Universal Journal of Applied Mathematics 12(4): 95-108, 2024 http://www.hrpub.org
DOI: 10.13189/ujam.2024.120402

Multimode Dispersion of Light Wave Propagation in Graded-index Cladded Fiber-optic Cable

Sujit K. Bose

S.N. Bose National Center for Basic Sciences, Salt Lake City, Kolkata 700106, India

Received May 13, 2024; Revised July 21, 2024; Accepted August 19, 2024

Cite This Paper in the Following Citation Styles

***(a)**: [1] Sujit K. Bose, "Multimode Dispersion of Light Wave Propagation in Graded-index Cladded Fiber-optic Cable," Universal Journal of Applied Mathematics, Vol.12, No.4, pp. 95-108, 2024. DOI: 10.13189/ujam.2024.120402*

***(b)**: SSujit K. Bose (2024). Multimode Dispersion of Light Wave Propagation in Graded-index Cladded Fiber-optic Cable, Universal Journal of Applied Mathematics, 12(4), 95-108. DOI: 10.13189/ujam.2024.120402*

Abstract An optical fiber essentially consists of a transparent core medium with a thin cladding of a slightly less refractive index for total internal reflection of a passing signal of light. The material is usually homogeneous, but lately graded-index fiber cables are gaining increased application for greater efficiency in the propagation characteristics. Technology has evolved to furnish grading of the refractive index of the core to have different profiles. A parabolic profile of some degree α is often mentioned in texts (Keiser [7], p .63), but lately profiles of various other shapes, including nonsmooth steps (Cvijetic [1], p.35) have come in to usage for better wave guide action. A theoretical study of light waves propagating through such fibers is presented in this article, based strictly on the Maxwell equations of electromagnetism solved in terms of the single Hertz vector $\mathbf{\Pi}$. The dispersion equation for the guided wave propagation is obtained in general terms from the theory. The method is first developed for the case of the parabolic profile and then extended to any general form of the refractive index. Numerical computation of the dispersion equation, for the first three modes in the parabolic case, with $\alpha = 2$, show interestingly enough that the dispersion curves are pairs of segmented curves having opposite curvatures. A study of a non-smooth index fiber, like that of NZDSF, is also carried out from the general method developed, for which the dispersion equation does not show any dispersion whatsoever for the fundamental mode $m = 0$ - the purpose for which it is designed for practical use.

Keywords Fiber Optic Cables, Graded Index Core, Multimode Propagation, Dispersion Characteristics

---

1 Introduction

Optical fiber cables in passive optical networks have a variety of structures for different kinds of usage (Cvijetic [1], p.35). A cable may consist of a single fiber or a number of them having a cladding of slightly less index of refraction for total internal reflection. The refractive index of the glass core can be of varied properties as well, with the objective of reducing the distortion of the guided light waves propagating along its length and reduction in its attenuation. In recent years, extremely large 100-μm core diameter for gigabit transmission of data are also being developed (Bourdin et. al. [2]). A theoretical study for a uniform index core (or a step index fiber cable) can be traced to the treatise of Stratton [3], p. 524 based rigorously on the solution of Maxwell equations of electromagnetism. Several other books study the same problem in detail, deriving the dispersion equation of the propagating waves (Marcuse [4], p.289, Snyder and Love [5], p. 248, Agrawal [6], p.32, Keiser [7], p.54, and Saleh and Teich [8], p.397. In this important particular case, the propagating waves shows dispersion of the guided waves at all frequencies, which leads to formation of wave groups of narrower wave lengths, resulting in broadening pulse width during transmission of information data. As was shown by this author (Bose [9]) such propagation may lead to bursting of the pulse for some frequencies with serious consequences. In order to reduce the dispersion of the guided waves, several variable index fibers have been suggested (Cvijetic [1], p. 32) where the index of refraction has graded structure of different profiles in the radial direction. One such proposed profile is of parabolic shape of some degree α (Keiser [7], p. 68), while others have sections of rectilinear or some slightly curved shapes. Dispersion characteristics of such optical fibers remain to be theoretically investigated in a systematic manner.

Most of the earlier theories carried out in the 1970's centered around treating the scalar and vector Helmholtz's equation $\nabla^2\mathbf{E} + n^2(x, y)\mathbf{E} = 0$, where n is the variable refractive index. Due to variability of n across sections of a fiber, approximate methods of solution of the equation have been adopted by several authors (Hansen and Nicolassen [10], Feit and Fleck [11], Olshensky and Keck [12], Feit and Fleck [13], Almawagani et. al. [14], Savotchenko [15]). Azafraniec and Brily [16] develop geometric analysis of the propagation problem. In a recent paper, Agrawal [17] has treated the modal expansion of the Helmholtz's equation in the form of an integral containing a kernel similar to one in diffraction. The above cited papers ignore any cladding of the fiber involving internal reflection and discussion of chromatic dispersion.

The theoretical analysis presented here, is based on the other hand, strictly on the Maxwell equations of electromagnetism, where the Helmholtz equation is an approximated approach, because as is well known, that light waves are of that type (Born and Wolf [18], p. xxx of the Introduction), when the frequency pf the electromagnetic wave ranges between 175 to 375 THz (wave length between 770 to 1675 nm) (Keiser [6], p. 6). The electric and the magnetic field intensities $\mathbf{E}$ and $\mathbf{H}$ that govern the field involve two parameters, viz. permittivity and permeability and the variability of the refractive index n of the medium is assumed to be due to permittivity alone (Marcuse [4], p. 11). In the next section, following Mohosen [19], the electromagnetic field is represented in terms of a single field quantity - the Hertz vector $\mathbf{\Pi}$ that satisfies a wave equation modified due to inhomgeneity of the medium, such that $\mathbf{E}$ and $\mathbf{H}$ are expressed in terms of the components of $\mathbf{\Pi}$. The general formalism is next applied to the case of parabolic profile of the refractive index of the core, having a degree α. The formal modal solutions are then obtained for the components of $\mathbf{E}$ and $\mathbf{H}$ in the core region $0 \leq r \leq a$ (a = fiber radius), with a cladding over the region $r > a$ as insignificant energy penetrates it. Using the corresponding solution for the cladding region, the dispersion equation for the inhomogeneous wave guide propagation is obtained by purely classical methods of mathematical analysis. The equation is then treated for a second degree profile shape of $\alpha = 2$ for the refractive index of the core having suitable values and other parameters involved. The computations carried out for the first three modes $m = 0,\ 1,\ 2$, reveal that the guided phenomenon of dispersion is significantly bifurcated into two finite frequency ranges. These dispersion curves are shown in figures 1, 2 and 3. In the case of the fundamental mode $m = 0$, the two frequency ranges are comparatively significant, while that for the cases of $m = 1$ and $m = 2$ the frequency ranges are less extended. It may be mentioned here that the case $\alpha = 2$ is studied in great detail by Marcuse [4], p. 263, but without any guiding cladding mechanism leading to a different solution from that presented here. In the ensuing section, a general profile type $f(r/a)$ of the refractive index of the core is taken up similarly by expanding the function $f(\cdot)$ in a general Dini series (Watson [20], p. 577) permissible because of the boundedness of the function. The dispersion equation is then developed as in the parabolic case. That equation is tested for a stepped refractive index profile of an NZDSF (nonzero dispersion shifted fiber) for a single mode fiber ($m = 0$ case) and it was found that the equation has no roots what so ever, indicating that there is no dispersion for a fiber with a cladding, so that a wave would propagate without any change of shape. In this manner many other profiles of the refractive index can be examined for wave guide dispersion.

