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Root Numbers of Asai L-Functions

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Let E/F be a quadratic extension of p-adic fields. We compute the value of $\epsilon(\frac{1}{2}, \pi, r, \psi)$ for a square integrable representation π of $GL_n(E)$, which is (Galois) conjugate self-dual, where r denotes the Asai representation. This is the twisted version of a well-known result due to Bushnell and Henniart. The proof makes use of a result on the corresponding global root number, which is proved by a method conceived by Lapid and Rallis.

1 Introduction

A representation (π, V) of a group G is said to be distinguished with respect to a character χ of a subgroup H if there exists a linear form ℓ of V such that $\ell(\pi(h)v) = \chi(h)\ell(v)$ for all $v \in V$ and $h \in H$. The representation is said to be distinguished with respect to H if χ is the trivial character. The concept of distinction is especially important when H is the fixed group of an involution on G. In this case, according to the philosophy of Jacquet, H-distinguished representations of G are often functorial lifts from another group G' [18].

In the global context of automorphic representations, distinction is defined in terms of the nonvanishing of the period integral. Suppose F is a number field and $\mathbb{A}_F$ its ring of adeles. Let G be a reductive algebraic group over F and H a reductive subgroup of G over F. Assume that the center Z_H of H is contained in the center Z_G of G. Let χ be a one-dimensional representation of $H(\mathbb{A}_F)$ trivial on $H(F)$ such that $Z_H(\mathbb{A}_F)$ acts trivially

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on $\phi(h)\chi^{-1}(h)$. Then a cuspidal representation π of $G(\mathbb{A}_F)$ is said to be χ-distinguished if the period integral

$$\int_{H(F)Z_H(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \phi(h)\chi^{-1}(h)dh$$

is nonzero for some $\phi \in \pi$.

In this paper, we are interested in the pair $G = GL_n(E)$ and $H = GL_n(F)$, where E/F is a quadratic extension of p-adic fields (or the corresponding adelic situation) [8]. A cuspidal (resp. square integrable) representation π of $GL_n(\mathbb{A}_E)$ (resp. $GL_n(E_w)$) being distinguished is characterized by the global (resp. local) Asai L-function having a pole at $s = 1$ (resp. $s = 0$) [2, 7]. Thus it follows from the following factorization of L-functions [12, 19]

$$L(s, \pi \times \pi^\sigma) = L(s, \pi, r)L(s, \pi \otimes \kappa, r)$$

that $\pi^\vee \simeq \pi^\sigma$, i.e. π is conjugate self-dual, if and only if π is distinguished or distinguished with respect to the quadratic character $\omega_{E/F}$ (but not both). Here κ is an extension of $\omega_{E/F}$ to $\mathbb{A}_E^*$ (or E_w^*) and r is the Asai representation [7, 12].

We note that the characterization of distinction in terms of poles of Asai L-functions reduces to well-known results in the case when $E = F \oplus F$. In this case, $G = GL_n(F) \times GL_n(F)$ and a representation π of G is of the form $\pi_1 \otimes \pi_2$. Now the Asai representation is just the tensor product and consequently the Asai L-function $L(s, \pi, r)$ is nothing but the Rankin–Selberg L-function $L(s, \pi_1 \times \pi_2)$. The representation π is distinguished precisely when π_1 and π_2 are duals of each other and this is characterized by $L(s, \pi_1 \times \pi_2)$ having a pole at $s = 0$ (or $s = 1$ in the global case).

We now turn our attention to the epsilon factor $\epsilon(s, \pi, r, \psi)$, where ψ is an additive character of F [36]. We want to compute the value of $\epsilon(\frac{1}{2}, \pi, r, \psi)$ when π is distinguished. Bushnell and Henniart have computed the corresponding epsilon value in the case when $E = F \oplus F$ [4]. For an irreducible admissible representation π of $GL_n(F)$, they show that $\epsilon(\frac{1}{2}, \pi \times \pi^\vee, \psi) = \omega_\pi(-1)^{n-1}$, where ω_π is the central character of π. That the epsilon value is ± 1 is clear from the local functional equation and the theorem of Bushnell and Henniart fixes this sign. The local result of course would imply the global result $\epsilon(\frac{1}{2}, \pi \times \pi^\vee) = 1$ for a cuspidal representation π of $GL_n(\mathbb{A}_F)$.

Now let E/F be a quadratic extension of p-adic fields and let π be an irreducible admissible representation of $GL_n(E)$. The epsilon factor $\epsilon(s, \pi, r, \psi)$ satisfies the following identity [36]:

$$\epsilon(s,\pi,r,\psi)\epsilon(1-s,\pi^\vee,r,\psi^{-1})=1.$$

Thus if $\pi^\vee \simeq \pi^\sigma$, it can be deduced (cf. 2.1, (d)) that

$$\epsilon(1/2,\pi,r,\psi)^2=\begin{cases}\omega_{E/F}(-1)^{\binom{n}{2}} & \text{if } \pi \text{ is distinguished}\\ \omega_{E/F}(-1)^{\binom{n+1}{2}} & \text{if } \pi \text{ is } \omega_{E/F}\text{-distinguished,}\end{cases}$$

where $\omega_{E/F}$ is the quadratic character of F^* associated to E/F. Our main theorem computes this epsilon value when π is a square integrable representation.

Let $\lambda(E/F,\psi)$ denote Langlands' λ-factor. This is the root number of the quadratic character $\omega_{E/F}$ and thus $\lambda(E/F,\psi)^2=\omega_{E/F}(-1)$.

Theorem 1.1. Let E/F be a quadratic extension of p-adic fields and let π be an irreducible square integrable representation of $GL_n(E)$ such that $\pi^\vee \simeq \pi^\sigma$. Then

$$\epsilon(1/2,\pi,r,\psi)=\begin{cases}\omega_\pi(\delta)^{n-1}\lambda(E/F,\psi)^{\binom{n}{2}} & \text{if } \pi \text{ is distinguished}\\ \omega_\pi(\delta)^{n-1}\lambda(E/F,\psi)^{\binom{n+1}{2}}\omega_{E/F}(-1)^{\binom{n}{2}} & \text{if } \pi \text{ is } \omega_{E/F}\text{-distinguished,}\end{cases}$$

where δ is an element of E^* of trace zero. □

Note that in the above theorem, $\omega_\pi(\delta)^{n-1}$ does not depend on the choice of δ. Indeed, if π is distinguished, ω_π has trivial restriction to F^* and hence $\omega_\pi(\delta)$ itself does not depend on the choice of δ. If π is $\omega_{E/F}$-distinguished, $\omega_\pi|_{F^*}=1$ or $\omega_{E/F}$ depending on whether n is even or odd. Again the claim follows.

The proof of Theorem 1.1 is by global means, making use of the following global theorem.

Theorem 1.2. Let E/F be a quadratic extension of number fields. Let π be an irreducible cuspidal representation of $GL_n(\mathbb{A}_E)$ such that $\pi^\vee \simeq \pi^\sigma$. Then $\epsilon(\frac{1}{2},\pi,r)=1$. □

We note that Theorem 1.2 should be viewed as an instance of an automorphic analogue of the well-known result of Fröhlich and Queyrut according to which the Artin root number of an orthogonal Galois representation is always 1 [6, 10]. In general, one expects the global root number $\epsilon(\frac{1}{2},\pi,\rho)$ to be 1 if the Langlands parameter of π composed with ρ is orthogonal. Recently Lapid and Rallis verified this expectation for root numbers of symmetric square and exterior square L-functions [23] (see [25], and more generally

[26], for another instance of this). For a precise conjecture about the root numbers of automorphic L-functions, we refer to [30].

