Polynomial Formula for Sums of Powers of Integers

K B Athreya and S Kothari

In this article, it is shown that for any positive integer $k \geq 1$, there exist unique real numbers $a_{kr}, \quad r = 1, 2, \ldots, (k+1)$, such that for any integer $n \geq 1$

$$S_{k,n} \equiv \sum_{j=1}^{n} j^k = \sum_{r=1}^{(k+1)} a_{kr} n^r.$$

The numbers a_{kr} are computed explicitly for $r = k+1, k, k-1, \ldots, (k-10)$. This fully determines the polynomials for $k = 1, 2, \ldots, 12$. The cases $k = 1, 2, 3$ are well known and available in high school algebra books.

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1. Introduction

Let k and n be positive integers and $S_{k,n} = \sum_{j=1}^{n} j^k$. A well-known story in the history of mathematics is that the great German mathematician Carl Friedrich Gauss, while in elementary school, noted that for $k = 1$, $S_{1,n} \equiv 1 + 2 + \ldots + n$ is, when written in reverse order, equal to $n + (n-1) + \quad \ldots + 1$. And when they are added one gets $2S_{1,n} = (1+n) + (2+(n-1)) + (3+(n-2)) + \ldots + (n+1) = (n+1)n$. This yields $S_{1,n} = \frac{n(n+1)}{2}$. Shailesh Shirali has told the authors that the great Indian mathematician of ancient times, Aryabhata [1], has explicitly mentioned in one of his verses the formulas for the cases $k = 2$ and $k = 3$. The three cases, $k = 1, 2, 3$, are now in high school algebra texts. Proofs of these formulas are given using the principle of induction. In this article, we establish that for any integer $k \geq 1$ there exist unique real numbers a_{kr}, $r = 1, 2, \ldots, (k+1)$, such that for any positive integer $n \geq 1$,

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$$S_{k,n} \equiv \sum_{j=1}^{n} j^k = \sum_{r=1}^{k+1} a_{kr} n^r.$$

In particular, it is shown that for any $k \geq 1$,

$$a_{k(k+1)} = \frac{1}{(k+1)}, a_{kk} = \frac{1}{2};$$

for $k \geq 2, a_{k(k-1)} = \dfrac{k}{12};$

for $k \geq 3, a_{k(k-2)} = 0;$

for $k \geq 4, a_{k(k-3)} = -\dfrac{k(k-1)(k-2)}{720};$

for $k \geq 5, a_{k(k-4)} = 0;$

for $k \geq 6, a_{k(k-5)} = \dfrac{k(k-1)(k-2)(k-3)(k-4)}{6 \times 7!};$

for $k \geq 7, a_{k(k-6)} = 0;$

for $k \geq 8, a_{k(k-7)} = \dfrac{-3k(k-1)\dots(k-6)}{6!7!};$

for $k \geq 9, a_{k(k-8)} = 0;$

for $k \geq 10, a_{k(k-9)} = 10\dfrac{k(k-1)\dots(k-8)}{12!};$

and for $k \geq 11, a_{k(k-10)} = 0.$

It is to be noted that once a formula is proposed, its validity may be verified by the principle of induction. The main problem is to guess what form the formula takes. We guessed that it should be a polynomial (based on the known formulas for $k = 1, 2, 3$) and found the explicit polynomial by a recurrence relation.

Sury [2] has noted that Euler had shown that for any positive integers k and n the sum $S_{k,n}$ can be expressed in terms of Bernoulli polynomials. Sury has also pointed out that Euler's work implies that for each positive integer k the sum $S_{k,n}$ is a polynomial in n. The main contribution of the present article is to determine this polynomial explicitly. We do so from elementary methods, thus making the article accessible to our undergraduate students.

2. An Explicit Formula

Fix a positive integer $k \geq 1$. For a positive integer $n \geq 1$, let

$$f(n) \equiv S_{k,n} \equiv \sum_{j=1}^{n} j^k$$

and set $f(0) = 0$. Then $\{f(n)\}_{n\geq 0}$ satisfies the recurrence relation

$$f(n+1) = f(n) + (n+1)^k,\ n = 0, 1, 2, \qquad (1)$$

with initial condition

$$f(0) = 0. \qquad (2)$$

It is clear that $\{f(n)\}_{n\geq 0}$ is uniquely determined by (1) and (2). Thus, if we find real numbers $a_{kj}, j = 1, 2 \ldots, (k+1)$ such that

$$g_k(n) \equiv \sum_{r=1}^{(k+1)} a_{kr} n^r, \quad n \geq 0 \qquad (3)$$

satisfies (1) and (2), then $g_k(n)$ has to equal S_{kn} for all $n \geq 1$. Since $g_k(0) = 0$ and hence (2) holds, to ensure that $g_k(\cdot)$ satisfies (1) it suffices to ensure that the coefficients of powers of n on both sides of the following equation

$$g_k(n+1) = g_k(n) + (n+1)^k,\ \text{i. e.,} \qquad (4)$$

$$\sum_{r=1}^{(k+1)} a_{kr}(n+1)^r = \sum_{r=1}^{(k+1)} a_{kr} n^r + (n+1)^k, \qquad (5)$$

are equal.

