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Iwahori–Hecke model for supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

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In this paper, we realize a regular supersingular representation π of $GL_2(\mathbb{Q}_p)$ as a quotient of a representation induced from the Iwahori subgroup of $GL_2(\mathbb{Q}_p)$. We also show that this realization provides a uniform way of looking at all the self-extensions of π which have a four dimensional space of $I(1)$-invariants.

1. Introduction

The theory of mod p smooth representations of p-adic reductive groups started with the seminal work of Barthel and Livné [3,4] which classified smooth irreducible $\bar{\mathbb{F}}_p$-representations of $GL_2(F)$, where F is a finite extension of $\mathbb{Q}_p$, with a central character. These representations fall into four disjoint classes: (i) one dimensional characters,

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(ii) twists of the Steinberg representation, (iii) irreducible principal series, and (iv) the so-called supersingular representations, which, it turns out, cannot be obtained as a subquotient of any principal series representation. In the work of Barthel and Livné, both the Steinberg and the principal series representations are explicitly realized. For $F = \mathbb{Q}_p$, the classification of supersingular representations has been carried out by Breuil [6], where it is proved that the relevant quotients for supersingulars

$$\frac{\mathrm{ind}_{GL_2(\mathcal{O}_F)\cdot F^*}^{GL_2(F)} \mathrm{Sym}^r \bar{\mathbb{F}}_p^2}{(T)}$$

considered by Barthel and Livné are in fact irreducible. When $F \neq \mathbb{Q}_p$, the corresponding quotient representations are of infinite length and are not admissible [8, Theorem 3.3], and the classification of supersingular representations is quite complicated [8,5].

When $F = \mathbb{Q}_p$, the extensions between these representations are understood by the work of Paškūnas [16]. For a supersingular representation π of $GL_2(\mathbb{Q}_p)$, it is known that (for $p > 3$)

$$\dim \mathrm{Ext}^1_{GL_2(\mathbb{Q}_p)}(\pi, \pi) = 3,$$

where Ext is computed in the category of smooth representations with a given central character [16, Theorem 1.1].

Let $G = GL_2(\mathbb{Q}_p)$ and $K = GL_2(\mathbb{Z}_p)$ be its standard maximal compact subgroup. Let Z be the center of G. Let

$$\pi_r = \frac{\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \bar{\mathbb{F}}_p^2}{(T)},$$

where $0 \leq r \leq p-1$, be a supersingular representation of $G = GL_2(\mathbb{Q}_p)$. Here, the Hecke operator T is such that [3, Proposition 8]

$$\mathrm{End}_G\big(\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \bar{\mathbb{F}}_p^2\big) \cong \bar{\mathbb{F}}_p[T].$$

We have the following isomorphism (cf. [6, Theorem 1.3], Theorem 5.1, and Remark 5):

$$\frac{\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \bar{\mathbb{F}}_p^2}{(T)} \cong \frac{\mathrm{ind}_{KZ}^G \mathrm{Sym}^{p-1-r} \bar{\mathbb{F}}_p^2}{(T)} \otimes \det{}^r. \tag{1}$$

Before we come to our results, it is instructive to draw a consequence of the above isomorphism to the question of finding self-extensions of π_r. One important property of the Hecke operator T is that it is injective, and this has the corollary that we have an exact sequence:

$$0 \to \pi_r \to \frac{\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \bar{\mathbb{F}}_p^2}{(T^2)} \to \pi_r \to 0.$$

Thanks to (1), we get one more non-split self-extension of π_r given by

$$\frac{\operatorname{ind}_{KZ}^{G} \operatorname{Sym}^{p-1-r} \bar{\mathbb{F}}_p^2}{(T^2)} \otimes \det^r.$$

It is known that the two dimensional subspace of the three dimensional $\operatorname{Ext}^1_G(\pi_r, \pi_r)$ spanned by the classes of these two self-extensions consists of all the self-extensions of π_r which have a four dimensional space of $I(1)$-invariants, where $I(1)$ denotes the pro-p-Iwahori subgroup of G (cf. Subsection 7.4).

Our focus in this paper is to give an alternative description of π_r as a quotient of a representation induced from the Iwahori subgroup of G by the images of certain Iwahori–Hecke operators. We carry this out only when π_r is regular; i.e., when $0 < r < p-1$. Via this model, which we call the Iwahori–Hecke model, the self-extensions of π_r belonging to the above mentioned two dimensional space can all be realized in a natural manner.

To this end, consider the unique nontrivial element in

$$\mathbb{P}^1\big(\operatorname{Ext}^1_{GL_2(\mathbb{F}_p)}\big(\operatorname{Sym}^{p-1-r}\bar{\mathbb{F}}_p^2 \otimes \det^r, \operatorname{Sym}^r\bar{\mathbb{F}}_p^2\big)\big)$$

given by

$$0 \to \operatorname{Sym}^r\bar{\mathbb{F}}_p^2 \to \operatorname{ind}_{B(\mathbb{F}_p)}^{GL_2(\mathbb{F}_p)} d^r \to \operatorname{Sym}^{p-1-r}\bar{\mathbb{F}}_p^2 \otimes \det^r \to 0, \tag{2}$$

where d^r denotes the character [7, Proof of Proposition 4.7] (see also [9, §7] and [15, §3.1])

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto d^r.$$

Let d^r continue to denote the character of I, the Iwahori subgroup of G, obtained by inflating d^r under reduction modulo p. We then have [3, Proposition 13]:

$$\operatorname{End}_G\big(\operatorname{ind}_{IZ}^G d^r\big) \cong \frac{\bar{\mathbb{F}}_p[T_{-1,0}, T_{1,2}]}{(T_{-1,0}T_{1,2}, T_{1,2}T_{-1,0})},$$

where $T_{-1,0}$ and $T_{1,2}$ are certain Iwahori–Hecke operators (cf. Subsection 2.4).

The following theorem gives the Iwahori–Hecke model of a supersingular representation.

Theorem 1.1. *For $0 < r < p-1$, the representation*

$$\frac{\operatorname{ind}_{IZ}^G d^r}{(T_{-1,0}, T_{1,2})}$$

is isomorphic to $\pi_r = \frac{\operatorname{ind}_{KZ}^G \operatorname{Sym}^r \bar{\mathbb{F}}_p^2}{(T)}$, and thus it is an irreducible supersingular representation.

In Section 4, we in fact prove the stronger assertion that (cf. Theorem 4.1)

$$\mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r \bar{\mathbb{F}}_p^2 \cong \frac{\mathrm{ind}_{IZ}^{G} d^r}{(T_{-1,0})},$$

from which Theorem 1.1 is immediate. We may also remark here that Theorem 4.1 and the first part of Theorem 1.1 hold true for any totally ramified extension of $\mathbb{Q}_p$ as well (cf. Remark 1 and Remark 3).

The two non-split self-extensions mentioned earlier can now be realized as follows (cf. Section 6):

$$\frac{\mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r \bar{\mathbb{F}}_p^2}{(T^2)} \cong \frac{\mathrm{ind}_{IZ}^{G} d^r}{(T_{-1,0}, T_{1,2}^2)}, \tag{3}$$

$$\frac{\mathrm{ind}_{KZ}^{G} \mathrm{Sym}^{p-1-r} \bar{\mathbb{F}}_p^2}{(T^2)} \otimes \det{}^r \cong \frac{\mathrm{ind}_{IZ}^{G} d^r}{(T_{-1,0}^2, T_{1,2})}. \tag{4}$$

And hopefully by taking a larger reducible representation in the numerator instead of the irreducible symmetric powers, we have created more space to carve in different directions and to quotient out, in order to realize linear combinations of (3) and (4) as well. Indeed, in Section 6, we show that

$$\frac{\mathrm{ind}_{IZ}^{G} d^r}{(\lambda_1^{-1} T_{-1,0} - \lambda_2^{-1} T_{1,2})}, \tag{5}$$

where $\lambda_1 \lambda_2 \neq 0$, is a non-split self-extension of π_r, which is in fact

$$\lambda_1 \frac{\mathrm{ind}_{IZ}^{G} d^r}{(T_{-1,0}^2, T_{1,2})} + \lambda_2 \frac{\mathrm{ind}_{IZ}^{G} d^r}{(T_{-1,0}, T_{1,2}^2)},$$

where $+$ indicates the addition of associated short exact sequences in $\mathrm{Ext}_G^1(\pi_r, \pi_r)$ (cf. Section 6).

We summarize the above discussion as our next theorem.