Theorem 1.2 will be proved by the method of Lapid and Rallis [23]. Consider the quasi-split unitary group $G = U(n, n)$ in $2n$ variables with a maximal parabolic $P = MU$, where we identify M with $\mathrm{R}_{E/F} GL_n$. Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $GL_n(\mathbb{A}_E)$, which is distinguished with respect to $GL_n(\mathbb{A}_F)$. From the inner product formula for residues of Eisenstein series associated to the representation $I(\pi, s)$ of G parabolically induced from $\pi \otimes |\,|_E^s$, it follows that the form $(M_{-1}\phi_1, \phi_2)$ defined on $I(\pi, \frac{1}{2})$ is positive semi-definite, where M_{-1} is the residue of the intertwining operator $M(s)$: $I(\pi, s) \longrightarrow I(\pi, -s)$ at $s = \frac{1}{2}$ (here π is identified with its conjugate dual). Now consider the normalized intertwining operator $R(s) = \otimes_v R_v(s)$ given by $M(s) = m(s)R(s)$, where

$$m(s) = \frac{L(2s, \pi^\vee, r)}{\epsilon(2s, \pi^\vee, r)L(2s + 1, \pi^\vee, r)}.$$

From the properties of the normalized intertwining operator, it follows that the form $(R(\frac{1}{2})\phi_1, \phi_2)$ on $I(\pi, \frac{1}{2})$ is semi-definite. Using a local argument originally due to Keys and Shahidi [20], the multiplicativity of the normalized intertwining operators and the theory of R-groups [13], one shows that the Hermitian form $(R_v(0)\phi_{1,v}, \phi_{2,v})$ on $I(\pi_v, 0)$ is positive definite on $I(\pi_v, 0)$. A continuity argument on K-types of $I(\pi_v, s)$ for $0 \le s \le \frac{1}{2}$, and the crux of the work of Lapid and Rallis is to make this continuity argument work, will show that the form $(R_v(\frac{1}{2})\phi_{1,v}, \phi_{2,v})$ on $I(\pi_v, \frac{1}{2})$ is positive semi-definite. It follows that the global form $(R(\frac{1}{2})\phi_1, \phi_2)$ is positive semi-definite, as the global form is a product of the local forms $(R_v(\frac{1}{2})\phi_{1,v}, \phi_{2,v})$. Therefore the residue of $m(s)$ at $s = \frac{1}{2}$ is positive and from this we can conclude that the Asai root number of π is 1 whenever π is distinguished with respect to $GL_n(\mathbb{A}_F)$. As already observed before, the local theorem of Bushnell and Henniart [4] implies that the global root number $\epsilon(s, \pi \times \pi^\sigma)$ is 1. Therefore, as we already know that one of the factors in

$$\epsilon(s, \pi \times \pi^\sigma) = 1 = \epsilon(s, \pi, r)\epsilon(s, \pi \times \kappa, r)$$

is 1, so is the other one too, proving Theorem 1.2.

At this point, it needs to be pointed out that it is more standard to define $I(\pi, s)$ as $\mathrm{Ind}_P^G(\pi \otimes |\,|_E^{s/2})$ in the Langlands–Shahidi paradigm. However, we make the transformation $s \to 2s$, and this explains the appearance of $2s$ in the normalizing factor $m(s)$, to be consistent with the work of Lapid and Rallis.

As for Theorem 1.1, we first prove it in the case of a supercuspidal distinguished representation. Assuming that we have proved Theorem 1.2, the idea to prove Theorem 1.1

is to globalize the supercuspidal distinguished representation π_v of $GL_n(E_v)$ to a cuspidal distinguished representation π of $GL_n(\mathbb{A}_E)$ in a suitable way by appealing to the globalization theorem of Hakim and Murnaghan [16]. In fact, we will make use of a refined form of the globalization theorem due to Prasad and Schulze-Pillot [33], so that we can assume that the finite components of π outside v are unramified. The local epsilon values of these unramified representations as well as the archimedean components will then be computed by making use of the theorem of Bushnell and Henniart [4] and by using properties of the Asai representation r. These computations together with Theorem 1.2 will establish Theorem 1.1 in this case. Now the root number of an $\omega_{E/F}$-distinguished supercuspidal representation can be computed by employing a somewhat similar strategy. The square integrable case is then dealt with by making use of the multiplicativity of γ-factors. Of crucial importance in carrying out this step is the following theorem, which says when the generalized Steinberg representation is distinguished (cf. Section 4.4 of [3]).

Theorem 1.3. Let τ be an irreducible supercuspidal representation of $GL_a(E)$ and let $\pi = \mathrm{St}(\tau)$ be the unique square integrable constituent of the representation parabolically induced from $\tau|\ |_E^{\frac{1-b}{2}} \otimes \cdots \otimes \tau|\ |_E^{\frac{b-1}{2}}$. If b is odd (resp. even), then π is distinguished with respect to $GL_n(F)$, where $n = ab$, if and only if τ is distinguished (resp. $\omega_{E/F}$-distinguished) with respect to $GL_a(F)$. $\square$

For other instances of local epsilon factor computations of similar flavor, see [32].

In the final section of this paper, we make a conjecture about the value of $\epsilon(\frac{1}{2}, \pi_1 \times \pi_2, \psi')$, where π_i are irreducible admissible representations of $GL_{n_i}(E)$ distinguished with respect to $GL_{n_i}(F)(i = 1, 2)$. We also indicate how Theorem 1.1 provides some evidence towards this conjecture. The interest in these epsilon values stems out of the fact that they characterize distinction as well as functorial lifts from unitary groups in the case when $n_1 = 2,\ n_2 = 1$ [1, 15], thus leading to a proof of the Flicker–Rallis conjecture, which links distinction with lifting from the unitary groups [8]. Assuming that a distinguished representation descents to $U(n)$ and that this descent preserves the epsilon factor, we also demonstrate that our conjecture is in agreement with the calculation of the local root number, of a representation of $U(n)$, defined via the doubling method of Piatetski-Shapiro and Rallis [26]. We refer to [24] for a concrete situation in a similar setting where the descent method plays an important role. Motivated by the analysis of representations of "G-type" as in the work of Lapid and Rallis [23] as well as by certain computations on the Langlands parameters by D. Prasad (cf. p. 431, [31]), in this section we also conjecturally relate distinction with intertwining operators on induced representations.

It should be noted that the local L-functions and the ϵ-factors can be defined in three distinct ways: via the Langlands formalism on the Galois side of the Langlands correspondence, via the Langlands–Shahidi method applied to a suitable unitary group [12, 36], and via the Rankin–Selberg approach [9, 19]. The first and second definitions of the L-function match and the corresponding definition of epsilon factors match up to a root of unity by the work of Henniart [17]. Similar matching holds true in the second and third definitions too, but this is known only when the representation is square integrable [3]. The L-function and the ϵ-factor of this paper are the ones defined by the Langlands–Shahidi method. Computing the local Asai root number defined via the Rankin–Selberg integrals is an interesting problem, which we do not address in this paper (cf. Remark 4.4).

Finally, a word about the notation: we will index local constituents of global objects by places of F. For instance, π_v denotes π_w if v is a place of F that is inert in E and w is the unique place of E lying over v and it denotes $\pi_{w_1} \otimes \pi_{w_2}$ if v splits as $w_1 w_2$. Thus, $L(s, \pi_v, r)$ signifies either the Asai L-function $L(s, \pi_w, r)$ or the Rankin–Selberg L-function $L(s, \pi_{w_1} \times \pi_{w_2})$.

2 Preliminaries

2.1 The Asai representation

The Asai representation r is a representation of

$$^{L}[\mathrm{R}_{E/F} GL(n)] = GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) \rtimes \mathrm{Gal}(E/F)$$

(where σ acts by swapping) on $\mathbb{C}^n \otimes \mathbb{C}^n$ is given by

$$r((a, b))(x \otimes y) = ax \otimes by,$$
$$r(\sigma)(x \otimes y) = y \otimes x.$$

This representation r is equivalent to the adjoint action of $^{L}[\mathrm{R}_{E/F} GL(n)]$ on the Lie algebra $^{L}\mathfrak{n}$ of the unipotent radical of the parabolic subgroup of $U(n, n)$ with Levi component $^{L}[\mathrm{R}_{E/F} GL(n)]$ [7]. Given an n-dimensional representation ρ of the Weil–Deligne group W_E

of E, via r, we get an n^2-dimensional representation, say $r(\rho)$, of the Weil–Deligne group W_F of F.