2 The basic equations of light wave propagation

A light wave propagating principally through the core of the cable is governed by the Maxwell equations of electromagnetism for certain frequency range f of the wave. The Maxwell equations govern the implied electric and magnetic intensities $\mathbf{E}$ and $\mathbf{H}$ of the (optical) field by the equations

$$\nabla \cdot \mathbf{D} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{4}$$

in which $\mathbf{D}$ and $\mathbf{B}$ denote the "displacement current" and the "magnetic induction" respectively that are related to $\mathbf{E}$ and $\mathbf{H}$ by the equations

$$\mathbf{D} = \epsilon \mathbf{E} \tag{5}$$

and

$$\mathbf{B} = \mu \mathbf{H} \tag{6}$$

where ϵ is the electric permittivity and μ the magnetic permeability of the transparent material. Let the latter two quantities for the core of the cable be respectively ϵ_1 and μ_1; then the variable refractive index n of the medium is related to $\epsilon_1,\ \mu_1$ by the relation (Keiser [6], p.68)

$$\epsilon_1 \mu_1 = \frac{n^2}{c_0^2}, \quad 0 \le r \le a \tag{7}$$

where c_0 is the velocity of light in the vacuum. Similarly, for the cladding material is $\epsilon_2,\ \mu_2$ are the permittivity and permeability respectively, then

$$\epsilon_2 \mu_2 = \frac{n_2^2}{c_0^2}, \quad r \ge a \tag{8}$$

where n_2 denotes the constant refractive index of the cladding material. The variability of n in Eq. (7) is due to variability of the permittivity ϵ_1 alone where μ_1 is regarded as invariable in the dielectric medium (Marcuse [4], p. 11).

In general, the Eqs. (3) - (8) are more compactly represented by a single Hertz vector $\mathbf{\Pi}$ (Mohsen [16]). Eq. (4) implies that the magnetic induction $\mathbf{B}$ is a solenoidal field represented by the curl of a vector field $\partial \mathbf{\Pi}/\partial t$, and so using Eq. (6),

$$\mathbf{B} = \nabla \times \frac{\partial \mathbf{\Pi}}{\partial t}, \quad \mathbf{H} = \frac{1}{\mu} \nabla \times \frac{\partial \mathbf{\Pi}}{\partial t} \tag{9}$$

Hence, by Eqs. (4) and (5), since μ is not a variable

$$\epsilon \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\mu} \nabla \times \nabla \times \frac{\partial \mathbf{\Pi}}{\partial t} \tag{10}$$

or, by integration with respect to time t,

$$\mathbf{E} = \frac{1}{\mu \epsilon} \nabla \times \nabla \times \mathbf{\Pi} \tag{11}$$

and so the displacement current given by Eq. (5) yields

$$\mathbf{D} = \frac{1}{\mu} \nabla \times \nabla \times \mathbf{\Pi} \tag{12}$$

automatically satisfying Eq. (1). Thus all the field quantities are determined in terms of $\mathbf{\Pi}$. In order to find the equation governing $\mathbf{\Pi}$, one has from Eqs. (3) and (9),

$$\nabla \times \left(\frac{1}{\mu \epsilon} \nabla \times \nabla \times \mathbf{\Pi} \right) = -\nabla \times \frac{\partial^2 \mathbf{\Pi}}{\partial t^2} \tag{13}$$

which is satisfied if

$$\frac{1}{\mu\epsilon}\,\nabla\times\nabla\times\mathbf{\Pi}+\frac{\partial^2\mathbf{\Pi}}{\partial t^2}=\nabla\psi \tag{14}$$

where ψ is an arbitrary scalar field. Hence, using the well known vector identity for curl curl,

$$\nabla(\nabla\cdot\mathbf{\Pi})-\nabla^2\mathbf{\Pi}+\mu\epsilon\,\frac{\partial^2\mathbf{\Pi}}{\partial t^2}=\mu\epsilon\,\nabla\psi \tag{15}$$

or,

$$\mu\epsilon\,\frac{\partial^2\mathbf{\Pi}}{\partial t^2}-\nabla^2\mathbf{\Pi}=\mu[\nabla(\epsilon\psi)-\psi\nabla\epsilon]-\nabla(\nabla\cdot\mathbf{\Pi}) \tag{16}$$

Now since ψ is arbitrary, choosing

$$\psi=\frac{1}{\mu\epsilon}\,\nabla\cdot\mathbf{\Pi} \tag{17}$$

Eq. (16) reduces to the wave equation for the varying index medium as

$$\nabla^2\mathbf{\Pi}+\frac{\nabla\epsilon}{\epsilon}\,(\nabla\cdot\mathbf{\Pi})=\mu\epsilon\,\frac{\partial^2\mathbf{\Pi}}{\partial t^2} \tag{18}$$

Evidently, for a homogeneous medium, such as the cladding material, $\nabla\epsilon=0$ so that Eq. (18) reduces to a wave equation propagating with a velocity $1/\sqrt{\mu\epsilon}$.