Theorem 1.2. *Let* $\pi_r = \frac{\mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r \bar{\mathbb{F}}_p^2}{(T)}$, $0 < r < p-1$, *be an irreducible supersingular representation of* $GL_2(\mathbb{Q}_p)$. *Then a basis for the subspace of* $\mathrm{Ext}_G^1(\pi_r, \pi_r)$ *consisting of isomorphism classes of short exact sequences*

$$0 \to \pi_r \to \tau \to \pi_r \to 0,$$

where the representation τ *has a four dimensional space of* $I(1)$*-invariants, is given by isomorphism classes of*

(i) $0 \to \pi_r \to \frac{\mathrm{ind}_{IZ}^G\, d^r}{(T_{-1,0}, T_{1,2}^2)} \cong \frac{\mathrm{ind}_{KZ}^G \,\mathrm{Sym}^r\, \bar{\mathbb{F}}_p^2}{(T^2)} \to \pi_r \to 0,$

(ii) $0 \to \pi_r \to \frac{\mathrm{ind}_{IZ}^G\, d^r}{(T_{-1,0}^2, T_{1,2})} \cong \frac{\mathrm{ind}_{KZ}^G \,\mathrm{Sym}^{p-1-r}\, \bar{\mathbb{F}}_p^2}{(T^2)} \otimes \det^r \to \pi_r \to 0.$

Furthermore, for $\lambda_1, \lambda_2 \in \bar{\mathbb{F}}_p^\times$, the linear combination of the above elements, namely $\lambda_1 \cdot (i) + \lambda_2 \cdot (ii)$, in $\mathrm{Ext}_G^1(\pi_r, \pi_r)$ is given by the class of

$$0 \to \pi_r \to \frac{\mathrm{ind}_{IZ}^G\, d^r}{(\lambda_1^{-1} T_{-1,0} - \lambda_2^{-1} T_{1,2})} \to \pi_r \to 0. \tag{6}$$

In the last section, Section 7, we analyze the pro-p-Iwahori Hecke module structure on the space of pro-p-Iwahori invariants of these representations. Appealing to a result of Ollivier on the equivalence between the category of smooth representations, with a given central character, of $GL_2(\mathbb{Q}_p)$ which are generated by their $I(1)$-invariants and the category of pro-p-Iwahori–Hecke algebra modules [14], we provide a second proof of (6) (cf. Proposition 7.1). Yet another, perhaps more conceptual, approach to this question is indicated in Subsection 7.3, and the argument there is supplied to us by V. Paškūnas.

We end this introduction by noting that the Iwahori–Hecke model appears to be more amenable to carrying out certain computations. For instance, the authors have used the methods of this paper to give a short proof of the K-socle filtration of a supersingular representation of $GL_2(\mathbb{Q}_p)$ and to determine the structure of invariants under the principal congruence subgroups of K as a K-module [1]. These results are originally due to S. Morra [12,13]. We hope that the idea of considering quotients of compact inductions from groups smaller than the maximal compact group may prove to be useful in more general situations.

2. Preliminaries

For most part of this section, we follow [3, §2, §3] and [6, §2] closely.

2.1. Notations

Let $p \geq 3$ be a prime number. Let $\mathbb{F}_p$ denote the field with p elements. We fix an algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$. Let $\mathbb{Q}_p$ denote the field of p-adic numbers, and $\mathbb{Z}_p$, its ring of integers. Let $G = GL_2(\mathbb{Q}_p)$. The standard maximal compact subgroup of G is $K = GL_2(\mathbb{Z}_p)$. The center of G is denoted by Z. Let I (resp. $I(1)$) denote the Iwahori (resp. pro-p-Iwahori) subgroup of G. We have:

$$I = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}, \qquad I(1) = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix}.$$

We also define:

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \qquad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mathcal{A}$ be the tree of $SL_2(\mathbb{Q}_p)$. The vertices (resp. oriented edges) of $\mathcal{A}$ are in G-equivariant bijection with G/KZ (resp. G/IZ). Put $I_0 = \{0\}$, and for $n > 0$, let

$$I_n = \left\{[\lambda_0] + [\lambda_1]p + \ldots + [\lambda_{n-1}]p^{n-1}, \lambda_i \in \mathbb{F}_p\right\} \subset \mathbb{Z}_p,$$

where $[\cdot]$ denotes the multiplicative representative. For $n \in \mathbb{N}$ and $\lambda \in I_n$, define:

$$g^0_{n,\lambda} = \begin{pmatrix} p^n & \lambda \\ 0 & 1 \end{pmatrix}, \qquad g^1_{n,\lambda} = \begin{pmatrix} 1 & 0 \\ p\lambda & p^{n+1} \end{pmatrix}.$$

Note that

$$g^0_{0,0} = \mathrm{Id}, \qquad g^1_{0,0} = \alpha, \qquad \beta g^0_{n,\lambda} = g^1_{n,\lambda} w.$$

The $g^0_{n,\lambda}$ and $g^1_{n,\lambda}$ together form a set of representatives for G/KZ:

$$G = \left(\coprod_{n,\lambda} g^0_{n,\lambda} KZ\right) \sqcup \left(\coprod_{n,\lambda} g^1_{n,\lambda} KZ\right).$$

Similarly, a set of coset representatives for G/IZ is given by

$$\left\{ g^0_{n,\lambda}, g^1_{n,\lambda} w, g^0_{n,\lambda} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} w, g^1_{n,\lambda} w \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} w \right\}$$

where $\mu \in I_1$.

For $0 \le m \le n$, let $[\cdot]_m : I_n \to I_m$ denote the truncation map given by

$$\sum_{i=0}^{n-1} [\lambda_i] p^i \mapsto \sum_{i=0}^{m-1} [\lambda_i] p^i.$$

2.2. Generalities on Hecke algebras

Let H be an open subgroup of G and let (σ, V_σ) be a smooth representation of H over $\bar{\mathbb{F}}_p$. Let $(\mathrm{ind}_H^G \sigma, S(G,\sigma))$ denote the compactly supported induction of σ:

$$S(G,\sigma) = \left\{ f : G \to V_\sigma \;\middle|\; f(hg) = \sigma(h)\big(f(g)\big), h \in H, g \in G \right\},$$

where f is locally constant and compactly supported modulo H on the left. The group G acts on $S(G,\sigma)$ on the right:

$$(gf)\big(g'\big) = f\big(g'g\big).$$

For $g \in G$ and $v \in V_\sigma$, let $[g, v]$ denote the element of $\operatorname{ind}_H^G \sigma$ defined by

$$[g, v]\big(g'\big) = \begin{cases} \sigma(g'g) \cdot v & \text{if } g'g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$g\big([g', v]\big) = [gg', v]$$

and

$$[gh, v] = \big[g, \sigma(h)v\big],$$

for $h \in H$. Any element in $\operatorname{ind}_H^G \sigma$ is a finite sum $\sum_i [g_i, v_i]$ with $g_i \in G$, $v_i \in V_\sigma$.

The Hecke algebra associated to (H, σ) is by definition $\operatorname{End}_G(\operatorname{ind}_H^G \sigma)$. By Frobenius reciprocity, the Hecke algebra is isomorphic to the algebra of functions $\phi : G \to \operatorname{End}_{\bar{\mathbb{F}}_p}(V_\sigma)$ with compact support modulo H such that

$$\phi(h_1 g h_2) = \sigma(h_1) \circ \phi(g) \circ \sigma(h_2)$$

for $h_1, h_2 \in H$ and $g \in G$. If ϕ is such a function, the corresponding intertwining operator T_ϕ is defined by

$$T_\phi\big([g, v]\big) = \sum_{g'H \in G/H} \big[gg', \phi\big({g'}^{-1}\big)(v)\big].$$

If ϕ is supported on one double coset $H\gamma^{-1}H$ for some $\gamma \in G$, then if $H\gamma H = \coprod_i h_i \gamma H$, we may write the above formula as

$$T_\phi\big([g, v]\big) = \sum_i \big[g h_i \gamma, \phi\big(\gamma^{-1}\big)\sigma\big(h_i^{-1}\big)(v)\big].$$

2.3. The spherical Hecke algebra

Let $\operatorname{Sym}^r \bar{\mathbb{F}}_p^2$ denote the representation of K obtained by inflating the rth symmetric power of the standard representation of $GL_2(\mathbb{F}_p)$. Recall that

$$\left(\operatorname{Sym}^r \bar{\mathbb{F}}_p^2, \bigoplus_{i=0}^{r} \bar{\mathbb{F}}_p x^{r-i} y^i\right)$$

is given by

$$\left(\operatorname{Sym}^r \bar{\mathbb{F}}_p^2\right)\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\left(x^{r-i}y^i\right) = (ax+cy)^{r-i}(bx+dy)^i.$$

Then, $\operatorname{Sym}^r \bar{\mathbb{F}}_p^2$, $0 \le r \le p-1$, are irreducible, and moreover these are all the irreducible representations of K, up to twisting by one dimensional characters.

Taking $H = KZ$ and considering $\operatorname{Sym}^r \bar{\mathbb{F}}_p^2$ as a representation of KZ by making $\operatorname{diag}(p,p)$ act trivially, the formula for T_ϕ can be written in terms of the specific coset representatives introduced earlier in Subsection 2.1, where ϕ is supported in $KZ\alpha^{-1}K$. To this end, for $\lambda \in \mathbb{Z}_p$, let $w_\lambda = \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}$. Then (cf. [6, §2.5])

$$T_\phi\left([g,v]\right) = \sum_{\lambda \in I_1}\left[gg^0_{1,\lambda}, \sigma(w)\phi\left(\alpha^{-1}\right)\sigma(w_\lambda)(v)\right] + \left[g\alpha, \phi\left(\alpha^{-1}\right)(v)\right],$$

where $\sigma = \operatorname{Sym}^r \bar{\mathbb{F}}_p^2$.