For general properties of r, we refer to [29]. We, however, list a few properties that we will need. Suppose ρ is a representation of W_E. Then,

(a) $r(\rho)|_{W_E} = \rho \otimes \rho^\sigma$
(b) $r(\rho_1 \oplus \rho_2) = r(\rho_1) \oplus r(\rho_2) \oplus \mathrm{Ind}_{W_E}^{W_F}(\rho_1 \otimes \rho_2^\sigma)$
(c) $r(\rho) = r(\rho^\sigma)$
(d) $\det r(\rho) = r(\det \rho)^n \omega_{E/F}^{\binom{n}{2}}$.

We start with a small lemma in view of which a result of the type of Theorem 1.2 should be expected.

Lemma 2.1. If ρ is an irreducible representation of W_E such that $\rho^\vee \cong \rho^\sigma$, then $r(\rho)$ is orthogonal. □

Proof. First, $r(\rho)$ is self-dual by (c). Now fix a nondegenerate bilinear form, unique up to scaling, $B : \rho \times \rho \longrightarrow \mathbb{C}$ such that $B(gv, g^\sigma w) = B(v, w)$. Then there exists a constant $c(\rho) \in \{\pm 1\}$ such that $B(v, w) = c(\rho) B(w, v)$. We can define a nondegenerate invariant bilinear form on $r(\rho)$ by sending $(v_1 \times v_2, w_1 \times w_2)$ to $B(v_1, w_2) B(w_1, v_2)$. ■

Remark 2.2. The constant $c(\rho)$ is introduced by Rogawski (cf. 15.1, [34]). If $\rho^\vee \cong \rho^\sigma$, then $r(\rho)$ contains the trivial representation if and only if $c(\rho) = 1$. □

The factorization of the Rankin–Selberg L-function $L(s, \pi \times \pi^\sigma)$ given in the introduction is formally a consequence of (a). The factorization

$$L\big(s, \pi_v \times \pi_v^\sigma\big) = L(s, \pi_v, r) L(s, \pi_v \otimes \kappa_v, r)$$

holds at all the finite places v of F. This is proved by Goldberg via the Langlands–Shahidi method [12]. Alternatively, one can deduce this from the formal property (a) by appealing to Henniart's result [17]. At the archimedean places, the L-function that we consider coincides with the Langlands L-function on the Galois side [36] and hence property (a) can be employed again. Hence we have the global identity

$$L(s, \pi \times \pi^\sigma) = L(s, \pi, r) L(s, \pi \otimes \kappa, r),$$

where π is a cuspidal representation of $GL_n(\mathbb{A}_E)$. From the global functional equations [36]

$$L(s,\pi,r) = \epsilon(s,\pi,r)L(1-s,\pi^\vee,r)$$

and

$$L(s,\pi_1\times\pi_2) = \epsilon(s,\pi_1\times\pi_2)L\left(1-s,\pi_1^\vee\times\pi_2^\vee\right),$$

it follows that we have the following global identity:

$$\epsilon(s,\pi\times\pi^\sigma) = \epsilon(s,\pi,r)\epsilon(s,\pi\otimes\kappa,r). \tag{1}$$

Similarly, starting with property (c), we get

$$L(s,\pi,r) = L(s,\pi^\sigma,r) \;\&\; \epsilon(s,\pi,r) = \epsilon(s,\pi^\sigma,r). \tag{2}$$

Remark 2.3. The theorem, quoted in Section 1, due to Bushnell and Henniart [4] implies that $\epsilon(\frac{1}{2},\pi\times\pi^\vee) = 1$ for a cuspidal representation of $GL_n(\mathbb{A}_E)$. Thus if π is such that $\pi^\vee \simeq \pi^\sigma$, it follows from the identity (1) that

$$\epsilon\left(\tfrac{1}{2},\pi,r\right) = \epsilon\left(\tfrac{1}{2},\pi\otimes\kappa,r\right).$$

This is because $\pi^\vee \simeq \pi^\sigma$ implies that both these epsilon values are ± 1 as can be seen from the global functional equation of Asai L-functions. $\square$

Lemma 2.4. Let E/F be a quadratic extension of p-adic fields. Let π be an irreducible admissible generic representation of $GL_n(E)$. Then,

$$L(s,\pi,r) = L(s,\pi^\sigma,r),$$

and

$$\epsilon(s,\pi,r,\psi) = \epsilon(s,\pi^\sigma,r,\psi). \qquad \square$$

Proof. The equality of L-functions is a formal consequence of property (c), as by Henniart's work [17], the Langlands–Shahidi L-function is the same as the L-function on the Galois side.

If π is a (unitary) supercuspidal representation, then the equality of epsilon factors once again follows from the work of Henniart (cf. 1.8, [17]). This is because it is possible to realize a supercuspidal representation as a component of a global cuspidal representation in such a way that at all the other finite places, the local representation is unramified (Proposition 5.1, [36]).

Now the general result follows from the inductive property of the γ-factors (cf. (3.13), Theorem 3.5, [36]), the knowledge of how the γ-factor behaves under twisting by unramified characters (cf. (3.12), Theorem 3.5, [36]), and by the first part of the lemma. ■

Remark 2.5. A general method to establish an equality for two possibly different definitions of L-functions at "bad" places applies once there are global functional equations, equality of the L-functions at all the "good" places, and regularity in the region $\mathrm{Re}(s) \geq \frac{1}{2}$ for the "bad" L-functions. Thus the knowledge of equality of two L-functions at all the unramified places and the split places leads to the equality of the L-functions for a square integrable representation. If multiplicativity is known for the corresponding γ-factors, the equality can be extended to all irreducible admissible representations. For more details, see the remarks in Section 3 of [3] and the references therein. □

Remark 2.6. Henniart's work on equality of Asai L-functions also implies that the corresponding epsilon factors are equal up to a root of unity. □

2.2 Intertwining operators

Let J_n be the $n \times n$ matrix whose (i, j) entry is $(-1)^{n-i}\delta_{i,n-j+1}$. Let $G = U(n, n)$ be the quasi-split unitary group in $2n$ variables associated to the quadratic extension E/F of number fields defined with respect to the form J_{2n}.

Let $B = TU$ denote the Borel subgroup of G over F corresponding to the upper triangular matrices. Let A_0 be the maximal F-split torus in T. Then,

$$A_0(F) = \mathrm{diag}\big(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}\big).$$

The restricted root system $\Phi(G, A_0)$ is of type C_n. We identify the set of positive roots determined by U with $\{e_i \pm e_j, 2e_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}$, where the e_i's denote the standard basis vectors of $\mathbb{R}^n$. Let $\Delta = \{e_i - e_{i+1}, 2e_n \mid 1 \leq i \leq n-1\}$ be a subset of simple roots.

Let $P = MN$ be a maximal parabolic of G generated by $\theta = \Delta - \{\alpha\}$, where $\alpha = 2e_n$. We identify $M(\mathbb{A}_F)$ with $GL_n(\mathbb{A}_E)$ and $N(\mathbb{A}_F)$ is given by

$$N(\mathbb{A}_F) = \left\{ \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix} \mid X \in M_n(\mathbb{A}_E);\, {}^tX^\sigma = J_n X J_n \right\}.$$

Let $\tilde{w}$ be the unique element of the Weyl group of A_0 in G such that $\tilde{w}(\theta) \subset \Delta$ and $\tilde{w}(\alpha) < 0$. Then $\tilde{w}$ corresponds to the $n \times n$ matrix with -1 on its antidiagonal and zeros elsewhere. Note that P is self-conjugate; i.e. $\tilde{w}(\theta) = \theta$. Let us also choose a representative w of $\tilde{w}$ in G following the recipe given in [20]. Thus,

$$w = \begin{pmatrix} 0 & (-1)^{n-1} I_n \\ -I_n & 0 \end{pmatrix},$$

and hence $w^2 = (-1)^n I_{2n}$. With our identification of M with $\mathrm{R}_{E/F} GL(n)$, w becomes the involution $J_n {}^t g^{-\sigma} J_n^{-1}$.