In the case of a light wave propagating along the axis of a cable, there is no transversal component of the Hertz vector, so that if $\Pi_1^{(1)}$ and $\Pi_3^{(1)}$ are the radial and longitudinal components of $\mathbf{\Pi}^{(1)}$ in the core

$$\mathbf{\Pi}^{(1)}=\Pi_1^{(1)}\mathbf{e}_1+\Pi_3^{(1)}\mathbf{e}_3 \tag{19}$$

where $\mathbf{e}_1$ and $\mathbf{e}_3$ are unit vectors in the radial and longitudinal directions. Thus Eq.(18) splits into the scalar equations

$$\nabla^2\Pi_1^{(1)}+\frac{1}{\epsilon_1}\,\frac{\partial\epsilon_1}{\partial r}\,\frac{1}{r}\,\frac{\partial\,(r\Pi_1^{(1)})}{\partial r}-\mu_1\epsilon_1\,\frac{\nabla^2\Pi_1^{(1)}}{\partial t^2}$$

$$=-\frac{1}{\epsilon_1}\,\frac{\partial\epsilon_1}{\partial r}\,\frac{\partial\Pi_3^{(1)}}{\partial z} \tag{20}$$

and

$$\nabla^2\Pi_3^{(1)}=\mu_1\epsilon_1\,\frac{\nabla^2\Pi_3^{(1)}}{\partial t^2} \tag{21}$$

It is to be noted that while Eq. (21) is a standard form wave equation with variable propagation velocity $1/\sqrt{\mu_1\epsilon_1}$, Eq. (21) is a modified form with a forcing function due to the axial component $\Pi_3^{(1)}$
The corresponding electric and magnetic intensities in the core following Eqs. (11) and (90 are thus

$$E_r^{(1)}=\frac{1}{\mu_1\epsilon_1}\Big[-\frac{1}{r^2}\,\frac{\partial^2\Pi_1^{(1)}}{\partial\theta^2}-\frac{\partial^2\Pi_1^{(1)}}{\partial z^2}+\frac{\partial^2\Pi_3}{\partial z\partial r}\Big] \tag{22}$$

$$E_\theta^{(1)}=\frac{1}{\mu\epsilon_1}\Big[\frac{\partial}{\partial r}\Big(\frac{1}{r}\,\frac{\partial\Pi_1^{(1)}}{\partial\theta}+\frac{1}{r}\,\frac{\partial^2\Pi_3^{(1)}}{\partial\theta\partial z}\Big] \tag{23}$$

$$E_z^{(1)}=\frac{1}{\mu_1\epsilon_1}\Big[\frac{1}{r}\,\frac{\partial}{\partial r}\Big\{r\Big(\frac{\partial\Pi_1^{(1)}}{\partial z}-\frac{\partial\Pi_3^{(1)}}{\partial r}\Big)\Big\}-\frac{1}{r^2}\,\frac{\partial^2\Pi_3^{(1)}}{\partial\theta^2}\Big] \tag{24}$$

$$H_r^{(1)}=\frac{1}{\mu_1}\,\frac{\partial}{\partial\theta}\Big(\frac{\partial\Pi_3^{(1)}}{\partial t}\Big) \tag{25}$$

$$H_\theta^{(1)}=-\frac{1}{\mu_1}\,\Big[\frac{\partial}{\partial r}\Big(\frac{\partial\Pi_3^{(1)}}{\partial t}\Big)-\frac{\partial}{\partial z}\Big(\frac{\partial\Pi_1^{(1)}}{\partial t}\Big)\Big] \tag{26}$$

$$H_z^{(1)} = -\frac{1}{\mu_1}\frac{\partial}{r\partial\theta}\Big(\frac{\partial\Pi_1^{(1)}}{\partial t}\Big) \tag{27}$$

In the cladding material, which is homogeneous, the radial component of the Hertz vector $\Pi_1^{(2)}$ vanishes, so that the electric and magnetic field intensities in the medium are given by the single wave equation for $\Pi_3^{(2)}$ with constant wave velocity of propagation $1/\sqrt{\mu_2\epsilon_2}$. Hence, in this case the field intensities $E_r^{(2)}$. $E_\theta^{(2)}$, $E_z^{(2)}$, $H_r^{(2)}$, $H_\theta^{(2)}$, $H_z^{(2)}$ reduce to the same form as Eqs. (22) - (27) with $\Pi_1^{(2)}$ set to zero and μ_1, ϵ_1 replaced by μ_2, ϵ_2 respectively. It is to be noted that in this case $H_z^{(2)} = 0$.

3 Modal propagation through parabolic index core

Here the index of refraction n of the core material is assumed to be parabolic of degree α such that

$$n^2 = n_1^2\Big[1 - 2\Delta\Big(\frac{r}{a}\Big)^\alpha\Big], \quad 0 \le r \le a \tag{28}$$

where

$$\Delta = \frac{n_1^2 - n_2^2}{2\,n_1^2} << 1 \tag{29}$$

in which the index n_1 of the core is slightly greater than n_2 (Keiser [7], p.68). In this case, using Eq. (28), one has

$$\frac{1}{\epsilon_1}\frac{d\epsilon_1}{dr} = -\frac{2\alpha\Delta}{a}\Big(\frac{r}{a}\Big)^{\alpha-1} \tag{30}$$

This coefficient is of order Δ appearing in Eq. (20). However, it is generally neglected altogether on physical ground (Marcuse [4], p.11), but is retained here without much difficulty.

The propagation of light waves in an optical fiber can take place in different modes as in the case of uniform homogeneous core step-index cable, essentially because of smallness of Δ. In such propagation, the solution of Eq. (21) is of the form (Marcuse [4], p. 293, Keiser [7], p. 54)

$$\Pi_3^{(1)} = f_3(r)\,e^{im\theta}\,e^{i(kz-\omega t)}, \quad 0 \le r \le a \tag{31}$$

where $f_3(r)$ is a function of the radial coordinate r only, $i = \sqrt{-!}$, m the mode, k the wave number and ω the circular frequency of the propagating wave. Inserting Eq.(31) in (21) and using the profile of the refractive index given by Eq. (28), one gets the equation

$$f_3''(r) + \frac{1}{r}f_3'(r) - \frac{m^2}{r^2}f_3(r) + \Big[\frac{\omega^2}{c_0^2}n_1^2\Big\{1 - 2\,\Delta\Big(\frac{r}{a}\Big)^\alpha\Big\}$$

$$-k^2\Big]\,f_3(r) = 0 \tag{32}$$

where the primes denote differentiation with respect to the argument. Re-arranging the equation for a perturbation solution,

$$r^2 f_3''(r) + r\,f_3'(r) + \Big[\Big(\frac{\omega^2}{c_0^2}\,n_1^2 - k^2\Big)\,r^2 - m^2\Big]\,f_3(r)$$

$$= \frac{2\,\omega^2 n_1^2\,\Delta}{c_0^2}\,r^2\Big(\frac{r}{a}\Big)^\alpha\,f_3(r) \tag{33}$$