Let $T = T_{\phi_1}$ denote the intertwining operator corresponding to ϕ_1 which has support on $KZ\alpha^{-1}K$ and such that

$$\phi_1\left(\alpha^{-1}\right) = \begin{bmatrix} & & & & 0 \\ & & & & \cdot \\ & & 0 & & \cdot \\ & & & & \cdot \\ & & & & 0 \\ 0 & \cdot & \cdot & \cdot & 0 \quad 1 \end{bmatrix}_{(r+1)\times(r+1)},$$

in the basis $\{x^{r-i}y^i\}_{0\le i\le r}$ of $\operatorname{Sym}^r \bar{\mathbb{F}}_p^2$. Now,

$$T\left(\left[\mathrm{Id}, x^{r-i}y^i\right]\right) = \begin{cases} \sum_{\lambda\in I_1}(-\lambda)^i[g^0_{1,\lambda}, x^r] & \text{if } i \ne r, \\ [\alpha, y^r] + \sum_{\lambda\in I_1}(-\lambda)^r[g^0_{1,\lambda}, x^r] & \text{if } i = r. \end{cases} \tag{7}$$

Moreover, the spherical Hecke algebra is the polynomial algebra in one variable [3, Proposition 8]:

$$\operatorname{End}_G\left(\operatorname{ind}_{KZ}^G \operatorname{Sym}^r \bar{\mathbb{F}}_p^2\right) \cong \bar{\mathbb{F}}_p[T].$$

We know that

$$\pi_r = \frac{\operatorname{ind}_{KZ}^G \operatorname{Sym}^r \bar{\mathbb{F}}_p^2}{(T)}$$

is irreducible [6, Theorem 1.1]. As already recalled in the introduction, we have the following isomorphism [6, Theorem 1.3]:

$$\frac{\operatorname{ind}_{KZ}^G \operatorname{Sym}^r \bar{\mathbb{F}}_p^2}{(T)} \cong \frac{\operatorname{ind}_{KZ}^G \sigma_{p-1-r}}{(T)} \otimes \det{}^r.$$

2.4. The Iwahori–Hecke algebra

Let d^r, $0 < r < p-1$, denote the character

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto d^r$$

of the Borel subgroup of $GL_2(\mathbb{F}_p)$, and let d^r continue to denote the character of IZ, obtained on I by inflating d^r under reduction modulo p, and by making $\operatorname{diag}(p,p)$ act trivially. We are interested in the Iwahori–Hecke algebra of (IZ, d^r). For this, let $\phi_{n,n+1}$ (resp. $\phi_{n+1,n}$) be the element of the associated convolution algebra, which is supported on $IZ\alpha^{-n}I$ (resp. $IZ\beta\alpha^{-n}I$) and is determined by $\phi_{n,n+1}(\alpha^{-n}) = 1$ (resp. $\phi_{n+1,n}(\beta\alpha^{-n}) = 1$). The corresponding intertwining operators are denoted by $T_{n,n+1}$ and $T_{n+1,n}$. Among the various properties of these Hecke operators, we will have occasion to use the following:

$$T_{-1,0} \circ T_{1,2} = 0 \quad \& \quad T_{1,2} \circ T_{-1,0} = 0. \tag{8}$$

In fact the Iwahori–Hecke algebra is the following commutative algebra:[3, Proposition 13]:

$$\operatorname{End}_G\big(\operatorname{ind}_{IZ}^G d^r\big) \cong \frac{\bar{\mathbb{F}}_p[T_{-1,0}, T_{1,2}]}{(T_{-1,0}T_{1,2}, T_{1,2}T_{-1,0})}.$$

The cases $r = 0, p-1$ have to be dealt with separately [4], but we do not get into the details as we restrict ourselves to the regular case $0 < r < p-1$ throughout this paper.

Let us denote a typical element of $\operatorname{ind}_{IZ}^G d^r$ by $[\![g, 1]\!]$. We have the following explicit formulas

$$T_{-1,0}\big([\![g, 1]\!]\big) = \sum_{\lambda \in I_1} [\![g g^0_{1,\lambda}, 1]\!]. \tag{9}$$

$$T_{1,2}\big([\![g, 1]\!]\big) = \sum_{\lambda \in I_1} \left[\!\!\left[g\beta \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} w, 1 \right]\!\!\right], \tag{10}$$

obtained by substituting $n = 1$ in [3, (16) and (17)].

Note that in our identification of oriented edges of the tree $\mathcal{A}$ of $SL_2(\mathbb{Q}_p)$ with G/IZ, the edge corresponding to $[\![g, 1]\!]$ has the edge corresponding to $[\![g\beta, 1]\!]$ as its opposite edge. Now for an oriented edge e of $\mathcal{A}$, let $o(e)$ (resp. $t(e)$) denote its origin (resp. terminus). Then,

$$T_{-1,0}(e) = \sum_{t(e') = o(e)} a(e')e',$$

and

$$T_{1,2}(e) = \sum_{o(e')=t(e)} b(e')e',$$

where $a(e'), b(e') \in \mathbb{F}_p$ can be easily worked out using (9) and (10). Visualizing the Iwahori–Hecke operators this way will come in handy in investigating their properties in Section 3.

3. The generators of the Iwahori–Hecke algebra

In this section, we prove a couple of properties of the Iwahori–Hecke operators $T_{-1,0}$ and $T_{1,2}$ which we will use in the sequel.

We know from (8) that $\operatorname{Im} T_{1,2} \subseteq \operatorname{Ker} T_{-1,0}$ and $\operatorname{Im} T_{-1,0} \subseteq \operatorname{Ker} T_{1,2}$. The next proposition asserts that they are in fact equal.

Proposition 3.1. *We have:*

$$\operatorname{Im} T_{-1,0} = \operatorname{Ker} T_{1,2} \quad \& \quad \operatorname{Im} T_{1,2} = \operatorname{Ker} T_{-1,0}.$$

Proof of Proposition 3.1. We need to prove only one of the two identities in the statement of Proposition 3.1, as the proof of the other one proceeds exactly the same way. Thus, we are going to prove that

$$\operatorname{Ker} T_{1,2} \subseteq \operatorname{Im} T_{-1,0}.$$

Let $f \in \operatorname{Ker} T_{1,2}$. Write $f = f^0 + f^1$ where f^0 is a linear combination of vectors of the form

$$[\![g^0_{n,\lambda}, 1]\!], \left[\!\!\left[g^0_{n-1,[\lambda]_{n-1}} \begin{pmatrix} 1 & \lambda_{n-1} \\ 0 & 1 \end{pmatrix} w, 1 \right]\!\!\right], [\![\beta, 1]\!], \tag{11}$$

and f^1 is a linear combination of vectors of the form

$$[\![g^1_{n,\lambda} w, 1]\!], \left[\!\!\left[g^1_{n-1,[\lambda]_{n-1}} w \begin{pmatrix} 1 & \lambda_{n-1} \\ 0 & 1 \end{pmatrix} w, 1 \right]\!\!\right], [\![\mathrm{Id}, 1]\!], \tag{12}$$

where, $n \geq 1$ is an integer, $\lambda = [\lambda_0] + [\lambda_1]p + \ldots + [\lambda_{n-1}]p^{n-1} \in I_n$, and $[\cdot]$ is the truncation map (cf. Subsection 2.1). It is useful to visualize the vectors in (11) as edges on one side of the tree $\mathcal{A}$, say on the left, and the vectors in (12) as edges on the other side of the tree, say on the right. Observe that the decomposition $f = f^0 + f^1$ is unique. Now, by (10), $T_{1,2}(f^0)$ (resp. $T_{1,2}(f^1)$) is again a linear combination of edges on the left (resp. right). Thus, it follows that both f^0 and f^1 are in $\operatorname{Ker} T_{1,2}$. Since βf^1 is a linear combination of vectors of the form (11), and since $T_{-1,0}$ and $T_{1,2}$ are intertwining operators, without loss of generality we can assume that f itself is a linear combination of vectors of the form (11).

Now write f uniquely as

$$f = f_m + f_{m-1} + \ldots + f_1,$$

where f_i $(i \geq 2)$ is a linear combination of vectors of the form

$$\llbracket g^0_{i,\lambda}, 1 \rrbracket, \left\llbracket g^0_{i-2,[\lambda]_{i-2}} \begin{pmatrix} 1 & \lambda_{i-2} \\ 0 & 1 \end{pmatrix} w, 1 \right\rrbracket \tag{13}$$

and f_1 is a linear combination of $\llbracket g^0_{1,\lambda}, 1 \rrbracket$ and $\llbracket \beta, 1 \rrbracket$. In our description of vectors in $\operatorname{ind}_{IZ}^G d^r$ as edges of the tree $\mathcal{A}$, the edges in (13) are the edges in f with terminus, i.e., oriented towards, the vertices of $\mathcal{A}$ associated to the cosets

$$g^0_{i-1,[\lambda]_{i-1}} KZ \in G/KZ,$$

for various

$$[\lambda]_{i-1} = [\lambda_0] + [\lambda_1]p + \cdots + [\lambda_{i-2}]p^{i-2}.$$

Now $T_{1,2}(f_i)$ consists of linear combinations of edges which constitute edges of the form f_{i+1} and f_{i-1} with their origins at vertices $g^0_{i-1,[\lambda]_{i-1}} KZ$. We conclude that $T_{1,2}(f_i)$ and $T_{1,2}(f_j)$ have no oriented edge in common for $i \neq j$. It follows that each f_i $(i = 1, 2, \ldots, m)$ is in $\operatorname{Ker} T_{1,2}$.