We fix a nontrivial additive character $\psi = \otimes_v \psi_v$ of $\mathbb{A}_F$, which is trivial on F. Then,

$$(x_{i,j}) \longrightarrow \psi\big((x_{1,2} + x_{1,2}^{\sigma}) + \cdots + (x_{n-1,n} + x_{n-1,n}^{\sigma}) + x_{n,n+1}\big)$$

defines a nondegenerate character of $U(\mathbb{A}_F)/U(F)$, which we continue to denote by ψ.

If π is a cuspidal representation of $GL_n(\mathbb{A}_E)$, let $I(\pi, s)$ denote the normalized induced representation

$$\mathrm{Ind}_P^G\big(\pi \otimes |\ |_E^s\big).$$

We define the global intertwining operator $M(s,\pi)$ by

$$M(s,\pi)\phi_s(g) = \int_{N(\mathbb{A}_F)} \phi_s(w^{-1}ng)dn,$$

where $g \in G$ and $\phi_s \in I(\pi, s)$. Then $M(s,\pi)$ maps $I(\pi, s)$ to $I(w\pi, -s) = I(\pi^{\vee\sigma}, -s)$. In a similar fashion, in the local case we can define the local intertwining operators between the local induced representations. We have

$$M(s,\pi) = \otimes_v M(s, \pi_v),$$

where $\pi = \otimes \pi_v$.

Consider the Eisenstein series

$$E(s,\phi,g) = \sum_{\gamma \in P(F)\backslash G(F)} \phi_s(\gamma g),$$

where ϕ_s is a K-finite vector in $I(\pi, s)$. It is known that the series converges for Re s large, it extends to a meromorphic function of s in $\mathbb{C}$, with only a finite number of poles in the plane Re $s \geq 0$, all simple and on the real axis, and that the poles of the Eisenstein series coincide with those of its constant terms.

Let $R(s,\pi)$ and $R(s,\pi_v) = R^{\psi_v}(s,\pi_v)$ denote Shahidi's normalized intertwining operators. In our situation, the adjoint action r of ${}^LM$ on ${}^L\mathfrak{n}$ is irreducible and in the notation of Shahidi [36], $r = r_1$. Thus, we have $R(s,\pi_v) = m_v(s,\pi_v)M(s,\pi_v)$, where the normalization factor $m_v(s,\pi_v) = m_v^{\psi_v}(s,\pi_v)$ is given by

$$\frac{\lambda(E_v/F_v,\psi_v)^{\binom{n}{2}} L\left(2s,\pi_v^\vee,r\right)}{\epsilon\left(2s,\pi_v^\vee,r,\psi_v^{-1}\right) L\left(2s+1,\pi_v^\vee,r\right)}. \tag{3}$$

Remark 2.7. Note that we have the λ-factor coming in the normalization. The λ-factor is thrown in in order to prove that the normalized operator is Hermitian when π_v is conjugate self-dual (cf. Proposition 2.10). The appearance of this factor does not affect the analytic properties of the intertwining operator. Also note that the factor $\lambda(E_v/F_v,\psi_v)^{\binom{n}{2}}$ is precisely the constant $\lambda_G(\psi_v,w)$ that appears in the formula for the local coefficient in Shahidi's paper (cf. (3.11), p. 289, [36]). □

The so-called Assumption (A) of Kim [21] is known in our case (for instance, see Proposition 5.2 of [22]):

Proposition 2.8. $R(s,\pi_v)$ is holomorphic and nonvanishing for Re $s \geq \frac{1}{2}$ for any v. □

Remark 2.9. In fact, Kim and Krishnamurthy obtain holomorphy and nonvanishing for Re $s > -\frac{1}{2}$ (cf. Proposition 9.4, [22]). □

2.3 The inner product formula

Let $R(s,\pi) = \otimes_v R(s,\pi_v)$. Then $M(s,\pi) = m(s,\pi)R(s,\pi)$, where

$$m(s,\pi) = \frac{L(2s,\pi^\vee,r)}{\epsilon(2s,\pi^\vee,r)L(2s+1,\pi^\vee,r)}.$$

Thanks to Proposition 2.8, it follows that the singularities of $E(s,\phi,g)$ in Re $s \geq 0$ coincide with those of $L(2s,\pi^\vee,r)$ (cf. Proposition 2.2, [11]) and hence the only possible singularity of $E(s,\phi,g)$ in Re $s \geq 0$ is a simple pole at $s = \frac{1}{2}$. Further, one knows that this pole occurs if and only if π is distinguished with respect to $GL_n(\mathbb{A}_F)$ [7]. Thus, if π is distinguished with respect to $GL_n(\mathbb{A}_F)$, then $m(s,\pi)$ will have a pole at $s = \frac{1}{2}$ and the residue there is given by

$$m_{-1} = \frac{1}{2}\cdot\frac{\mathrm{res}_{s=1}L(s,\pi^\vee,r)}{\epsilon(1,\pi^\vee,r)L(2,\pi^\vee,r)} = \frac{1}{2}\cdot\frac{\mathrm{res}_{s=1}L(s,\pi,r)}{\epsilon(1,\pi,r)L(2,\pi,r)}, \tag{4}$$

since $\pi^\vee \simeq \pi^\sigma$ and L and ϵ are invariant under the Galois action (cf. Lemma 2.4).

Let $E_{-1}(\phi, \cdot)$ denote the residue of $E(s, \phi, g)$ at $s = \frac{1}{2}$. In particular, it is identically zero unless $\pi^\vee \cong \pi^\sigma$, since this condition is necessary for π to be distinguished. The inner product formula for two residues of Eisenstein series [23, 27] is given, up to a positive constant, by

$$\int_{G(F)\backslash G(\mathbb{A}_F)} E_{-1}(\phi_1, g)\overline{E_{-1}(\phi_2, g)}\, dg = \int_K \int_{\mathbb{A}_E^* GL_n(E)\backslash GL_n(\mathbb{A}_E)} M_{-1}\phi_1(mk)\overline{\phi_2(mk)}\, dm\, dk, \quad (5)$$

where K is the standard maximal compact of $G(\mathbb{A}_F)$ and M_{-1} is the residue of the intertwining operator $M(s, \pi)$ at $s = \frac{1}{2}$.

Now suppose $\pi = \otimes_v \pi_v$ is distinguished with respect to $GL_n(\mathbb{A}_F)$, where the restricted tensor product is taken with respect to a choice of unramified vectors e_v almost everywhere. Let $(\cdot, \cdot)_\pi$ be the invariant positive definite Hermitian form on π. This gives rise to the invariant nondegenerate sesqui-linear form

$$(\cdot, \cdot)_s : I(\pi, -s) \times I(\pi, \bar{s}) \longrightarrow \mathbb{C}$$

given by

$$(\phi_1, \phi_2)_s = \int_K (\phi_1(k), \phi_2(k))_\pi \, dk.$$

Note that the right-hand side of (5) viewed as a positive definite invariant Hermitian form on $I(\pi, \frac{1}{2})$ is $(M_{-1}\phi_1, \phi_2)_{\frac{1}{2}}$. Choose positive definite Hermitian forms $(\cdot, \cdot)_{\pi_v}$ on π_v for all v so that $(e_v, e_v)_{\pi_v} = 1$ almost everywhere. We get sesqui-linear forms $(\cdot, \cdot)_{v,s}$ as above. Then, $(\cdot, \cdot)_\pi = \otimes_v (\cdot, \cdot)_{\pi_v}$ and $(\cdot, \cdot)_s = \otimes_v (\cdot, \cdot)_{v,s}$ up to a positive constant.