It is now convenient to rescale the independent variable r by the nondimensional variable s defined by the relation

$$r = \Big(\frac{a}{u}\Big)\,s \tag{34}$$

in which u is a nondimensional parameter defined by the equation

$$u = ak\sqrt{\frac{c_p^2\, n_1^2}{c_0^2} - 1} \tag{35}$$

where $c_p = \omega/k$ is the phase velocit of propagation in the core. Thus Eq. (33) transforms into the equation

$$s^2 f_3''(s) + s\, f_3'(s) + (s^2 - m^2)\, f_3(s) = B_1\, s^{\alpha+2} f_3(s) \tag{36}$$

where the nondimensional constant B_1 is defined as

$$B_1 = \frac{2\, c_p^2\, a^2 k^2\, n_1^2}{c_0^2\, u^{\alpha+2}} \tag{37}$$

If the right hand side of the Eq. (36) is set to zero, the solution is $f_3(s) = A\, J_m(s)$, where A is a constant and $J_m(\cdot)$ the Bessel function of order m. Hence, inserting this first approximation in to the right hand side of Eq. (36), the general solution of that equation can be assumed to be of the form

$$f_3(s) = A\left[J_m(s) + B_1\, \Delta\, s^{\lambda}(C_0 + C_1 s^2 + C_2 s^4 + \cdots)\right] \tag{38}$$

where the coefficients C_0, C_1, $C_2 \cdots$ are certain constants to be determined. Inserting the expression for f_3 in Eq. (36) with f_3 replaced by $A\, J_m(s)$ on the right hand side, one obtains

$$\begin{aligned} s^{\lambda}[(\lambda^2 - m^2)\, C_0 &+ \{[(\lambda+2)^2 - m^2]\, C_1 + C_0\}\, s^2 \\ &+\{[(\lambda+4)^2 - m^2\}\, C_2 + C_1\}\, s^4 + \cdots] \\ &= s^{\alpha+2} J_m(s) = \frac{s^{\alpha+m+2}}{2^m} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(s^2/4)^{\nu}}{\nu!\,(m+\nu)!} \end{aligned} \tag{39}$$

Hence, equating coefficients of the powers of s on the two sides of Eq. (39), one obtains

$$\lambda = \alpha + m + 2 \tag{40}$$

$$C_0 = \frac{1}{2^m\, m!}\, \frac{1}{\lambda^2 - m^2} \tag{41a}$$

with the recurrence relation for the determination of the coefficients C_{ν} as

$$\begin{aligned} [(\lambda + 2\nu)^2 - m^2]\, C_{\nu} + C_{\nu-1} &= \frac{1}{2^m}\, \frac{(-1)^{\nu}}{\nu!\,(m+\nu)!\, 4^{\nu}}, \\ \nu &= 1,\, 2,\, 3,\, \cdots \end{aligned} \tag{41b}$$

yielding all the coefficients C_0, C_1, C_2, $\cdots$ of the solution (38)

For the complete description of the electromagnetic field, the radial component $\Pi_1^{(1)}$, using Eq. (24) is obtained from the solution of the equation

$$\begin{aligned} \nabla^2\Pi_1^{(1)} &- \mu_1\epsilon_1 \frac{\partial^2 \Pi_1^{(1)}}{\partial t^2} \\ &= \frac{2\alpha\Delta}{a}\left(\frac{r}{a}\right)^{\alpha-1}\left[\frac{\partial \Pi_3^{(1)}}{\partial z} + \frac{1}{r}\,\frac{\partial}{\partial r}(r\Pi_1^{(1)})\right] \end{aligned} \tag{42}$$

in which $\Pi_3^{(1)}$ is given by Eq. (38). The solution of Eq. (42) is of the modal form

$$\Pi_3^{(1)} = f_1(r)\, e^{im\theta}\, e^{i(kz-\omega t)} \tag{43}$$

where the function $f_1(r)$ satisfies the equation to the first order of Δ as,

$$r^2 f_1''(r) + r\, f_1'(r) + \left(\frac{u^2}{a^2}\, r^2 - m^2\right) f_1(r)$$

$$= \frac{2\alpha\Delta}{a}\, r^2 \left(\frac{r}{a}\right)^{\alpha-1} ikA\, J_m\left(\frac{a}{u}\, r\right) \tag{44}$$

or, resorting to the independent variable s defined by Eq. (34), the Eq. (44) becomes

$$s^2 f_1''(s) + s\, f_1'(s) + (s^2 - m^2)\, f_1(s)$$

$$= B_2\, \Delta A\, s^{\alpha+1} J_m(s) \tag{45}$$

where the nondimensional constant B_2 is given by the equation

$$B_2 = \frac{2i\, ak\, \alpha}{u^{\alpha+1}} \tag{46}$$

If the Bessel function $J_m(s)$ is expanded as a power series, the solution of the Eq. (45) is of the form

$$f_1(s) = B_2\, \Delta A\, (D_0 + D_1 s^2 + D_2 s^4 + \cdots) \tag{47}$$

where $D_0,\, D_1,\, D_2,\, \cdots$ are certain constants. Inserting the form (47) in Eq. (45) and equating the coefficients of the different powers of s, it again follows that

$$\mu = \alpha + m + 1 \tag{48}$$

$$D_0 = \frac{1}{2^m m!}\, \frac{1}{\mu^2 - m^2} \tag{49a}$$

and the recurrence relation for the rest of the coefficie nts D_ν as

$$[(\mu + 2\nu)^2 - m^2]\, D_\nu + D_{\nu-1} = \frac{1}{2^m}\, \frac{(-1)^\nu}{\nu!(m+\nu)!\, 4^\nu},$$

$$\nu = 1,\, 2,\, 3,\, \cdots \tag{49b}$$

yielding the values of $D_0,\, D_1,\, D_2,\, \cdots$ in the solution Eq.(47). Using the forms (31), (38) and (43), (47) for the field components of the Hertz vector $\Pi_3^{(1)}$ and $\Pi_1^{(1)}$ respectively in the Eqs. (22) - (27), the expressions for the electromagnetic field components within the fiber core can be explicitly written down. The required components for the satisfaction of the components are:

$$E_z^{(1)} = \frac{c_0^2}{n_1^2}\left(1 + \frac{2\Delta}{u^\alpha}\, s^\alpha\right) \frac{uA}{a^2}\Big[u\Big\{J_m(s)$$

$$-B_1\, \Delta s^{\lambda-2}[C_0(\lambda^2 - m^2) + C_1\{(\lambda+2)^2 - m^2\}$$

$$+C_2\{(\lambda+4)^2 - m^2\}\, s^4 + \cdots]\Big\}$$

$$+i\, ak\, B_2\, \Delta s^{\mu-1}\Big\{D_0(\mu+1) + D_1(\mu+3)\, s^2$$

$$+D_2(\mu+5)\, s^4 + \cdots\Big\}\Big] \tag{50}$$

and

$$H_\theta^{(1)} = \frac{i\omega u\, A}{a\mu_1}\Big[J_m'(s) + B_1\, \Delta s^{\lambda-1}\{\lambda C_0 + (\lambda+2)\, s^2$$

$$+(\lambda+4)\, C_2 s^4 + \cdots\}$$

$$-\frac{iak}{u}\, B_2\, \Delta s^\mu (D_0 + D_1 s^2 + D_2 s^4 + \cdots)\Big] \tag{51}$$