We do one more reduction. Write each f_i as

$$f_i = f_{i,1} + \ldots + f_{i,p^{i-1}}$$

where the j's in $f_{i,j}$'s are indexed by $(\lambda_0, \lambda_1, \ldots, \lambda_{i-2}) \in \mathbb{F}_p^{i-1}$, with each $f_{i,j}$ a linear combination of vectors of the form (13) for a fixed $(\lambda_0, \lambda_1, \ldots, \lambda_{i-2})$. Observe that for a fixed $(\lambda_0, \lambda_1, \ldots, \lambda_{i-2})$, we may think of

$$f_{i,(\lambda_0,\lambda_1,\ldots,\lambda_{i-2})}$$

as that part of f_i consisting of edges oriented towards the vertex

$$g^0_{i-1,[\lambda_0]+[\lambda_1]p+\cdots+[\lambda_{i-2}]p^{i-2}} KZ.$$

Once again, $T_{1,2}(f_{i,j})$ and $T_{1,2}(f_{i,k})$ have no edge in common for $j \neq k$, and therefore we conclude that each $f_{i,j} \in \operatorname{Ker} T_{1,2}$.

In order to finish the proof, we claim that

$$f_{i,j} \in \operatorname{Im} T_{-1,0}$$

for all (i, j). If j corresponds to $(\lambda_0, \lambda_1, \ldots, \lambda_{i-2})$, note that $(g^0_{i-1,[\lambda]_{i-1}})^{-1} f_{i,j}$ is a linear combination of $[\![g^0_{1,\lambda_{i-1}}, 1]\!]$ and $[\![\beta, 1]\!]$. Thus, Proposition 3.1 follows from Lemma 3.2. $\square$

Lemma 3.2. *Let $T_{-1,0}, T_{1,2} \in \operatorname{End}_G(\operatorname{ind}^G_{IZ} d^r)$ as in Subsection 2.4. Assume that $r \neq 0$, $p-1$. Then,*

$$z[\![\beta, 1]\!] + \sum_{\mu=0}^{p-1} z_\mu \left[\!\!\left[\begin{pmatrix} p & \mu \\ 0 & 1 \end{pmatrix}, 1\right]\!\!\right] \in \operatorname{Ker} T_{1,2}$$

$$\implies \quad z[\![\beta, 1]\!] + \sum_{\mu=0}^{p-1} z_\mu \left[\!\!\left[\begin{pmatrix} p & \mu \\ 0 & 1 \end{pmatrix}, 1\right]\!\!\right] \in \operatorname{Im} T_{-1,0}.$$

Proof of Lemma 3.2. Suppose

$$z[\![\beta, 1]\!] + \sum_{\mu=0}^{p-1} z_\mu \left[\!\!\left[\begin{pmatrix} p & \mu \\ 0 & 1 \end{pmatrix}, 1\right]\!\!\right] \in \operatorname{Ker} T_{1,2}.$$

Applying the formula (10) for $T_{1,2}$, we see that:

$$z \sum_{\lambda \in I_1} \left[\!\!\left[\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} w, 1\right]\!\!\right] + \sum_{\mu \in I_1} \sum_{\lambda \in I_1} z_\mu \left[\!\!\left[\begin{pmatrix} p & \mu \\ 0 & 1 \end{pmatrix} \beta \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} w, 1\right]\!\!\right] = 0.$$

Now,

$$\begin{aligned}
&\sum_{\mu \in I_1} \sum_{\lambda \in I_1} z_\mu \left[\!\!\left[\begin{pmatrix} p & \mu \\ 0 & 1 \end{pmatrix} \beta \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} w, 1\right]\!\!\right] \\
&= \sum_{\mu \in I_1} \sum_{\lambda \in I_1} z_\mu \left[\!\!\left[\begin{pmatrix} \mu\lambda + 1 & \mu \\ \lambda & 1 \end{pmatrix}, 1\right]\!\!\right] \\
&= \sum_{\mu \in I_1} z_\mu \left[\!\!\left[\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, 1\right]\!\!\right] + \sum_{\mu \in I_1} \sum_{\lambda \neq 0} z_\mu \left[\!\!\left[\begin{pmatrix} 1 & \mu + \frac{1}{\lambda} \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} \lambda & 1 \\ 0 & -\frac{1}{\lambda} \end{pmatrix}, 1\right]\!\!\right] \\
&= \sum_{\mu \in I_1} z_\mu [\![\mathrm{Id}, 1]\!] + \sum_{\mu \in I_1} \sum_{\lambda \neq 0} z_\mu \left(-\frac{1}{\lambda}\right)^r \left[\!\!\left[\begin{pmatrix} 1 & \mu + \frac{1}{\lambda} \\ 0 & 1 \end{pmatrix} w, 1\right]\!\!\right] \\
&= \sum_{\mu \in I_1} z_\mu [\![\mathrm{Id}, 1]\!] + \sum_{\mu \in I_1} \sum_{\lambda \neq \mu} z_\mu (\mu - \lambda)^r \left[\!\!\left[\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} w, 1\right]\!\!\right] \\
&= \left(\sum_{\mu \in I_1} z_\mu\right) [\![\mathrm{Id}, 1]\!] + \sum_{\lambda \in I_1} \left(\sum_{\mu \in I_1} z_\mu (\mu - \lambda)^r\right) \left[\!\!\left[\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} w, 1\right]\!\!\right].
\end{aligned}$$

Note that we have made use of the condition $r \neq 0$, $p-1$, in the very last step.

Thus, we get the following linear system of equations:

$$\sum_{\mu\in I_1} z_\mu = 0, \tag{14}$$

$$\sum_{\mu\in I_1} z_\mu(\mu-\lambda)^r = -z. \tag{15}$$

The space of solutions of the corresponding homogeneous system is spanned by the following $p-1-r$ vectors:

$$\left(\mu^i\right)_{0\le\mu\le p-1}, \quad 0\le i\le p-2-r.$$

Indeed, these $p-1-r$ vectors are linearly independent and they do satisfy the homogeneous system of equations, and by plugging in the $r+1$ vectors $(\mu^i)_{0\le\mu\le p-1}$, $p-1-r\le i\le p-1$, we see that the rank of the associated matrix is at least $r+1$, thus showing that the space of solutions is of dimension $p-1-r$ (and that the rank of the associated matrix is $r+1$). Clearly, a particular solution of the system is given by $(z\mu^{p-1-r})_{0\le\mu\le p-1}$.

Hence, to prove the lemma, we need to check that the $p-1-r$ vectors

$$\sum_{\mu=0}^{p-1}\mu^i \left[\!\left[\begin{pmatrix} p & \mu \\ 0 & 1\end{pmatrix}, 1\right]\!\right], \quad 0\le i\le p-2-r,$$

and

$$[\![\beta,1]\!] + \sum_{\mu=0}^{p-1}\mu^{p-1-r}\left[\!\left[\begin{pmatrix} p & \mu \\ 0 & 1\end{pmatrix}, 1\right]\!\right]$$

are in $\operatorname{Im} T_{-1,0}$.

To this end, we start by computing

$$T_{-1,0}\left(z[\![\mathrm{Id},1]\!] + \sum_{\mu=0}^{p-1} z_\mu \left[\!\left[\begin{pmatrix} 1 & \mu \\ 0 & 1\end{pmatrix} w, 1\right]\!\right]\right).$$

Applying the formula (9) for $T_{-1,0}$, we see that the image of the second summand above under $T_{-1,0}$ is:

$$\begin{aligned}&\sum_{\mu\in I_1}\sum_{\lambda\in I_1} z_\mu \left[\!\left[\begin{pmatrix} 1 & \mu \\ 0 & 1\end{pmatrix} w \begin{pmatrix} p & \lambda \\ 0 & 1\end{pmatrix}, 1\right]\!\right] \\ &\quad = \sum_{\mu\in I_1}\sum_{\lambda\in I_1} z_\mu \left[\!\left[\begin{pmatrix} \mu p & \mu\lambda+1 \\ p & \lambda\end{pmatrix}, 1\right]\!\right]\end{aligned}$$

$$= \sum_{\mu \in I_1} z_\mu \left[\!\!\left[\beta \begin{pmatrix} \mu & 0 \\ \mu p & 1 \end{pmatrix}, 1 \right]\!\!\right] + \sum_{\mu \in I_1} \sum_{\lambda \neq 0} z_\mu \left[\!\!\left[\begin{pmatrix} p & \mu + \frac{1}{\lambda} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\lambda} & 0 \\ p & \lambda \end{pmatrix}, 1 \right]\!\!\right]$$

$$= \sum_{\mu \in I_1} z_\mu [\![\beta, 1]\!] + \sum_{\mu \in I_1} \sum_{\lambda \neq 0} z_\mu \lambda^r \left[\!\!\left[\begin{pmatrix} p & \mu + \frac{1}{\lambda} \\ 0 & 1 \end{pmatrix}, 1 \right]\!\!\right]$$

$$= \sum_{\mu \in I_1} z_\mu [\![\beta, 1]\!] + \sum_{\mu \in I_1} \sum_{\lambda \neq \mu} z_\mu (\lambda - \mu)^{p-1-r} \left[\!\!\left[\begin{pmatrix} p & \lambda \\ 0 & 1 \end{pmatrix}, 1 \right]\!\!\right]$$

$$= \left(\sum_{\mu \in I_1} z_\mu \right) [\![\beta, 1]\!] + \sum_{\lambda \in I_1} \left(\sum_{\mu \in I_1} z_\mu (\lambda - \mu)^{p-1-r} \right) \left[\!\!\left[\begin{pmatrix} p & \lambda \\ 0 & 1 \end{pmatrix}, 1 \right]\!\!\right].$$

Once again, the justification for the last step is that $r \neq 0, p-1$.