Proposition 2.10. Suppose π_v is such that $\pi_v^\vee \cong \pi_v^\sigma$. Then, the normalized operator $R^{\psi_v}(s, \pi_v)$ is Hermitian for $s \in \mathbb{R}$. □

Proof. The adjoint of

$$M(s, \pi_v, w) : I(\pi_v, s) \longrightarrow I(w\pi_v, -s)$$

is the operator

$$M(\bar{s}, w\pi_v, w^{-1}) : I(w\pi_v, \bar{s}) \longrightarrow I(\pi_v, -\bar{s})$$

(cf. Proposition 2.4.2, [35]). Since $w^2 = (-1)^n I_{2n}$, it follows that $M(s, \pi_v, w)^\star = \omega_{\pi_v}(-1)^n M(s, \pi_v, w)$ for s real. Since π_v is unitary, we have,

$$\overline{L(2s, \pi_v^\vee, r)} = L(2\bar{s}, \pi_v, r),$$

and

$$\overline{\epsilon\left(2s, \pi_v^\vee, r, \psi_v^{-1}\right)} = \epsilon(2\bar{s}, \pi_v, r, \psi_v).$$

Therefore we see that

$$\overline{m_v^{\psi_v}(s, \pi_v)} = \omega_{\pi_v}(-1)^n m_v^{\psi_v}(s, \pi_v)$$

for $s \in \mathbb{R}$, by Lemma 2.4, property (d) of Section 2.1, and Remark 2.6. The proposition follows. ■

3 Proof of Theorem 1.2

3.1 Idea of the proof

We proceed as in Lapid–Rallis. Let π be a cuspidal representation of $GL_n(\mathbb{A}_E)$ distinguished with respect to $GL_n(\mathbb{A}_F)$. We consider the form $J(\pi, s)$ on $I(\pi, s)$ defined by $(R(s)\phi_1, \phi_2)_s$, where $s \in \mathbb{R}$ and $R(s) = R(s, \pi)$ is the normalized intertwining operator. From (5), we know that $(M_{-1}\phi, \phi)_{\frac{1}{2}} \geq 0$. Now $M_{-1} = m_{-1}R(\frac{1}{2})$, and therefore it follows that $J(\pi, \frac{1}{2})$ is semi-definite with the same sign as m_{-1}. As in [23], the idea is to show that $J(\pi, \frac{1}{2})$ is positive semi-definite and to conclude that $m_{-1} > 0$.

We know that $L(s, \pi, r)$ is holomorphic and nonzero for $\mathrm{Re}\, s > 1$ [7]. Since π is unitary, $\overline{L(s, \pi, r)} = L(\bar{s}, \pi^\vee, r)$. Since $s \in \mathbb{R}$ and $\pi^\vee \cong \pi^\sigma$ as π is distinguished, we see that $L(s, \pi, r)$ is real and nonzero for $s > 1$ by (2). Therefore we get

$$\frac{\mathrm{res}_{s=1} L(s, \pi, r)}{L(2, \pi, r)} > 0.$$

Hence (4) implies that $\epsilon(1, \pi, r) > 0$. Similarly, $\epsilon(s, \pi, r)$ is real (and nonzero) and so $\epsilon(\frac{1}{2}, \pi, r) > 0$. Now from the functional equation, we have

$$\epsilon(s, \pi, r)\epsilon(1 - s, \pi^\vee, r) = 1$$

and hence $\epsilon(\frac{1}{2}, \pi, r) = \pm 1$ again by (2). This proves that $\epsilon(\frac{1}{2}, \pi, r) = 1$ if π is distinguished. If $\pi^\vee \cong \pi^\sigma$, then either π or $\pi \otimes \kappa$ is distinguished by the factorization of global L-functions given in the introduction. Now Theorem 1.2 follows from (1).

3.2 The continuity argument

We need to show that $J(\pi, \frac{1}{2})$ is positive semi-definite. The form $J(\pi, s)$ admits a local analogue and we have $J(\pi, s) = \otimes_v J^{\psi_v}(\pi_v, s)$ up to a positive constant. We know that

$J^{\psi_v}(\pi_v, \frac{1}{2})$ is Hermitian (cf. Proposition 2.10). It is also semi-definite. We will prove that each $J^{\psi_v}(\pi_v, \frac{1}{2})$ is positive semi-definite. Observe that in the unramified situation, that is when both E_v/F_v and π_v are unramified, this latter claim is obvious as $R(s, \pi_v)$ fixes the spherical vector for all s.

We can follow [23] verbatim. We will only summarize the key points here. The first step is to reduce to the case of tempered π_v. There is a continuous path $\pi_{v,t}$, $t \in [0, 1]$, inside unitary generic conjugate self-dual representations, from $\pi_v = \pi_{v,1}$ to the tempered representation $\pi_{v,0}$. As in [23], one can prove that if $J^{\psi_v}(\pi_v, \frac{1}{2})$ is semi-definite then $J^{\psi_v}(\pi_{v,0}, \frac{1}{2})$ is semi-definite with the same sign, by applying the following elementary lemma to the family of Hermitian forms $\{J(\pi_{v,t}, \frac{1}{2}), 0 < t \leq 1\}$ on K-types of $I(\pi_{v,t}, \frac{1}{2})$. The proof of the fact that the rank of $J(\pi_{v,t}, \frac{1}{2})$ is constant on a K-type requires a subtle analysis of reducibility of certain induced representations (cf. p. 22, [23]).

Lemma 3.1. Let $\{J_s\}_{a \leq s \leq b}$ be a continuous family of Hermitian forms on a finite-dimensional vector space. Suppose that $\mathrm{rank}(J_s)$ is constant for $a \leq s \leq b$. Then the positive and negative parts of the signature of J_s are also constant in $[a, b]$. □

Since $J(\pi, 1/2)$ is known to be semi-definite, each $J(\pi_v, 1/2)$ is semi-definite. We may also assume that π_v is tempered. Employing the theory of R-groups as in [13], it can be shown that semi-definiteness of $J(\pi_v, 1/2)$ implies that π_v is of "G-type" (cf. Section 5.3). In that case, the multiplicativity of normalized intertwining operators implies that $R(0, \pi_v)$ is a scalar (cf. Lemma 6 of [23]).

In the next section, following an argument due to Keys and Shahidi, we will show that

- $R(0, \pi_v)$ fixes the ψ_v-generic irreducible constituent of $I(\pi_v, 0)$.

Therefore it follows that $R(0, \pi_v) = I$ on $I(\pi_v, 0)$ and that the Hermitian form $J(\pi_v, 0)$ is positive definite on K-types of $I(\pi_v, 0)$. By the results of Casselman–Shahidi and Muić [5, 28], it follows that $I(\pi_v, s)$ is irreducible for $0 < s < \frac{1}{2}$. Thus $J(\pi_v, s)$ is nondegenerate on a given K-type. Applying Lemma 3.1 again, we see that $J(\pi_v, 1/2)$ is positive semi-definite. Therefore, $J(\pi, 1/2)$ is positive semi-definite.