In the extended medium of the cladding of the cable $r \geq a$, which is supposed to be of uniform index n_2 $(< n_1)$, the Hertz vector $\Pi_3^{(2)}$ following Eq. (20) satisfies the wave equation

$$\nabla^2 \Pi^{(2)} = \mu_2 \epsilon_2 \, \frac{\nabla^2 \Pi^{(2)}}{\partial t^2} \tag{52}$$

where μ_2 and ϵ_2 are respectively the values of the permeability and permittivity of the medium. In as much as the electromagnetic field is totally axial in the medium with no radial component, $\mathbf{\Pi}^{(2)} = \Pi_3^{(2)} \mathbf{e}_3$, where $\Pi_3^{(2))}$ satisfies an equation of the type (52). Since the electromagnetic waves diverge from the axis of the cable, the solution for $\Pi_3^{(2)}$ is of the form

$$\Pi_3^{(2)} = C\, K_m\Big(\frac{v}{a}\, r\Big)\, e^{im\theta}\, e^{i(kz-\omega t)} \tag{53}$$

(Bose [8]), where $K_m(\cdot)$ is the modified Bessel function of order m of the second kind, and v is defined by

$$v = ak\sqrt{1 - \frac{c_p^2 n_2^2}{c_0^2}} \tag{54}$$

As in the case of the core of the cable, the electromagnetic components $E_z^{(2)}$ and $H_\theta^{(2)}$ are given by the expressions

$$\begin{aligned} E_z^{(2(} &= -\frac{n_2^2}{c_0^2}\Big[\frac{1}{r}\,\frac{\partial}{\partial r}\Big(r\,\frac{\partial \Pi_3^{(2)}}{\partial r}\Big) + \frac{1}{r^2}\,\frac{\partial^2 \Pi_3^{(2)}}{\partial \theta^2}\Big] \\ &= -\frac{n_2^2}{c_0^2}\,\frac{v^2}{a^2}\, C\, K_m\Big(\frac{v}{a}\, r\Big)\, e^{im\theta}\, e^{i(kz-\omega t)} \end{aligned} \tag{55}$$

and

$$\begin{aligned} H_\theta^{(2)} &= -\frac{1}{\mu_2}\,\frac{\partial}{\partial r}\Big(\frac{\partial \Pi_3^{(2)}}{\partial t}\Big) \\ &= \frac{i\omega\, v}{a\mu_2}\, C\, K_m'\Big(\frac{v}{a}\, r\Big)\, e^{im\theta}\, e^{i(kz-\omega t)} \end{aligned} \tag{56}$$

4 The dispersion equation

The boundary conditions at the core and the cladding lead to the dispersion equation. At the interface of the two media, the tangential components of the electric and the magnetic intensities are equal, that is to say, $E_z^{(1)} = E_z^{(2)}$, and $H_\theta^{(1)} = H_\theta^{(2)}$ at $r = a$ or from Eq. (34), $s = u$. In satisfying these conditions, much simplification occurs in the equation by assuming $n_2 \approx n_1$, and $\mu_2 \approx \mu_1$ as the parameters hardly differ for the two media. In this way, by eliminating the coefficients A and C, the dispersion equation is obtained as

$$\begin{aligned} &(1 + 2\Delta)\,\Big[\Big\{J_m(u) - B_1\,\Delta\, u^{\lambda-2}\,[C_0(\lambda^2 - m^2) + C_1\{(\lambda+2)^2 - m^2\}\, u^2 + C_2\{(\lambda+4)^2 - m^2\}\, u^4 + \cdots]\Big\} \\ &\quad + i\, ak\, B_2\,\Delta u^{\mu-1}[D_0(\mu+1) + D_1(\mu+3)\, u^2 + D_2(\mu+5)\, u^4 + \cdots]\Big] \times K_m'(v) \\ &\quad + u\,\Big[J_m'(u) + B_1\,\Delta u^{\lambda-1}\Big\{\lambda C_0 + (\lambda+2)\, C_1 u^2 + (\lambda+4)\, C_2 u^4 + \cdots\Big\} \\ &\quad - i\, ak\, B_2\,\Delta u^{\mu-1}\,(D_0 + D_1\, u^2 + D_2\, u^4 + \cdots)\Big] \times v^2 K_m(v) = 0 \end{aligned} \tag{57}$$

The Eq. (57) can be solved numerically for prescribed data. As a representative example, let $n_1 = 1.5$, $n_2 = 1.48515$, and $\alpha = 2$ (second degree parabolic profile). The equation can be expressed in terms of familiar numerical frequency f (often denoted by V in the fiber-optic literature) and the normalized propagation constant b defined by the relations

$$f = \sqrt{u^2 + v^2} = \frac{a\omega}{c_0}\,\sqrt{n_1^2 - n_2^2} \tag{58}$$

and
$$b = \frac{c_0^2/c_p^2 - n_2^2}{n_1^2 - n_2^2} = \frac{v^2}{f^2} \tag{59}$$

This means that
$$u = \sqrt{1-b}\, f \quad \text{and} \quad v = \sqrt{b}\, f \tag{60}$$

With the transformation (60), the left hand side of Eq. (57) becomes a function of f and b. The coefficients B_1 and B_2 appearing in the function are given by Eqs. (37) and (46) respectively, where the coefficient ak appearing in the two coefficients can be expressed as
$$ak = \sqrt{\frac{n_1^2\, v^2 + n_2^2\, u^2}{2n_1\,(n_1 - n_2)}} \tag{61}$$

where u and v are as defined in Eq. (60). Thus expressed in terms of f and b, the zeros of the left hand sided of Eq. (57) are first isolated in the domains $b = (0,\, 1)$, and $f = (0,\, 10)$ with increments of 0.1 and 0.25 respectively. Next, the zeros are refined by simple computation of the left hand side of Eq. (57). In this manner the dispersion curves for any mode m can be searched. The curves obtained in this manner for the first three modes $m = 0,\, 1,\, 2$ are plotted in figures 1, 2 and 3 respectively with f as the abscissa and b as the ordinate. Remarkably, the curves bifurcate into two segments over limited ranges of frequency. For the fundamental mode $m = 0$ (figure 1), the ranges of dispersion are more pronounced compared to the other two cases, lying in the frequency intervals $(0.95,\, 2.43)$ and $(3.10,\, 7.19)$. For the mode $m = 1$, the ranges are very short, lying between $(1.33,\, 1.78)$ and $(2.34,\, 2.60)$; whereas for the mode $m = 2$ these are a bit more pronounced compared to the case of $m = 1$ lying in $(1.67,\, 1.77)$ and $(2.50,\, 3.12)$. The two branches of the dispersion curve in each of the case $m = 0$ and $m = 1$ have opposite curvature but not in the case of $m = 2$. In contrast to these features, the dispersion curves for the different modes in a step-index fiber, extend to all values of the frequency f (Keiser [7], p. 59). The ranges of frequency for which there is no dispersion of the light waves propagating with the specified frequency and wave length are more suitable for the transmission of packets of data through the fiber optic cable because of no distortion of the carrier wave due to dispersive effect in the propagating waves.