Thus,

$$T_{-1,0} \left(z [\![\mathrm{Id}, 1]\!] + \sum_{\mu=0}^{p-1} z_\mu \left[\!\!\left[\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} w, 1 \right]\!\!\right] \right)$$
$$= \left(\sum_{\mu \in I_1} z_\mu \right) [\![\beta, 1]\!] + \sum_{\lambda \in I_1} \left(z + \left(\sum_{\mu \in I_1} z_\mu (\lambda - \mu)^{p-1-r} \right) \right) \left[\!\!\left[\begin{pmatrix} p & \lambda \\ 0 & 1 \end{pmatrix}, 1 \right]\!\!\right].$$

In order to finish the proof of Lemma 3.2, we need to show that the system of equations

$$\sum_{\mu \in I_1} z_\mu = 0 \quad \& \quad \sum_{\mu \in I_1} z_\mu (\lambda - \mu)^{p-1-r} = \lambda^i \quad (0 \leq i \leq p-2-r) \tag{16}$$

and

$$\sum_{\mu \in I_1} z_\mu = 1 \quad \& \quad \sum_{\mu \in I_1} z_\mu (\lambda - \mu)^{p-1-r} = \lambda^{p-1-r} \tag{17}$$

can be solved. This is immediate from

$$\sum_{\mu \in I_1} \mu^{i+r} (\lambda - \mu)^{p-1-r} = \begin{cases} (-1)^{p-i-r} \binom{p-1-r}{i} \lambda^i & \text{if } 0 \leq i \leq p-1-r, \\ 0 & \text{if } p-r \leq i \leq p-1. \end{cases}$$

This proves the lemma. □

We also have:

Proposition 3.3.

$$\operatorname{Ker} T_{1,2} \cap \operatorname{Ker} T_{-1,0} = \{0\} = \operatorname{Im} T_{1,2} \cap \operatorname{Im} T_{-1,0}.$$

Proof of Proposition 3.3. Write $0 \neq f \in \operatorname{Ker} T_{1,2} \cap \operatorname{Ker} T_{-1,0}$ as

$$f = f(n) + \ldots + f(1) + f(0)$$

with $f(n) \neq 0$, where $f(i)$ is a linear combination of edges of the tree $\mathcal{A}$ of $SL_2(\mathbb{Q}_p)$ at distance i from the central edge corresponding to the trivial coset of G/IZ. That is, $f(i)$ is a linear combination of vectors of the form

$$[\![g^0_{i,\lambda}, 1]\!], \left[\!\!\left[g^0_{i-1,[\lambda]_{i-1}} \begin{pmatrix} 1 & \lambda_{i-1} \\ 0 & 1 \end{pmatrix} w, 1\right]\!\!\right], [\![g^1_{i,\lambda} w, 1]\!], \left[\!\!\left[g^0_{i-1,[\lambda]_{i-1}} w \begin{pmatrix} 1 & \lambda_{i-1} \\ 0 & 1 \end{pmatrix} w, 1\right]\!\!\right].$$

Now if $f(n)$ has edges of the form

$$[\![g^0_{n,\lambda}, 1]\!] \quad \text{or} \quad [\![g^1_{n,\lambda} w, 1]\!],$$

then $T_{-1,0}(f) \neq 0$, since $T_{-1,0}(f)$ will then have edges at distance $n+1$ from the central edge of $\mathcal{A}$ which will not get canceled with any other edge. If $f(n)$ has edges of the form

$$\left[\!\!\left[g^0_{n-1,\mu} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} w, 1\right]\!\!\right] \quad \text{or} \quad \left[\!\!\left[g^1_{n-1,\mu} w \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} w, 1\right]\!\!\right],$$

then $T_{1,2}(f)$ will have non-zero edges at distance $n+1$ from the central vertex of the tree, and thus $T_{1,2}(f) \neq 0$. Since a non-zero $f(n)$ has to have an edge of the form considered above, we get a contradiction. □

Remark 1. Observe that the arguments in this section do not use the fact that $F = \mathbb{Q}_p$ and our proofs go through verbatim in the case when the residue field of F is $\mathbb{F}_p$. Thus the results of this section hold true for totally ramified extensions of $\mathbb{Q}_p$ as well.

4. Iwahori–Hecke model of a supersingular representation

In this section, we prove the following theorem from which it is easy to deduce Theorem 1.1.

Theorem 4.1. *Let $0 < r < p-1$. We have a G-equivariant isomorphism*

$$\operatorname{ind}^G_{KZ} \operatorname{Sym}^r \bar{\mathbb{F}}_p^2 \cong \frac{\operatorname{ind}^G_{IZ} d^r}{(T_{-1,0})}$$

which sends $[\mathrm{Id}, x^r]$ to $\overline{[\![\beta, 1]\!]} = [\![\beta, 1]\!] + (T_{-1,0})$. Under this isomorphism the Hecke operator T is mapped to $T_{1,2}$.

We start with the following lemma.

Lemma 4.2. *We have*

$$\mathrm{Sym}^r \bar{\mathbb{F}}_p^2 \hookrightarrow \frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0})}$$

given by

$$x^r \mapsto \overline{[\![\beta, 1]\!]} = [\![\beta, 1]\!] + (T_{-1,0}).$$

Proof of Lemma 4.2. By (10),

$$\begin{aligned}
&T_{1,2}\Big(\sum_{\lambda \in I_1} \lambda^i [\![g_{1,\lambda}^0, 1]\!] \Big) \\
&= \sum_{\lambda \in I_1} \sum_{\mu \in I_1} \lambda^i \left[\!\!\left[\begin{pmatrix} 1+\lambda\mu & \lambda \\ \mu & 1 \end{pmatrix}, 1 \right]\!\!\right] \\
&= \sum_{\lambda \in I_1} \lambda^i [\![\mathrm{Id}, 1]\!] + \sum_{\lambda \in I_1} \sum_{\mu \neq 0} \lambda^i \left[\!\!\left[\begin{pmatrix} 1 & \lambda + \frac{1}{\mu} \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} \mu & 1 \\ 0 & -\frac{1}{\mu} \end{pmatrix}, 1 \right]\!\!\right] \\
&= \Big(\sum_{\lambda \in I_1} \lambda^i \Big) [\![\mathrm{Id}, 1]\!] + \sum_{\lambda \in I_1} \sum_{\mu \neq 0} \lambda^i \Big(-\frac{1}{\mu} \Big)^r \left[\!\!\left[\begin{pmatrix} 1 & \lambda + \frac{1}{\mu} \\ 0 & 1 \end{pmatrix} w, 1 \right]\!\!\right] \\
&= \Big(\sum_{\lambda \in I_1} \lambda^i \Big) [\![\mathrm{Id}, 1]\!] + \sum_{\lambda \in I_1} \sum_{\mu \neq 0} \lambda^i (-\mu)^r \left[\!\!\left[\begin{pmatrix} 1 & \lambda + \mu \\ 0 & 1 \end{pmatrix} w, 1 \right]\!\!\right] \\
&= \Big(\sum_{\lambda \in I_1} \lambda^i \Big) [\![\mathrm{Id}, 1]\!] + \sum_{\lambda \in I_1} \sum_{\mu \in I_1} \lambda^i (-\mu)^r \left[\!\!\left[\begin{pmatrix} 1 & \lambda + \mu \\ 0 & 1 \end{pmatrix} w, 1 \right]\!\!\right] \\
&= \Big(\sum_{\lambda \in I_1} \lambda^i \Big) [\![\mathrm{Id}, 1]\!] + \sum_{\mu \in I_1} \Big(\sum_{\lambda \in I_1} \lambda^i (\lambda - \mu)^r \Big) \left[\!\!\left[\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} w, 1 \right]\!\!\right] \\
&= \begin{cases} 0 & \text{if } 0 \le i \le p-2-r, \\ -(-1)^{i-(p-1-r)} \binom{r}{i-(p-1-r)} \sum_{\mu \in I_1} \mu^{i-(p-1-r)} & \\ \quad \times [\![\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right) w, 1]\!] & \text{if } p-1-r \le i \le p-2, \\ -[\![\mathrm{Id}, 1]\!] + (-1)^{r+1} \sum_{\mu \in I_1} \mu^r \, [\![\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right) w, 1]\!] & \text{if } i = p-1. \end{cases}
\end{aligned}$$

Note that we have made use of the condition $r \neq 0$ in the above. Observe also that

$$T_{1,2}\Big(\sum_{\lambda \in I_1} \lambda^{p-1-r} [\![g_{1,\lambda}^0, 1]\!] \Big) = T_{1,2}[\![\beta, 1]\!],$$

and hence

$$\sum_{\lambda \in I_1} \lambda^{p-1-r} \left[\!\left[g^0_{1,\lambda}, 1 \right]\!\right] = [\![\beta, 1]\!]$$

modulo the image of $T_{-1,0}$ by Proposition 3.1.