3.3 The Keys–Shahidi argument

We need to show that $R(0, \pi_v)$ fixes the ψ_v-generic part of $I(\pi_v, 0)$. To this end, let $\lambda_{\psi_v}(s, \pi_v)$ be the Whittaker functional on $I(\pi_v, s)$ and let $\lambda_{\psi_v}(-s, w\pi_v) = \lambda_{\psi_v}(-s, \pi_v^{\vee\sigma})$ be the Whittaker functional associated to $I(\pi_v^{\vee\sigma}, -s)$. By [35], λ_{ψ_v} is entire and nonvanishing. Moreover, there exists a complex number $C_{\psi_v}(s, \pi_v, w)$, called the local coefficient, such that

$$\lambda_{\psi_v}(s, \pi_v) = C_{\psi_v}(s, \pi_v, w)\lambda_{\psi_v}(-s, w\pi_v)M(s, \pi_v, w).$$

By (3.11) of [36], the local coefficient is given by

$$C_{\psi_v}(s, \pi_v, w) = \frac{\epsilon\big(2s, \pi_v^\vee, r, \psi_v^{-1}\big)L(1-2s, \pi_v, r)}{\lambda(E/F, \psi_v)^{\binom{n}{2}} L\big(2s, \pi_v^\vee, r\big)}.$$

Therefore, from (3), we get that

$$\lambda_{\psi_v}(s, \pi_v)(\phi_v) = \frac{L(1-2s, \pi_v, r)}{L(1+2s, \pi_v, r)} \cdot \lambda_{\psi_v}(-s, \pi_v)(R(s, \pi_v)(\phi_v)) \tag{6}$$

by appealing to Lemma 2.4, since $\pi_v^\vee \cong \pi_v^\sigma$. Since $L(s, \pi_v, r)$ is holomorphic at $s = 1$ (cf. Proposition 7.2, [36]), it follows from (6) that $R(0, \pi_v)$ fixes the ψ_v-generic irreducible constituent of $I(\pi_v, 0)$.

4 Proof of Theorem 1.1

We start with the following globalization result [16, 33].

Proposition 4.1. Let E/F be a quadratic extension of p-adic fields. Let π be a supercuspidal representation of $GL_n(E)$, which is distinguished with respect to $GL_n(F)$. Then the following hold.

1. There exist a number field $\tilde{F}$, a quadratic extension $\tilde{E}$ of $\tilde{F}$, and a place v_0 of $\tilde{F}$ inert in $\tilde{E}$ such that $\tilde{F}_{v_0} \simeq F$ and $\tilde{E}_{w_0} \simeq E$, where w_0 is the unique place of $\tilde{E}$ dividing v_0. Moreover, the archimedean places of $\tilde{F}$ split in $\tilde{E}$.
2. There exists a cuspidal representation Π of $GL_n(\mathbb{A}_{\tilde{E}})$, which is distinguished with respect to $GL_n(\mathbb{A}_{\tilde{F}})$ such that $\Pi_{w_0} \simeq \pi$. Moreover, Π can be taken to be unramified at all the finite places outside w_0. □

Let Π be a cuspidal representation as in Proposition 4.1. Since Π is distinguished, $\Pi^\vee \simeq \Pi^\sigma$ and therefore by Theorem 1.2, we get $\epsilon(\frac{1}{2}, \Pi, r) = 1$. Let Ψ be an additive character of $\mathbb{A}_F/F$ such that $\Psi_{v_0} = \psi$.

We have

$$\epsilon(s, \Pi, r) = \prod_{w|v} \epsilon(s, \Pi_w, r, \Psi_v) \cdot \prod_{v=w_1w_2} \epsilon\big(s, \Pi_{w_1} \times \Pi_{w_2}, \Psi_v\big). \tag{7}$$

Since Π is distinguished, we will have that each Π_w is distinguished and that Π_{w_1} and Π_{w_2} are duals of each other. By the result of Bushnell and Henniart, we get:

$$\prod_{v=w_1w_2} \epsilon\left(\frac{1}{2}, \Pi_{w_1} \times \Pi_{w_2}, \Psi_v\right) = \prod_v \epsilon\left(\frac{1}{2}, \Pi_v \times \Pi_v^\vee, \Psi_v\right) = \prod_v \omega_{\Pi_v}(-1)^{n-1}.$$

Next we do a local unramified computation.

Proposition 4.2. Let $\pi = I(\chi_1, \chi_2, \ldots, \chi_n)$ be an unramified representation of $GL_n(E)$, which is distinguished with respect to $GL_n(F)$. Then $\epsilon(\frac{1}{2}, \pi, r, \psi) = \omega_\pi(\delta)^{n-1}\lambda(E/F, \psi)^{\binom{n}{2}}$. $\square$

Proof. Since π is unramified and distinguished,

$$\pi^\vee \simeq \pi^\sigma \simeq \pi$$

and $\omega_\pi|_{F^*} = 1$. Therefore without loss of generality we conclude that there exists $1 \leq l \leq n$ such that

$$\chi_1^{-1} = \chi_2, \chi_3^{-1} = \chi_4, \ldots, \chi_{2l-1}^{-1} = \chi_{2l}$$

and

$$\chi_i^{-1} = \chi_i \quad \text{for } i = 2l+1, 2l+2, \ldots, n.$$

Since there is only one unramified quadratic character of E^*, call it μ, it follows that π is of the form

$$\pi = I\big(\chi_1, \chi_1^{-1}, \ldots, \chi_l, \chi_l^{-1}, 1, \ldots, 1, \mu, \ldots, \mu\big)$$

for characters χ_i of E^*. We also note that the identity (b) of the second section can be used to compute the epsilon value since we know that for an unramified π, the epsilon factor defined by the Langlands–Shahidi method matches the one defined on the Galois side of the Langlands correspondence [36]. Also note that $r(\chi) = \chi|_{F^*}$. Now

$$\epsilon\left(\tfrac{1}{2}, \chi_i|_{F^*}, \psi\right)\epsilon\left(\tfrac{1}{2}, \chi_i^{-1}|_{F^*}, \psi\right) = \chi_i(-1) = 1,$$

since χ_i is unramified. We observe that if the number of μ's coming in π is odd, then $\mu|_{F^*} = 1$ from the condition on the central character of π. If this number is even, the μ's can be paired up. Thus

$$\epsilon\left(\tfrac{1}{2}, \mu|_{F^*}, \psi\right) \times \cdots \times \epsilon\left(\tfrac{1}{2}, \mu|_{F^*}, \psi\right) = 1,$$

since $\mu(-1) = 1$. Now

$$\epsilon\left(\tfrac{1}{2}, \mathrm{Ind}_E^F(\chi), \psi\right) = \lambda(E/F, \psi)\epsilon\left(\tfrac{1}{2}, \chi, \psi_E\right),$$

where $\psi_E(x) = \psi(\mathrm{trace}_{\,E/F}(x))$. Note that

- $\epsilon(\frac{1}{2}, \mu, \psi_E) = \mu(\delta)$ if $\mu|_{F^*} = 1$, where δ is a trace zero element of E, by a theorem due to Fröhlich and Queyrut [10];
- $\epsilon(\frac{1}{2}, \mu^2, \psi_E) = 1$, since $\mu^2 = 1$;
- products of the form $\epsilon(\frac{1}{2}, \chi, \psi_E)\epsilon(\frac{1}{2}, \chi^{-1}, \psi_E) = \chi(-1) = 1$, since χ is unramified;
- products of the form $\epsilon(\frac{1}{2}, \chi\mu, \psi_E)\epsilon(\frac{1}{2}, \chi^{-1}\mu, \psi_E) = \chi\mu(-1) = 1$, since χ and μ are unramified.

This proves the proposition. ■

Remark 4.3. Note that when $E = F \oplus F$, the factor $\omega_\eta(\delta)^{n-1}\lambda(E/F, \psi)^{\binom{n}{2}}$ takes the form $\omega_\pi(-1)^{n-1}$, where $\eta = \pi \otimes \pi^\vee$. □

Now let δ be a trace zero element in $\tilde{E}$. We conclude that the right-hand side of (7) is

$$\epsilon\left(\frac{1}{2}, \pi, r, \psi\right) \cdot \prod_{v \neq v_0} \omega_{\Pi_v}(\delta)^{n-1}\lambda(E_v/F_v, \psi_v)^{\binom{n}{2}}.$$

On the other hand, the left-hand side of (7) is 1 by Theorem 1.2. Since

$$\prod_v \omega_{\Pi_v}(\delta)^{n-1}\lambda(E_v/F_v, \psi_v)^{\binom{n}{2}} = 1,$$

it follows that

$$\epsilon\left(\tfrac{1}{2}, \pi, r, \psi\right) = \omega_\pi(\delta)^{n-1}\lambda(E/F, \psi)^{\binom{n}{2}},$$

if π is a supercuspidal representation of $GL_n(E)$ distinguished with respect to $GL_n(F)$.