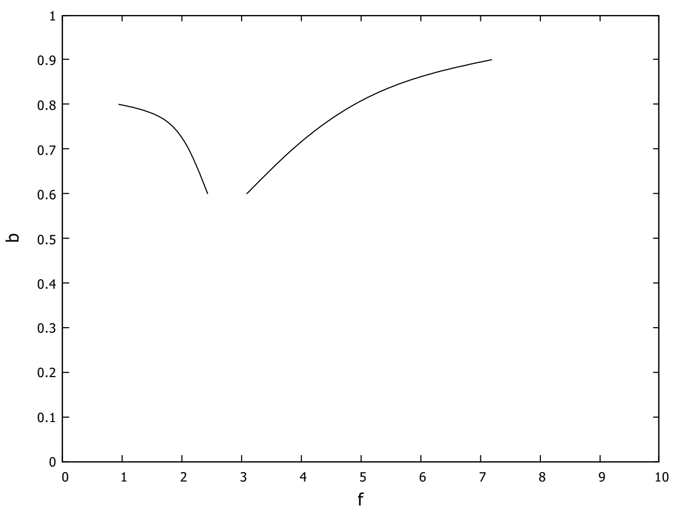

Figure 1. Dispersion Curve: Mode $m = 0$.

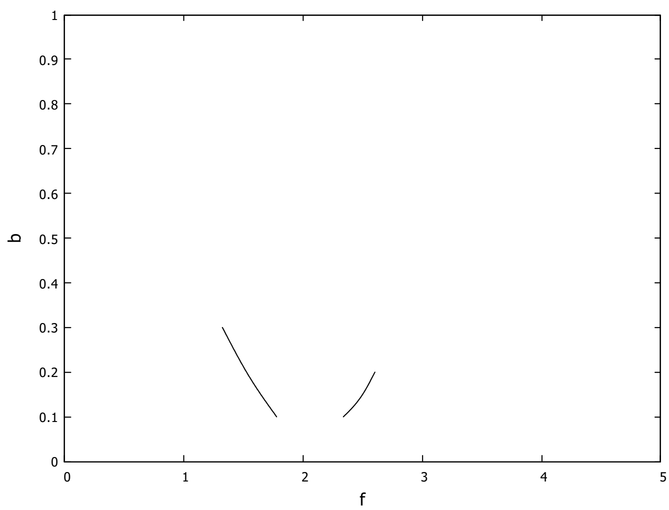

Figure 2. Dispersion Courve: Mode $m = 1$.

5 General profile core refractive index

Let the refractive index of the core medium be of the form
$$n^2 = n_1^2\Big[1 - 2\,\Delta\,\phi\Big(\frac{r}{a}\Big)\Big], \quad 0 \le r \le a \tag{62}$$

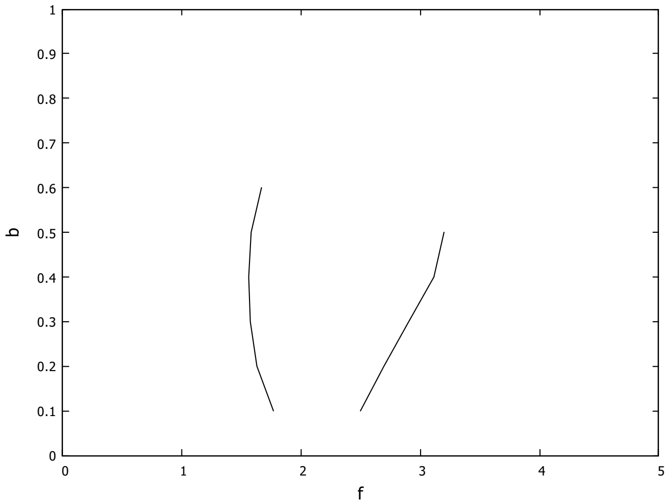

Figure 3. Dispersion Courve: Mode $m = 2$.

instead of the power law assumed in Eq. (28). In all practical cases the function $\phi(\cdot)$ is of bounded variation in $(0,\,1)$, so that $\int_0^1 \sqrt{t}\,\phi(t)\,dt$ exists, so that it is possible to expand $\phi(r/a)$ in a Dini series

$$\phi\Big(\frac{r}{a}\Big) = \sum_{\nu=0}^{\infty} b_\nu\, J_0\Big(\lambda_\nu\, \frac{r}{a}\Big) \tag{63}$$

(Watson [20], p. 577, 600), where $\lambda_0,\, \lambda_1,\, \lambda_2,\, \cdots$ are the zeros of $J_1(x)$ in ascending order of magnitude, and the coeffficients b_ν are given as

$$b_\nu = \frac{2}{J_0^2(\lambda_\nu)} \int_0^1 t f(t)\, J_0(\lambda_\nu t)\, dt \tag{64}$$

For the point $r = 0$, it atonce follows that

$$\sum_{\nu=0}^{\infty} b_\nu = \phi(0) \tag{65}$$

The values of λ_ν $(\nu = 1,\, 2,\, 3,\, \cdots)$ are tabulated in Abramowitz and Stegun [21], p.409. The value of λ_0 is evidently 0. It follows that Δ has now to be divided by $\phi(1) = \sum_{\nu=0}^{\infty} b_\nu J_0(\lambda_\nu)$ for continuity of the refractive index.

For modal propagation in the general type of core medium, the Hertz vector component

$$\Pi_3^{(1)} = f_3^{(g)}(r)\, e^{im\theta}\, e^{i(kz-\omega t)}, \quad 0 \le r \le a \tag{66}$$

where $f_3^{(g)}(\cdot)$ is of the form

$$f_3^{(g)}(s) = A\,[J_m(s) + B_1'\,\Delta\, s^{\lambda'}(C_0' + C_1'\, s^2 + C_2'\, s^4 + \cdots)] \tag{67}$$

and

$$B_1' = 2c_p^2\, a^2 k^2\, n_1^2/c_0^2 u^2 \tag{68}$$

Inserting the form (67) in Eq. (21), then as in the case of Eq. (39), one gets

$$\begin{aligned} s^{\lambda'}\,[(\lambda'^2 - m^2)\, C_0' + \{[(\lambda'+2)^2 - m^2]\, C_1' + C_0'\}\, s^2 \\ +\{[\lambda'+4)^2 - m^2]\, C_2' + C_1'\}\, s^4 + \cdots] \\ = s^2\, J_m(s) \sum_{\infty}^{\nu=0} b_\nu\, J_0\Big(\frac{\lambda_\nu s}{u}\Big) \end{aligned} \tag{69}$$

where the variable s is as before defined by Eq. (34). Now, the product of the two Bessel functions appearing in Eq. (69) is given by the power series expansion

$$J_m(s)\,J_0\Big(\frac{\lambda_\nu s}{u}\Big) = \Big(\frac{s}{2}\Big)^m \sum_{k=0}^{\infty} \frac{(-1)^k\,s^{2k}\,C_k^{(\nu)}}{4^k\,(k!)^2} \tag{70}$$

where

$$C_k^{(\nu)} = \begin{cases} \frac{k!}{(k+m)!}\,F(-k,\,-k-m,\,1;\,\lambda_\nu^2/u^2),\ \lambda_\nu < u \\ \frac{1}{m!}\left(\frac{\lambda_\nu}{u}\right)^{2k} F(-k,-k,m+1;u^2/\lambda_\nu^2), \lambda_\nu > u \end{cases} \tag{71}$$