By checking the action of the Iwahori subgroup of $GL_2(\mathbb{Z}_p)$ and the Weyl element w, it is easy to verify that the $r+1$ vectors

$$\left\{ \sum_{\lambda \in I_1} \lambda^i \left[\!\left[g^0_{1,\lambda}, 1 \right]\!\right] \;\middle|\; p-1-r \leq i \leq p-1 \right\}$$

form a copy of $\mathrm{Sym}^r \bar{\mathbb{F}}_p^2$ in $\frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0})}$ and

$$x^{r-i} y^i \mapsto \sum_{\lambda \in I_1} \lambda^{p-1-r+i} \left[\!\left[g^0_{1,\lambda}, 1 \right]\!\right]$$

for $0 \leq i \leq r$. $\square$

Proof of Theorem 4.1. By Frobenius reciprocity, the injective morphism given by

$$\begin{aligned} \mathrm{Sym}^r \bar{\mathbb{F}}_p^2 &\hookrightarrow \frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0})} \\ x^r &\mapsto \overline{[\![\beta, 1]\!]} \end{aligned}$$

of Lemma 4.2 extends to the injective morphism, say ϕ,

$$\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \bar{\mathbb{F}}_p^2 \hookrightarrow \frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0})}$$

given by

$$\left[\mathrm{Id}, x^r\right] \mapsto \overline{[\![\beta, 1]\!]}.$$

We now define a morphism, say ψ, in the opposite direction by

$$\begin{aligned} \mathrm{ind}_{IZ}^G d^r &\to \mathrm{ind}_{KZ}^G \mathrm{Sym}^r \bar{\mathbb{F}}_p^2 \\ [\![\beta, 1]\!] &\mapsto \left[\mathrm{Id}, x^r\right]. \end{aligned}$$

Observe that ψ factors through $\frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0})}$. Indeed,

$$\psi\big(T_{-1,0}\big([\![\mathrm{Id}, 1]\!]\big)\big) = \psi\left(\sum_{\lambda \in I_1} \left[\!\left[g^0_{1,\lambda}, 1 \right]\!\right] \right) = \sum_{\lambda \in I_1} \left[g^0_{1,\lambda}\beta, x^r \right] = \sum_{\lambda \in I_1} \left[\mathrm{Id}, (\lambda x + y)^r \right] = 0,$$

since $r \neq p-1$.

Since ϕ and ψ are obviously inverses of each other, the theorem follows. $\square$

Remark 2. Note that we have made use of $r \neq 0$ in the proof of Lemma 4.2 and $r \neq p-1$ in the proof of Theorem 4.1.

Remark 3. By Remark 1, Theorem 4.1, and hence the first part of Theorem 1.1, holds true for a totally ramified extension of $\mathbb{Q}_p$ as well.

5. Breuil's isomorphism

Recall from the introduction that we have the following isomorphism of Breuil [6, Theorem 1.3]:

$$\frac{\mathrm{ind}_{KZ}^{G} \operatorname{Sym}^r \bar{\mathbb{F}}_p^2}{(T)} \cong \frac{\mathrm{ind}_{KZ}^{G} \operatorname{Sym}^{p-1-r} \bar{\mathbb{F}}_p^2}{(T)} \otimes \det{}^r.$$

On the Iwahori–Hecke model, the above isomorphism takes the form:

Theorem 5.1. *Let d^r (resp. a^r) denote the character of the Iwahori subgroup of $GL_2(\mathbb{Z}_p)$ given by*

$$\begin{pmatrix} a & b \\ pc & d \end{pmatrix} \mapsto d^r \quad (\textit{resp. } a^r) \bmod p.$$

Then, we have

$$\frac{\mathrm{ind}_{IZ}^{G} d^r}{(T_{-1,0}, T_{1,2})} \cong \frac{\mathrm{ind}_{IZ}^{G} a^r}{(T_{-1,0}, T_{1,2})}.$$

Proof of Theorem 5.1. It is easy to see that we have an isomorphism

$$\mathrm{ind}_{IZ}^{G} d^r \cong \mathrm{ind}_{IZ}^{G} a^r$$

given by

$$[\![\beta, 1]\!] \leftrightarrow [\![\mathrm{Id}, 1]\!].$$

This is nothing but reversing the orientation of the edges on the tree of $SL_2(\mathbb{Q}_p)$. Under this map, $T_{-1,0} \leftrightarrow T_{1,2}$ and vice-versa. Thus, we get:

$$\frac{\mathrm{ind}_{IZ}^{G} d^r}{(T_{-1,0})} \cong \frac{\mathrm{ind}_{IZ}^{G} a^r}{(T_{1,2})},$$

and

$$\frac{\mathrm{ind}_{IZ}^{G} d^r}{(T_{-1,0}, T_{1,2})} \cong \frac{\mathrm{ind}_{IZ}^{G} a^r}{(T_{-1,0}, T_{1,2})}. \qquad \square$$

Remark 4. We also have the obvious isomorphism

$$\mathrm{ind}_{IZ}^{G} a^r \cong \mathrm{ind}_{IZ}^{G} d^{p-1-r} \otimes \det^r.$$

Thus, the isomorphism of Breuil's is the one in Theorem 5.1.

Remark 5. By Remark 3, Theorem 5.1 holds true for any totally ramified extension of $\mathbb{Q}_p$. This is Proposition 1.1 of [17].

6. Self-extensions for supersingular representations

In this section, we prove Theorem 1.2. We start by recalling the definition of Baer sum, the sum in $\mathrm{Ext}^1(\cdot,\cdot)$.

Let π_1, π_2 be two representations of a group and let

$$0 \longrightarrow \pi_2 \xrightarrow{f} \pi \xrightarrow{g} \pi_1 \longrightarrow 0$$

and

$$0 \longrightarrow \pi_2 \xrightarrow{f'} \pi' \xrightarrow{g'} \pi_1 \longrightarrow 0$$

be two extensions of π_1 by π_2. Let

$$\pi'' = \{(v, v') \in \pi \times \pi' \mid g(v) = g'(v')\}$$

be the pullback over π_1. Now consider the three copies of π_2, namely, $\pi_2 \times 0, 0 \times \pi_2$, and the skew-diagonal $\{(-w, w) \mid w \in \pi_2\}$ in π''. The quotient τ of π'' by the skew-diagonal in which the copies $\pi_2 \times 0$ and $0 \times \pi_2$ are identified; i.e.,

$$\tau = \frac{\pi''}{(f(w), 0) - (0, f'(w))},$$

with $w \in \pi_2$, is an extension of π_1 by π_2:

$$0 \longrightarrow \pi_2 \longrightarrow \tau \longrightarrow \pi_1 \longrightarrow 0,$$

where the first arrow is the map $w \mapsto [(f(w), 0)] = [(0, f'(w))]$, and the second arrow is the map $(v, v') \mapsto g(v) = g'(v)$. The class of this latter extension is the Baer sum of the first two extensions. It is commutative with the trivial element being the class of the split extension. A non-zero scalar multiple, say λ, of the extension

$$0 \longrightarrow \pi_2 \xrightarrow{f} \pi \xrightarrow{g} \pi_1 \longrightarrow 0$$

can be taken to be the class of the same extension with g replaced by $\lambda^{-1} g$.

We want to verify that the following three representations

$$\text{(i)} \quad \frac{\operatorname{ind}_{IZ}^G d^r}{(T_{-1,0}, T_{1,2}^2)}, \qquad \text{(ii)} \quad \frac{\operatorname{ind}_{IZ}^G d^r}{(T_{-1,0}^2, T_{1,2})}, \qquad \text{(iii)} \quad \frac{\operatorname{ind}_{IZ}^G d^r}{(\lambda_1^{-1} T_{-1,0} - \lambda_2^{-1} T_{1,2})} \tag{18}$$

belong to $\operatorname{Ext}_G^1(\pi_r, \pi_r)$. Here, $\lambda_1, \lambda_2 \in \bar{\mathbb{F}}_p^\times$. Let us clarify our notation in this section; we have:

$$\begin{aligned}
(T_{-1,0}^2, T_{1,2}) &= T_{-1,0}^2(\operatorname{ind}_{IZ}^G d^r) + T_{1,2}(\operatorname{ind}_{IZ}^G d^r) \\
&= \{T_{-1,0}^2(f) + T_{1,2}(g) \mid f, g \in \operatorname{ind}_{IZ}^G d^r\}, \\
(T_{-1,0}, T_{1,2}^2) &= T_{-1,0}(\operatorname{ind}_{IZ}^G d^r) + T_{1,2}^2(\operatorname{ind}_{IZ}^G d^r) \\
&= \{T_{-1,0}(f) + T_{1,2}^2(g) \mid f, g \in \operatorname{ind}_{IZ}^G d^r\}, \\
(\lambda_1^{-1} T_{-1,0} - \lambda_2^{-1} T_{1,2}) &= \{\lambda_1^{-1} T_{-1,0}(g) - \lambda_2^{-1} T_{1,2}(g) \mid g \in \operatorname{ind}_{IZ}^G d^r\}.
\end{aligned}$$

Remark 6. Note that

$$(\lambda_1^{-1} T_{-1,0} - \lambda_2^{-1} T_{1,2}) = (T_{-1,0}^2, \lambda_1^{-1} T_{-1,0} - \lambda_2^{-1} T_{1,2}, T_{1,2}^2),$$

since

$$T_{-1,0}^2(f) = T_{-1,0}(T_{-1,0}(f)) = (\lambda_1^{-1} T_{-1,0} - \lambda_2^{-1} T_{1,2})(\lambda_1 T_{-1,0}(f)),$$

and similarly for $T_{1,2}^2(f)$.

