebraische zyklen auf Hilbert-Blumenthal-Flächen*, J. Reine Angew. Math., **366** (1986), 53-120, MR 87k:11066, Zbl 0575.14004.

[10] H. Jacquet and R. Langlands, *Automorphic Forms on $GL(2)$*, Lect. Notes Math., **114**, Springer, 1970, MR 53 #5481, Zbl 0236.12010.

[11] H. Jacquet and Y. Ye, *Une remarque sur le changement de base quadratique*, C.R. Acad. Sci. Paris, **311**, Série 1, (1990), 671-676, MR 92j:11046, Zbl 0715.11026.

[12] ______, *Distinguished representations and quadratic base change for $GL(3)$*, Trans. Amer. Math. Soc., **348**(**3**) (1996), 919-939, MR 96h:11041, Zbl 0861.11033.

[13] D. Prasad, *Distinguished representations for quadratic extensions*, Compositio Math., **119** (1999), 335-345, MR 2001b:22016, Zbl 0969.22008.

[14] ______, *On a conjecture of Jacquet about distinguished representations of GL_n*, Duke Math J., **109**(**1**) (2001), 67-78, MR 2002g:22036.

[15] H. Saito, *On Tunnell's formula for characters of $GL(2)$*, Compositio Math., **85** (1993), 99-108, MR 93m:22021, Zbl 0795.22009.

[16] J. Tate, *Number theoretic background*, in 'Automorphic Forms, Representations, and L-functions' (Corvallis), AMS Proc. Symp. Pure Math., **33** (1979), 3-26, MR 80m:12009, Zbl 0422.12007.

[17] J. Tunnell, *Local epsilon factors and characters of* $GL(2)$, Amer. J. Math., **105** (1983), 1277-1307, MR 86a:22018 , Zbl 0532.12015.

[18] Y. Ye, *Kloosterman Integrals and Base Change*, Ph.D. thesis, Columbia University, 1986.

[19] ______, *Kloosterman integrals and base change for* $GL(2)$, J. Reine Angew. Math., **400** (1989), 57-121, MR 90i:11134, Zbl 0665.10020.

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