In as much as the first argument of the hypergeometric functions $F(\cdot)$ are $-k$, the functions reduce to polynomials of degree k in the last argument λ_ν^2/u^2 or u^2/λ_ν^2. (Gradshteyn and Ryzhik [22], p.960), where $F(\cdot)$ is the hypergeometric function defined by the well known power series expansion given in Gradshteyn and Ryzhik [22], p. 1089. The right hand side of Eq. (69) is then of the form

$$\frac{s^{m+2}}{2^m}\sum_{k=0}^{\infty} C_k'' s^{2k} \tag{72}$$

where

$$C_k'' = \frac{(-1)^k}{4^k (k!)^2}\sum_{\nu=0}^{\infty} b_\nu\, C_k^{(\nu)} \tag{73}$$

Inserting the expressions (70) and (72), (73) in Eq. (69) and equating the coefficients of different powers of s, one has

$$\lambda' = m+2 \tag{74}$$

$$C_0' = \frac{C_0''}{2^m}\,\frac{1}{\lambda'^2 - m^2} \tag{75a}$$

while the remaining coefficients are given by the recurrence relation

$$[(\lambda'+2k)^2 - m^2]\,C_k' + C_{k-1}' = \frac{C_k''}{2^m}, \quad k = 1,\,2,\,3,\,\cdots \tag{75b}$$

where in virtue of Eq. (65), $C_0'' = \phi(0)$.

The Hertz vector component $\Pi_1^{(1)}$ is governed by Eq. (20), in which the coefficient for the present case, to the first order of Δ is represented as

$$\frac{1}{\epsilon_1}\,\frac{\partial \epsilon_1}{\partial r} = -\frac{2\Delta}{a}\,\phi'\Big(\frac{s}{u}\Big) \tag{76}$$

The above expression follows from the definition of ϵ_1 in terms of the refractive index n given by Eq. (62). Thus, to a first order in Δ, Eq. (20) yields the equation

$$\nabla^2\Pi_1^{(1)} - \frac{1}{c_1^2}\,\frac{\partial^2 \Pi_1^{(1)}}{\partial t^2} = \frac{2\Delta}{a}\,\phi'\Big(\frac{s}{u}\Big)\,\frac{\partial \Pi_3^{(1)}}{\partial z}$$

$$= -\frac{2ik\Delta}{a}A\sum_{\nu=0}^{\infty} b_\nu\lambda_\nu\, J_m(s)\, J_1\Big(\frac{\lambda_\nu s}{u}\Big) e^{im\theta} e^{i(kz-\omega t)} \tag{77}$$

where the solution for $\Pi_3^{(1)}$ given by Eqs. (66) and (61) is used, with the relation $J_0'(\cdot) = -J_1(\cdot)$. Hence, the particular solution of Eq. (77) is of the form

$$\Pi_1^{(1)} = f_1^{(g)}(r)\,e^{im\theta}\,e^{i(kz-\omega t)}. \quad 0 \le r \le a \tag{78}$$

which satisfies the differential equation

$$s^2 {f_1^{(g)}}''(s) + s\,{f_1^{(g)}}'(s) + (s^2 - m^2)\,f_1^{(g)}(s)$$

$$= -\frac{2i\,ak\,\Delta}{u^2}\,A\,s^2 \sum_{\nu=0}^{\infty} b_\nu \lambda_\nu\, J_m(s)\, J_1\Big(\frac{\lambda_\nu s}{u}\Big) \tag{79}$$

The product of the Bessel functions on the right hand side of Eq. (79) is now expressed as

$$J_m(s)\, J_1\Big(\frac{\lambda_\nu s}{u}\Big) = \frac{s^{m+1}}{2^{m+1}}\,\frac{\lambda_\nu}{u} \sum_{k=0}^{\infty} \frac{(-1)^k s^{2k} D_k^\nu}{4^k (k!)^2} \tag{80}$$

where

$$D_k^{(\nu)} = \begin{cases} \frac{k!}{(k+m)!}\,F(-k,\,-k-m,\,2;\,\lambda_\nu^2/u^2), & \lambda_\nu < u \\ \frac{1}{m!(k+1)}\big(\frac{\lambda_\nu}{u}\big)^{2k} F(-k,-k-1,m+1;u^2/\lambda_\nu^2), & \lambda_\nu > u \end{cases} \tag{81}$$

(Gradshteyn and Rizhik [22], p.960) in which the hypergeometric functions reduce to polynomials of degree k. Thus, the solution of Eq. (20)) is of the form

$$f_1^{(g)}(s) = -B_2'\,\Delta\,A\,s^{\mu'}\,(D_0' + D_1' s^2 + D_2'\,s^4 + \cdots) \tag{82}$$

On substitution of the form (82) in Eq. (80) leads to the conclusion that

$$B_2' = \frac{2i\,ak}{u^3} \tag{83}$$

$$\mu' = m + 3 \tag{83}$$

$$D_0' = \frac{D_0''}{2^{m+1}}\,\frac{1}{\mu'^2 - m^2} \tag{85a}$$

and

$$[(\mu' + 2k)^2 - m^2]\,D_k' + D_{k-1}' = \frac{D_k''}{2^{m+1}}, \quad k = 1,\,2,\,3,\,\cdots \tag{85b}$$

where

$$D_k'' = \frac{(-1)^k}{4^k\,(k!)^2} \sum_{\nu=0}^{\infty} b_\nu\, \lambda_\nu^2\, D_k^\nu \tag{86}$$

It is apparent that the electromagnetic field expressed by Eqs. (66), (67) and (78), (82) are of the same form as in the power law case with the coefficients C_0, C_1, C_2, $\cdots$ and D_0, D_1, D_2, $\cdots$ respectively replaced by the coefficients C_0', C_1', C_2' $\cdots$ and D_0', D_1', D_2', $\cdots$ with the indices λ, μ changed to λ', μ' and the coefficients B_1, B_2 changed to B_1', B_2'. Therefore the dispersion equation in this general case, remains of the same form as that of (57) with corresponding changes in these coefficients.