Let us look at the kernel of the natural surjection from the representations listed in (18) to

$$\pi_r \cong \frac{\operatorname{ind}_{IZ}^G d^r}{(T_{-1,0}, T_{1,2})}.$$

The respective kernels are given by

$$\text{(i)} \quad \frac{(T_{-1,0}, T_{1,2})}{(T_{-1,0}, T_{1,2}^2)}, \qquad \text{(ii)} \quad \frac{(T_{-1,0}, T_{1,2})}{(T_{-1,0}^2, T_{1,2})}, \qquad \text{(iii)} \quad \frac{(T_{-1,0}, T_{1,2})}{(\lambda_1^{-1} T_{-1,0} - \lambda_2^{-1} T_{1,2})}.$$

We only need to claim that the above representations are isomorphic to π_r. To see this, in cases (i) and (ii), consider the surjection from $\operatorname{ind}_{IZ}^G d^r$ to these representations given by

$$f \mapsto T_{-1,0}(f) + T_{1,2}(f),$$

and in case (iii), consider

$$f \mapsto \lambda_1^{-1} T_{-1,0}(f) = \lambda_2^{-1} T_{1,2}(f).$$

It is straightforward to verify, by making use of Proposition 3.1 and Proposition 3.3, that these maps have $(T_{-1,0}, T_{1,2})$ as their kernel.

Thus, in order to finish the proof of Theorem 1.2, we need to prove (6). By the definition of Baer sum, we have:

$$\lambda_1 \frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0}^2, T_{1,2})} + \lambda_2 \frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0}, T_{1,2}^2)} = \frac{\{(f_1, f_2) \mid \lambda_1^{-1} f_1 - \lambda_2^{-1} f_2 \in (T_{-1,0}, T_{1,2})\}}{\{(T_{-1,0}(f), -T_{1,2}(f)) \mid f \in \frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0}, T_{1,2})}\}}. \tag{19}$$

Consider the surjective morphism from the numerator of the right hand side of (19) onto

$$\frac{\mathrm{ind}_{IZ}^G d^r}{(\lambda_1^{-1} T_{-1,0} - \lambda_2^{-1} T_{1,2})}$$

given by

$$(f_1, f_2) \mapsto \lambda_1^{-1} f_1 + \lambda_2^{-1} f_2. \tag{20}$$

Now,

$$\begin{aligned} \mathrm{Ker}((20)) &= \{(f_1, f_2) \mid \lambda_1^{-1} f_1 + \lambda_2^{-1} f_2 = \lambda_1^{-1} T_{-1,0}(f) - \lambda_2^{-1} T_{1,2}(f), \exists f\} \\ &= \{(f_1, f_2) \mid \lambda_1^{-1}(f_1 - T_{-1,0}(f)) = -\lambda_2^{-1}(f_2 + T_{1,2}(f)), \exists f\}. \end{aligned}$$

Since $f_1 \in \frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0}^2, T_{1,2})}$ and $f_2 \in \frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0}, T_{1,2}^2)}$, together with the knowledge that

$$\lambda_1^{-1} f_1 - \lambda_2^{-1} f_2 \in (T_{-1,0}, T_{1,2}),$$

we conclude that

$$f_1 \in \mathrm{Im}\, T_{-1,0} \quad \& \quad f_2 \in \mathrm{Im}\, T_{1,2}.$$

Since,

$$\lambda_1^{-1}(f_1 - T_{-1,0}(f)) = -\lambda_2^{-1}(f_2 + T_{1,2}(f)),$$

it follows from Proposition 3.3 that

$$\mathrm{Ker}((20)) = \left\{ (T_{-1,0}(f), -T_{1,2}(f)) \;\middle|\; f \in \frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0}, T_{1,2})} \right\},$$

showing that (20) is an isomorphism. This proves Theorem 1.2.

7. Pro-p-Iwahori Hecke module structure

7.1. Preliminaries on the pro-p-Iwahori Hecke modules

We briefly recap the relevant facts about the pro-p-Iwahori Hecke modules. Our reference here is [16, §9].

The pro-p-Iwahori Hecke algebra $\mathcal{H}$ is the Hecke algebra associated to $(I(1)Z, \zeta)$ where ζ is a smooth character $\zeta : Z \to \bar{\mathbb{F}}_p^\times$. For a smooth irreducible representation π of $GL_2(\mathbb{Q}_p)$, we take $\zeta = \omega$, the central character of π.[1] Thus, $\mathcal{H} = \mathrm{End}_G(\mathrm{ind}_{I(1)Z}^G \omega)$.

Let Rep_G denote the category of smooth representations of $GL_2(\mathbb{Q}_p)$ with central character ω, and let $\mathrm{Mod}_{\mathcal{H}}$ be the category of right $\mathcal{H}$-modules. Consider the functors $\mathcal{I} : \mathrm{Rep}_G \to \mathrm{Mod}_{\mathcal{H}}$ given by

$$\mathcal{I}(\pi) = \pi^{I(1)} \cong \mathrm{Hom}_G\big(\mathrm{ind}_{I(1)Z}^G \omega, \pi\big),$$

and $\mathcal{T} : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Rep}_G$ given by

$$\mathcal{T}(M) = M \otimes_{\mathcal{H}} \mathrm{ind}_{I(1)Z}^G \omega.$$

We have:

$$\mathrm{Hom}_{\mathcal{H}}\big(M, \mathcal{I}(\pi)\big) \cong \mathrm{Hom}_G\big(\mathcal{T}(M), \pi\big).$$

By a result of Vignéras, $\mathcal{I}$ and $\mathcal{T}$ induce a bijection between irreducible objects in Rep_G and $\mathrm{Mod}_{\mathcal{H}}$ [18, Theorem 5.4]. Ollivier shows that $\mathcal{I}$ and $\mathcal{T}$ induce an equivalence of categories between $\mathrm{Mod}_{\mathcal{H}}$ and the category of smooth $\bar{\mathbb{F}}_p$-representations of $GL_2(\mathbb{Q}_p)$, with central character ω, generated by their $I(1)$-invariants [14].

The algebra $\mathcal{H}$ has a basis indexed by the double cosets $I(1)\backslash G/I(1)Z$. Write T_g for an element corresponding to $I(1)gI(1)Z$. If $v \in \pi^{I(1)}$, the action of T_g is given by

$$T_g(v) = \sum_{u \in I(1)/(I(1)\cap g^{-1}I(1)g)} ug^{-1}v.$$

Given a character χ of the maximal split torus $T(\mathbb{F}_p)$ of $GL_2(\mathbb{F}_p)$, define $e_\chi \in \mathcal{H}$ by

$$e_\chi = \frac{1}{|T(\mathbb{F}_p)|} \sum_{t \in T(\mathbb{F}_p)} \chi(t) T_t.$$

Then the elements T_w, T_β, and e_χ for all χ, generate $\mathcal{H}$ as an algebra, and the following relations hold:

[1] The representation π has a central character by [2].

$$T_\beta^2 = 1, \qquad e_\chi T_w = T_w e_{\chi^w}, \qquad e_\chi T_\beta = T_\beta e_{\chi^w}, \qquad e_\chi T_w^2 = -e_\chi e_{\chi^w} T_w.$$

Also, $e_\chi e_{\chi^w} = e_\chi$ if $\chi = \chi^w$ and $e_\chi e_{\chi^w} = 0$ otherwise.