It is easy to see that if $\pi = I(\chi_1, \ldots, \chi_n)$ is unramified and distinguished as in Proposition 4.2, then

$$\epsilon\left(\tfrac{1}{2}, \pi \otimes \kappa, r, \psi\right) = \lambda(E/F, \psi)^n \epsilon\left(\tfrac{1}{2}, \pi, r, \psi\right). \tag{8}$$

It follows, by Proposition 4.1 and Remark 2.3, that the above identity holds true for a supercuspidal distinguished π.

Now let π be a supercuspidal representation of $GL_n(E)$, which is $\omega_{E/F}$-distinguished with respect to $GL_n(F)$. Then,

$$\begin{aligned}\epsilon\left(\tfrac{1}{2},\pi,r,\psi\right) &= \epsilon\left(\tfrac{1}{2},(\pi\otimes\kappa^{-1})\otimes\kappa,r,\psi\right)\\ &= \lambda(E/F,\psi)^n\epsilon\left(\tfrac{1}{2},\pi\otimes\kappa^{-1},r,\psi\right)\\ &= \lambda(E/F,\psi)^n\omega_{\pi\otimes\kappa^{-1}}(\delta)^{n-1}\lambda(E/F,\psi)^{\binom{n}{2}}\\ &= \omega_\pi(\delta)^{n-1}\lambda(E/F,\psi)^{\binom{n+1}{2}}\omega_{E/F}(-1)^{\binom{n}{2}},\end{aligned}$$

since $\omega_{\pi\otimes\kappa^{-1}}(\delta)^{n-1}=\omega_\pi(\delta)^{n-1}\kappa^{-n(n-1)/2}(\delta^2)=\omega_\pi(\delta)^{n-1}\omega_{E/F}(-1)^{\binom{n}{2}}$.

Thus Theorem 1.1 is proved in the supercuspidal case.

Let π be a square integrable representation of $GL_n(E)$, which is distinguished with respect to $GL_n(F)$. Let π be the unique irreducible quotient of the representation parabolically induced from $\tau|\,|_E^{\frac{1-b}{2}}\otimes\cdots\otimes\tau|\,|_E^{\frac{b-1}{2}}$, where τ is a supercuspidal representation of $GL_a(E)$ with $n=ab$. Since π is distinguished with respect to $GL_n(F)$, by Theorem 1.3, we know that τ is distinguished (resp. $\omega_{E/F}$-distinguished) with respect to $GL_a(F)$ if b is odd (resp. even).

If π is parabolically induced from $\tau_1\otimes\cdots\otimes\tau_b$ of $GL_{a_1}(E)\times\cdots\times GL_{a_b}(E)$ with $n=(\sum_i a_i)b$, then by the multiplicativity of γ-factors (cf. (3.13) of [36]), we get:

$$\gamma\left(\frac{1}{2},\pi,r,\psi\right)=\prod_{i<j}\lambda(E/F,\psi)^{a_ia_j}\prod_i\gamma\left(\frac{1}{2},\tau_i,r,\psi\right)\prod_{i<j}\gamma\left(\frac{1}{2},\tau_i,r,\psi\right),$$

where ψ_E is the additive character of E given by composing ψ with the trace map.

The γ-factor is related to the ϵ-factor by

$$\gamma(s,\pi,r,\psi)=\epsilon(s,\pi,r,\psi)\cdot\frac{L(1-s,\pi^\vee,r)}{L(s,\pi,r)}.$$

Since the behavior of ϵ-factors under twists by unramified characters is well understood, it is not hard to show that in our situation we get the following identity by making use of multiplicativity:

$$\epsilon\left(\tfrac{1}{2},\pi,r,\psi\right)=\lambda(E/F,\psi)^{\binom{b}{2}a^2}\cdot\epsilon\left(\tfrac{1}{2},\tau,r,\psi\right)^b\cdot\epsilon\left(\tfrac{1}{2},\pi\times\pi^\sigma,\psi_E\right)^{\binom{b}{2}}.$$

Considering the cases b odd (i.e. τ is distinguished) and b even (i.e. τ is $\omega_{E/F}$-distinguished), it is straightforward to show the first identity of Theorem 1.1. Similarly, by interchanging the role of b being odd or even, the second identity can be proved.

Remark 4.4. As mentioned in Section 1, another approach to the local Asai root number is via the theory of Rankin–Selberg integrals. The theory has been worked out by Flicker (cf. main theorem of the appendix, [9]) and Kable (cf. Theorem 3, [19]). It is natural to ask to compute the local root number defined this way for a conjugate self-dual representation of $GL_n(E)$. Moreover, computing the root number in this manner will be in the spirit of the proofs of the theorem of Bushnell and Henniart by Bushnell–Henniart [4] and Jacquet (cf. Lemma 4, [23]). The essential simplifying aspect of these proofs is that it is easy to make the zeta integrals for $\epsilon(\frac{1}{2}, \pi \times \pi^\vee, \psi)$ non-negative by choosing Whittaker functions $W \in \mathcal{W}(\pi)$ and $\overline{W} \in \mathcal{W}(\pi^\vee)$. In the Asai situation, it is easy to choose a Whittaker function that makes the corresponding zeta integrals real valued, but non-negativity is not obvious at all. Multiplicativity of the corresponding γ-factors, which is proved in general in the Langlands–Shahidi framework, is also not known in the Rankin–Selberg context. $\square$

5 Two Conjectures

5.1 Root number of a distinguished representation

Let π be an irreducible admissible representation of $GL_2(E)$ such that its central character restricts trivially to F^*. Then it is known that π is distinguished with respect to $GL_2(F)$ if and only if $\gamma(\frac{1}{2}, \pi \otimes \lambda, \psi') = 1$ for all characters λ of E^*, which have trivial restriction to F^* [15]. Here, ψ' is an additive character of E/F. Thus, if δ is any trace zero element, ψ' is given by

$$\psi'(x) = \psi_E(\delta x) = \psi(\text{trace}_{E/F}(\delta x)).$$

In particular, if π is distinguished with respect to $GL_2(F)$ and λ is distinguished with respect to $GL_1(F)$, then $\epsilon(\frac{1}{2}, \pi \times \lambda, \psi') = 1$. This is because the epsilon factor is the same as the gamma factor under the given conditions.

Let π_1 and π_2 be square integrable representations of $GL_{n_1}(E)$ and $GL_{n_2}(E)$, respectively. Let π be the representation of $GL_{n_1+n_2}(E)$ parabolically induced from $\pi_1 \otimes \pi_2^\sigma$.

From the inductive property of gamma factors (cf. (3.13), [36]), we get the following identity:

$$\epsilon(s, \pi, r, \psi) = \lambda(E/F, \psi)^{n_1 n_2} \epsilon(s, \pi_1, r, \psi) \epsilon\left(s, \pi_2^{\sigma}, r, \psi\right) \epsilon(s, \pi_1 \times \pi_2, \psi_E),$$

where the third epsilon factor on the right is the Rankin–Selberg epsilon factor.

Suppose π_i are distinguished with respect to $GL_{n_i}(F)$ $(i = 1, 2)$. Then Theorem 1.1 of the present paper asserts that

$$\epsilon\left(\tfrac{1}{2}, \pi_i, r, \psi\right) = \omega_{\pi_i}(\delta)^{n_i - 1} \lambda(E/F, \psi)^{\binom{n_i}{2}}$$

for $i = 1, 2$. If we assume that $\epsilon(\frac{1}{2}, \pi, r, \psi)$ also has a similar expression, then the above inductive property would imply that

$$\epsilon(s, \pi_1 \times \pi_2, \psi_E) = \omega_{\pi_1}(\delta)^{n_2} \omega_{\pi_2}(\delta)^{n_1}$$

and this implies

$$\epsilon(s, \pi_1 \times \pi_2, \psi') = 1,$$

since $\delta^2 \in F^*$ and ω_{π_i} have trivial restriction to F^* $(i = 1, 2)$.