As an application, consider a Nonzero Dispersive Shifted Fiber (NZDSF) (Cvijetic [1], p.32) portrayed by the index of refraction function of the form

$$\phi\Big(\frac{r}{a}\Big) = \begin{cases} = 1, & 0 \le r/a \le 1/4 \\ = -1/3, & 1/4 < r/a \le 1/2 \\ = 1/6, & 1/2 < r/a \le 3/4 \\ = -1/12, & 3/4 < r/a \le 1 \end{cases} \tag{89}$$

with $n_1 = 1.5$ and $n_2 = 1.48515$ as in the preceding case for the parabolic profile. Performing detailed computation of the left hand side of Eq. (57) for the relevant fundamental mode $m = 0$ of the fiber utilising Eqs. (64), (68), (74) - $(75b)$ and (83) - (86), it is found that the expresssion is nonzero for b ranging from 0 to 1 and f ranging from 0 to 10, indicating that there is absolutely no chromatic wave guide dispersion takes place. This means that a pulse of wave would pass without distortion of shape through core of the cable in the present case for the data considered.

6 Conclusion

A theory of light propagation through multimode graded-index fiber-optic cables with a cladding is carried out in this article for understanding the chromatic dispersion characteristics of the medium. The theory is based strictly on the Maxwell equations of electromagnetism that hold for such propagation (Born and Wolf [18]). A wave equation is then developed for the vector field by the Hertz vector $\mathbf{\Pi}$ alone, for propagation through the inhomogeneous medium. The modal solution of that equation is then consructed for the case of a parabolic shaped refractive index of some degree α (Eq. (28)) in the radial direction, cited widely in the literature (Keiser [6], p.54). Assuming the thin cladding material to be homogeneous, the dispersion equation for the phenomenon is then obtained using appropriate boundary conditions . The method of analysis is next extended to any general form of the refractive index (Eq. (62)) of the core, using similar classical methods. For evaluating the dispersion characteristics of the phenomenon, typical numerical examples are considered, firstly for a parabola of degree $\alpha = 2$ and then for a multi-stepped index core. Some interesting features have emerged from the computations that were carried out for m = 0, 1 and 2 for the parabolic case. In that case, it is significantly found that the dispersion curve is divided into two neatly monotonic limited segments of arcs; the two parts being close to each other. While they are oppositely curved for the first two values of m, it is not so for the case of $m = 2$. In the step-indexed case with a shape like that of a NZDSF fiber, there is no dispersion at all for the fundamental mode $m = 0$ for which the NZDSF fiber is used in applications. A dispersionless propagation is a desired property for data transmission through a fiber-optic cable. It is to be noted however, that in applications a high number of modes are employed for increased bandwidth. In the absence of systmatic theories elsewhere, the chromatic features of dispersion can be extracted from the general dispersion equation developed in this article for any type of graded-index fiber of given variable refractive index profiles.

Declarations: There are no conflicts of interest involved in the research reported of this paper. And no data were generated or analysed by AI or otherwise in the presented research.

Acknowledgement

The author is thankful to the referees for their minute observations leading to clarity in the presentation of this article.

REFERENCES

[1] Cvjetic M., “Optical Transmission Systems Engineering”, Artech House, Boston, 2004.

[2] Burdin V. A. et. al., “New silica laser-optimized multimode fibers with extremely enlarged 1–μm core diameter for gigabit onboard and industrial networks”, Fibers, vol. 8, pp. 1-13, 2020; https://doi.org/103390/fib8030018.

[3] Stratton J. A, “Electromagnetic Theory”, John Wiley & Sons, New York, 2007.

[4] Marcuse D., “Light transmission optics”, Van Nostrand Reinhold, New York, 1982.

[5] Snyder A. W., Love J. D., “Optical Wave Guide Theory, Chapman and Hall”, London, 1983.

[6] Agrawal G. P.,, “Fiber-Optic Communication Systems”, John Wiley & Sons, Hoboken, 2010.

[7] Keiser G.,“Optical Fiber Communication”, McGraw-Hill 2011.

[8] Saleh B. E. A., Teich, M. C., “Fundamentals of Photonics”, Wiley, Hoboken, 2019.

[9] Bose S. K.,“On the transmission of data packets through fiber-optic cables of uniform index”, J. Opt. Comun., 000010151520210031, 2021.

[10] Hansen J. R , E. Nicolassen, "Propagation in graded-indexfibers: Comparison between experiment and three theories", Applied Optics, vo;. 17, pp. 2831-2835, 1978.

[11] Feit, M. D., Fleck J. A., "Calculation of dispersion in graded-index multimode fibers by a propagating beam method", Applied Optics, vol.18, 2843-2851, 1979.

[12] R. Olshansky, R., Keck D. B., "Pulse propagating in guided-index optical fibers", Applied Optics, vol. 15, pp. 483-491, 1976.

[13] Feit, M. D., J. A. Fleck, "Light propagation in guided-index fibers", Applied Optics, vol. 17, pp. 3990-3998, 1978.

[14] Almawgani, A.H.M., Taya, S.A., Hussein, A.J., Colak, I., Dispersion properties of a slab waveguide with a graded-index core layer and a nonlinear cladding using the WKB approximation, J. Opt. Soc. Amer. B, vol.39, pp. 1606-1613, 2022.

[15] Savotchenko S.E., "Modes of a nonlinear fiber containing a parabolic graded-index core with light induced variable radius", Romanian J. Phys., vol. 68, pp. 1-13, 2023. https://doi.or/1059277/RomJPhys.2023.68.208.

[16] Azafraniec, B., Brily B. E., "Geometric optics analysis of modal propagation in graded-index cylindrical fiber", Applied Optics, vol. 29, pp.1772-1779, 1990.

[17] Agrawal G. P., "Invited papar: Self-imaging in multi-mode gradwd-index fibers and its impact on nonlinear phenomena", Optical fiber technology, vol. 50, pp. 309-316, 2019.

[18] M. Born M., Wolf E.,, "Principles of Optics", Cambridge University Press, Cambridge, 2005.

[19] Mohsen A., "Electromagnetic field representation in inhomogeneous anisotropic media", Appl. Phys., vol. 3, pp. 123-128, 1973.

[20] Watson G.N., "A Treatise on the Theory of Bessel Functions", Cambridge University Press, Cambridge, 1952.

[21] Abramowitz M., Stegun I. A., "Handbook of Mathematical Functions", Dover Publications, New York, 1972.

[22] Gradshteyn I. S., Ryzhik I. M., "Table of Integrals, Series and Products", Academic Press, Orlando, 1980.