Now let π be an irreducible supersingular representation of $GL_2(\mathbb{Q}_p)$. For $(\lambda_1, \lambda_2) \in \bar{\mathbb{F}}_p^2$, define an $\mathcal{H}$-module E_{λ_1,λ_2} as follows. It is a four dimensional vector space with basis $\{u_\chi, u_{\chi^w}, v_\chi, v_{\chi^w}\}$ with the action of $\mathcal{H}$ given on the generators by

$$T_w(v_\chi) = \lambda_1 u_{\chi^w}, \qquad T_w(v_{\chi^w}) = \lambda_2 u_\chi, \qquad T_w(u_\chi) = T_w(u_{\chi^w}) = 0,$$

and

$$T_\beta(v_\psi) = v_{\psi^w}, \qquad T_\beta(u_\psi) = u_{\psi^w}, \qquad e_\psi(v_\psi) = v_\psi, \qquad e_\psi(u_\psi) = u_\psi,$$

for $\psi \in \{\chi, \chi^w\}$. Then $\langle u_\psi, u_{\psi^w}\rangle$ is stable under the action of $\mathcal{H}$ and we have an exact sequence:

$$0 \to \mathcal{I}(\pi) \to E_{\lambda_1,\lambda_2} \to \mathcal{I}(\pi) \to 0.$$

The extension is split if and only if $(\lambda_1, \lambda_2) = (0, 0)$, and the map sending (λ_1, λ_2) to the equivalence class of the above exact sequence is an isomorphism of $\bar{\mathbb{F}}_p$-vector spaces between $\bar{\mathbb{F}}_p^2$ and $\mathrm{Ext}^1_{\mathcal{H}}(\mathcal{I}(\pi), \mathcal{I}(\pi))$.

We may also note that the embedding given by $\mathcal{T}$ below

$$0 \to \mathrm{Ext}^1_{\mathcal{H}}\big(\mathcal{I}(\pi), \mathcal{I}(\pi)\big) \to \mathrm{Ext}^1_G(\pi, \pi) \to \dots \tag{21}$$

has in its image precisely those self-extensions which have a four dimensional space of $I(1)$-invariants. If $0 \to \pi \to \tau \to \pi \to 0$, and if $\dim \tau^{I(1)} = 4$, then, $0 \to \pi^{I(1)} \to \tau^{I(1)} \to \pi^{I(1)} \to 0$, since $\dim \pi^{I(1)} = 2$, and the claim follows from [16, Proposition 9.1].

7.2. *Space of $I(1)$-invariants*

Now we compute the $\mathcal{H}$-module structure on the space of $I(1)$-invariants of the representations in Theorem 1.2. Before we do this, observe that

$$\left(\frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0}, T_{1,2})}\right)^{I(1)} = \big\langle [\![\mathrm{Id}, 1]\!], [\![\beta, 1]\!]\big\rangle \tag{22}$$

as can be seen from Theorem 1.1 together with [6, Theorem 3.2.4]. The dimension of the space of $I(1)$-invariants of the three representations in (18) is four. Indeed, we can exhibit four linearly independent $I(1)$-invariants, and this will do since a priori the dimension of the space of $I(1)$-invariants is at most four as these representations are self-extensions of π_r.

The spaces of $I(1)$-invariants in the three cases are as follows:

$$\left(\frac{\mathrm{ind}_{IZ}^{G} d^r}{(T_{-1,0}, T_{1,2}^2)}\right)^{I(1)} = \langle [\![\mathrm{Id}, 1]\!], [\![\beta, 1]\!], T_{1,2}([\![\mathrm{Id}, 1]\!]), T_{1,2}([\![\beta, 1]\!])\rangle, \tag{23}$$

$$\left(\frac{\mathrm{ind}_{IZ}^{G} d^r}{(T_{-1,0}^2, T_{1,2})}\right)^{I(1)} = \langle [\![\mathrm{Id}, 1]\!], [\![\beta, 1]\!], T_{-1,0}([\![\mathrm{Id}, 1]\!]), T_{-1,0}([\![\beta, 1]\!])\rangle, \tag{24}$$

and

$$\begin{aligned}\left(\frac{\mathrm{ind}_{IZ}^{G} d^r}{(\lambda_1^{-1}T_{-1,0} - \lambda_2^{-1}T_{1,2})}\right)^{I(1)} &= \langle [\![\mathrm{Id}, 1]\!], [\![\beta, 1]\!], \lambda_1^{-1}T_{-1,0}([\![\mathrm{Id}, 1]\!]) \\ &= \lambda_2^{-1}T_{1,2}([\![\mathrm{Id}, 1]\!]), \lambda_1^{-1}T_{-1,0}([\![\beta, 1]\!]) \\ &= \lambda_2^{-1}T_{1,2}([\![\beta, 1]\!])\rangle.\end{aligned} \tag{25}$$

Note that, in all the cases,

$$v_{d^r} = [\![\mathrm{Id}, 1]\!],$$

and

$$v_{a^r} = [\![\beta, 1]\!],$$

whereas

$$u_{d^r} = T_{-1,0}([\![\mathrm{Id}, 1]\!]) \quad \text{or} \quad T_{1,2}([\![\mathrm{Id}, 1]\!]) \quad \text{or} \quad \lambda_1^{-1}T_{-1,0}([\![\mathrm{Id}, 1]\!]) = \lambda_2^{-1}T_{1,2}([\![\mathrm{Id}, 1]\!]),$$

and

$$u_{a^r} = T_{-1,0}([\![\beta, 1]\!]) \quad \text{or} \quad T_{1,2}([\![\beta, 1]\!]) \quad \text{or} \quad \lambda_1^{-1}T_{-1,0}([\![\beta, 1]\!]) = \lambda_2^{-1}T_{1,2}([\![\beta, 1]\!]).$$

Also, it is easy to see that

$$T_w([\![\mathrm{Id}, 1]\!]) = T_{1,2}([\![\beta, 1]\!]), \tag{26}$$

and

$$T_w([\![\beta, 1]\!]) = T_{-1,0}([\![\mathrm{Id}, 1]\!]), \tag{27}$$

and that T_w takes the value 0 on all the other vectors.

Thus, with the choice of $\chi = a^r$, we have proved the following proposition.

Proposition 7.1. *The pro-p-Iwahori Hecke module structure of the three representations in Theorem 1.2 is as follows:*

(i) $(\frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0},T_{1,2}^2)})^{I(1)} \cong E_{0,1}$,
(ii) $(\frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0}^2,T_{1,2})})^{I(1)} \cong E_{1,0}$,
(iii) $(\frac{\mathrm{ind}_{IZ}^G d^r}{(\lambda_1^{-1}T_{-1,0}-\lambda_2^{-1}T_{1,2})})^{I(1)} \cong E_{\lambda_1,\lambda_2}$.

Note that Proposition 7.1 furnishes another proof of (6) in Theorem 1.2.

7.3. Another approach to Theorem 1.2

As mentioned in the introduction, the material in this section is communicated to us by V. Paškūnas.

Note that

$$\mathrm{End}_G\big(\mathrm{ind}_{IZ}^G \chi\big) = e_\chi \mathcal{H} e_\chi,$$

since $\mathrm{ind}_{IZ}^G \chi$ is the direct summand of $\mathrm{ind}_{I(1)Z}^G 1$ cut out by the idempotent e_χ.

We give a presentation of the $\mathcal{H}$-module $E_{1,0}$

$$e_\chi \mathcal{H} \oplus e_\chi \mathcal{H} \to e_\chi \mathcal{H} \to E_{1,0} \to 0 \tag{28}$$

as right $\mathcal{H}$-modules. The map $e_\chi \mathcal{H} \to E_{1,0}$ is defined by sending e_χ to v_χ. The relations defining $E_{1,0}$ imply that the images of $e_\chi, e_\chi T_w, e_\chi T_\beta, e_\chi T_w T_\beta$ form a basis of $E_{1,0}$ and $e_\chi T_\beta T_w \mapsto 0$.

The map $e_\chi \mathcal{H} \oplus e_\chi \mathcal{H} \to e_\chi \mathcal{H}$ is given by multiplication by $e_\chi T_\beta T_w e_\chi$ on the left on the first summand, and multiplication by $(e_\chi T_w T_\beta e_\chi)^2 = e_\chi T_w T_\beta T_w T_\beta$ on the left on the second summand. Thus, all elements of the form $e_\chi T_\beta T_w T_g$ or $e_\chi T_w T_\beta T_w T_\beta T_g$ lie in the image. Observe that the only elements which cannot be obtained this way are $e_\chi, e_\chi T_w, e_\chi T_\beta$, and $e_\chi T_w T_\beta$. This gives (28).

Applying the functor $\mathcal{T}$ from 7.1, we get:

$$\mathcal{T}(E_{1,0}) = \frac{\mathrm{ind}_{IZ}^G \chi}{(e_\chi T_\beta T_w e_\chi, (e_\chi T_w T_\beta e_\chi)^2)}.$$

7.4. A remark on $\mathrm{Ext}_G^1(\pi_r, \pi_r)$

The theorem of Paškūnas [16] referred to in the introduction asserts that

$$\dim \mathrm{Ext}_G^1(\pi_r, \pi_r) = 3.$$

The proof of this fact is quite nontrivial. In Paškūnas' proof, the dimension of $\mathrm{Ext}_G^1(\pi_r, \pi_r)$ is shown to be 3 in two major steps. In the first, it is proved that 3 is an upper bound, and this part of the proof uses only methods of mod p representation theory of $GL_2(\mathbb{Q}_p)$. In the second step, where 3 is shown to be a lower bound, Paškūnas uses an argument which is originally due to Colmez and Kisin [10,11]. The Colmez–Kisin argument is in the realm of the deformation theory of Galois representations. It seems a difficult problem to lay one's hands on a third self-extension which is linearly independent from the self-extensions considered in Theorem 1.2. Of course such a self-extension will have < 4 dimensional space of $I(1)$-invariants as the argument using (21) shows.

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