We are thus led to the following

Conjecture 5.1. Let π_i be irreducible admissible representations of $GL_{n_i}(E)$ which are distinguished with respect to $GL_{n_i}(F)$ $(i = 1, 2)$. Then, $\epsilon(\frac{1}{2}, \pi_1 \times \pi_2, \psi') = 1$. □

Remark 5.2. We have already remarked that when $n_1 = 2$ and $n_2 = 1$, the above conjecture is a theorem [15]. It may be of interest to note that when π is $\omega_{E/F}$-distinguished with respect to $GL_2(F)$ and $\lambda|_{F^*} = 1$, $\epsilon(\frac{1}{2}, \pi \times \lambda, \psi') = \pm 1$ and that this sign is $+1$ if and only if the character of $U(1)$ associated to λ appears in the restriction of the representation of $U(2)$ associated to π [1]. Thus, perhaps the value of $\epsilon(\frac{1}{2}, \pi_1 \times \pi_2, \psi')$, where π_1 is $\omega_{E/F}$-distinguished (resp. distinguished) with respect to $GL_n(F)$ and π_2 is distinguished (resp. $\omega_{E/F}$-distinguished) with respect to $GL_{n-1}(F)$, where n is even (resp. odd) might play a role in the Gross–Prasad conjecture in the context of unitary groups [14]. □

5.2 Root number for unitary groups

The purpose of this section is to compare Conjecture 5.1 with the computation of the ϵ-factor for unitary groups.

Let $U(n) = U(n, E/F)$ be the unitary group defined with respect to the form J_n, where J_n be the $n \times n$ matrix whose (i, j) entry is $(-1)^{n-i}\delta_{i,n-j+1}$. Let τ be an irreducible representation of $U(n)$. Let ψ be a character of $(F, +)$. For a character ω of E^* such that $\omega|_{F^*} = \omega_{E/F}^{n+1}$, Lapid and Rallis [26] compute the ϵ-factor of τ, defined via the so-called doubling method:

$$\epsilon\left(\frac{1}{2}, \tau \times \omega, \psi\right) = \omega_\tau(-1) \cdot \begin{cases} \omega(\delta^n) & n \text{ even} \\ \omega(\delta) & n \text{ odd.} \end{cases}$$

Suppose π is the (stable) base change of τ to $GL_n(E)$. Then it is conjectured that π is distinguished (resp. $\omega_{E/F}$-distinguished) with respect to $GL_n(F)$ if n is odd (resp. even). Thus for our choice of ω, $\pi \otimes \omega$ is always distinguished with respect to $GL_n(F)$. According to Conjecture 5.1,

$$\begin{aligned} \epsilon\left(\tfrac{1}{2}, \pi \times \omega, \psi_E\right) &= \omega_{\pi\otimes\omega}(\delta) \\ &= \omega_\pi(\delta)\omega^n(\delta) \\ &= \left(\omega_\tau^{-1}\omega_\tau^\sigma\right)(\delta)\omega^n(\delta) \\ &= \epsilon\left(\tfrac{1}{2}, \tau \times \omega, \psi\right), \end{aligned}$$

since $\omega(\delta^2) = 1$ when n is odd.

If $\omega|_{F^*} = \omega_{E/F}^n$, then $\pi \otimes \omega$ is always $\omega_{E/F}$-distinguished with respect to $GL_n(F)$. In this case, $\epsilon(\frac{1}{2}, \pi \times \omega, \psi_E)$ is a more subtle invariant (cf. Remark 5.2).

5.3 Distinguished representations and intertwining operators

Let $G = U(n, n)$ be the quasi-split unitary group in $2n$ variables defined with respect to E/F. Let P be a parabolic subgroup of G with a Levi component M isomorphic to $GL_{n_1}(E) \times \cdots \times GL_{n_t}(E)$ with $\sum_{i=1}^t n_i = n$. Let π_i, $1 \leq i \leq t$, be square integrable representations o $GL_{n_i}(E)$. Let $\pi = \pi_1 \otimes \cdots \otimes \pi_t$ be the associated square integrable representation of M. Then the R-group $R(\pi)$ is isomorphic to the product of r copies of $\mathbb{Z}/2$'s, where r is the number of inequivalent representations π_i, which are $\omega_{E/F}$-distinguished with respect to $GL_{n_i}(F)$ (cf. Theorem 1.3 of [3], see also Proposition 2.1 of [31]).

Now let π be an irreducible generic representation of $GL_n(E)$ such that $\pi^\vee \cong \pi^\sigma$. Then π can be written uniquely as a parabolically induced representation of the form

$$\pi_1 \times \pi_2 \times \pi_3,$$

where

- π_1 is of the form $\sigma_1 \times \cdots \times \sigma_s$, where the σ_i's are square integrable and distinguished;
- π_2 is of the form $\rho_1 \times \cdots \times \rho_r$, where the ρ_i's are square integrable, mutually inequivalent, and $\omega_{E/F}$-distinguished;
- π_3 is of the form $\tau_1^\vee \times \tau_1^\sigma \times \cdots \times \tau_t^\vee \times \tau_t^\sigma$, where the τ_i's are essentially square integrable and not distinguished, with $0 \leq e(\tau_i) < \frac{1}{2}$.

Here $e(\tau)$ is the exponent of τ. It is the unique real number so that $\tau \otimes |\ |_E^{-e(\tau)}$ has a unitary central character.

Certain computations involving the Rogwaski's constant mentioned in Remark 2.2 (cf. Remark 2.1 and Corollary 2.1 of [31]) lead us to believe that π is distinguished with respect to $GL_n(F)$ precisely when $\pi_2 = 0$.

Let us discuss the conjectural framework in a little bit more detail. To this end, let ϕ be a parameter for the unitary group $U(n) = U(n, E/F)$. Let ϕ_E be the Langlands parameter obtained via the (stable) base change lift. Write $\phi_E = \sum n_i \phi_i$, where n_i is the multiplicity of the irreducible representation ϕ_i. According to Corollary 2.1 of [31], an irreducible conjugate self-dual representation ϕ_i appearing with odd multiplicity in ϕ_E must have $c(\phi_i) = (-1)^{n-1}$. Here $c(\phi_i)$ is the Rogawski's constant. Recall that the irreducible admissible representation corresponding to a parameter ρ is supposed to be distinguished exactly when $c(\rho) = 1$ (cf. Remark 2.2). Now if π corresponds to ϕ_E, then n has to be odd since π is distinguished. Thus ϕ_i appearing in ϕ_E with odd multiplicity implies that the representation corresponding to ϕ_i has to be distinguished. If $\pi_2 \neq 0$, then there is an $\omega_{E/F}$-distinguished representation that comes with odd multiplicity, thus forcing $\pi_2 = 0$. On the other hand, if $\pi_2 = 0$, then π must be distinguished as it is natural to believe that a representation parabolically induced from distinguished representations is distinguished. If n is even, by applying similar arguments to $\pi \otimes \kappa$, we once again conclude that $\pi_2 = 0$.

In other words, the distinguished representations are supposed to coincide with the representations of "G-type" as defined by Lapid and Rallis (cf. p.17, [23]). Specifically, Lemma 6 of [23] motivates us to make the following conjecture.

Conjecture 5.3. Let π be an irreducible generic representation of $GL_n(E)$ such that $\pi^\vee \cong \pi^\sigma$. Then π is distinguished with respect to $GL_n(F)$ if and only if the normalized intertwining operator $R(0,\pi)$ on the parabolically induced representation $I(\pi,0)$ of G is a scalar. □

Note that if π is square integrable and distinguished, $I(\pi,0)$ is irreducible and hence $R(0,\pi)$ is a scalar. But in general, π can be distinguished with nontrivial π_3 such that some of the τ_i's are square integrable and $\omega_{E/F}$-distinguished and in that case $I(\pi,0)$ is reducible. The point of the conjecture is that even when $I(\pi,0)$ is reducible, $R(0,\pi)$ remains a scalar.

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