This leads to the following conditions:

$$(\text{coeff of } n^{k+1}) \quad a_{k(k+1)} = a_{k(k+1)} \qquad (6)$$

$$(\text{coeff of } n^k) \quad a_{k(k+1)} \binom{k+1}{1} + a_{kk} = a_{kk} + 1 \qquad (7)$$

$$(\text{coeff of } n^{k-1}) \qquad a_{k(k+1)}\binom{k+1}{2} + a_{kk}\binom{k}{1} + a_{k(k-1)} = a_{k(k-1)} + \binom{k}{1} \quad (8)$$

$$(\text{coeff of } n^{k-2}) \qquad a_{k(k+1)}\binom{k+1}{3} + a_{kk}\binom{k}{2} + a_{k(k-1)}\binom{k-1}{1} + a_{k(k-2)} = a_{k(k-2)} + \binom{k}{2} \quad (9)$$

and more generally,

$$(\text{coeff of } n^{k-r}) a_{k(k+1)} \qquad \binom{k+1}{r+1} + a_{kk}\binom{k}{r} + \ldots + a_{k_1(k-r+1)}\binom{k-r+1}{1} + a_{k(k-r)} = a_{k(k-r)} + \binom{k}{r}. \quad (10)$$

Now (6) provides no information but (7), (8), etc. do. Indeed (7) holds iff

$$a_{k(k+1)}\binom{k+1}{1} = 1 \quad (11)$$

and (8) holds iff

$$a_{kk}\binom{k}{1} = \binom{k}{1} - a_{k(k+1)}\binom{k+1}{2} \quad (12)$$

and (10) holds iff for $k \geq (r+1)$

$$a_{k(k-r+1)}\binom{k-r+1}{1} = \left(\binom{k}{r} - a_{k(k+1)}\binom{k+1}{r+1} - a_{k^k}\binom{k}{r}\ldots - a_{(k-r+2)}\binom{k-r+2}{2}\right). \quad (13)$$

for $r = 0, 1, 2, \ldots k$.

It follows from (7), (8), (9) and (10) that $a_{kr}, r = 1, 2, \cdots, k+1$ are uniquely determined as (10) provides a recursive determination of $a_{k(k-r+1)}$ from the knowledge of a_{kj} for $j = (k+1), \ldots, (k-r+2)$. Thus, (7)–(9) yield for any $k \geq 1$,

$$a_{k(k+1)} = \frac{1}{(k+1)}, \quad a_{kk} = \frac{k}{2};$$
$$\text{and for} \quad k \geq 2, a_{k(k-1)} = \frac{k}{12};$$

and (10) determines a_{kj} for $j \leq (k-2)$.

This proves the theorem given in the abstract. Let us call it Theorem A:

Theorem A. *Let k be a positive integer. Then, there exist unique numbers a_{kr}, $r = 1, 2, \ldots, (k+1)$ such that for any integer $n > 1$,*

$$S_{k,n} \equiv \sum_{j=1}^{n} j^k = \sum_{r=1}^{(k+1)} a_{kr} n^r.$$

3. Another Proof of Theorem A

We provide another proof of Theorem A at the end of this equation by first establishing the following:

Theorem B. *For positive integers k and n let $g_k(n) \equiv \sum_{j=1}^{n} j^k$. Then*

(i) $g_1(n) = \frac{n(n+1)}{2}$

(ii) *For $k \geq 2$, $n \geq 1$,*

$$(k+1)g_k(n) = \sum_{r=1}^{k-1} \binom{k+1}{r} (n^r - g_r(n)) + n^{k+1} + (k+1)n^k - n\,.$$

Proof.

(i) This result follows from Gauss's argument:

$$g_1(n) = 1 + 2 + \ldots + n = n + (n-1) + \ldots + 1$$

implying $2g_1(n) = (1+n) + (2 + (n-1)) + \ldots + (n+1) = (n+1)n.$

(ii) For $k \geq 2,\ n \geq 1$,

$$\begin{aligned} &\sum_{r=1}^{(k-1)} \binom{k+1}{r}(n^r - g_r(n)) \\ &= \sum_{r=1}^{(k-1)} \binom{k+1}{r}(n^r - \sum_{j=1}^{n} j^r) \\ &= -\sum_{r=1}^{(k-1)} \binom{k+1}{r} \sum_{j=1}^{n-1} j^r \end{aligned}$$

(where, if $n = 1$, we set $\sum_{j=1}^{n-1} j^r = 0$ for $r \geq 1$),

$$\begin{aligned} &= -\sum_{j=1}^{n-1} \left(\sum_{r=0}^{(k+1)} \binom{k+1}{r} j^r - 1 - (k+1)j^k - j^{k+1} \right) \\ &= -\sum_{j=1}^{n-1} (j+1)^{k+1} + (n-1) + (k+1) \sum_{j=1}^{n-1} j^k \\ &\quad + \sum_{j=1}^{n-1} j^{k+1} \\ &= -(g_{k+1}(n) - 1) + (n-1) + (k+1)(g_k(n) - n^k) \\ &\quad + g_{k+1}(n) - n^{k+1} \\ &= n + (k+1)g_k(n) - (k+1)n^k - n^{k+1} \end{aligned}$$

yielding (ii).

Proof of Theorem A. By Theorem B(i), $g_1(n)$ is a polynomial of degree two. Then Theorem B (ii) implies that $g_2(n)$ is a polynomial of degree three and by induction $g_k(n)$ is a polynomial of degree $(k+1)$. Further, Theorem B (ii) also shows that the leading coefficient in $g_k(n)$ is $(k+1)^{-1}$. One can use the same formula to show that the coefficient of n^k in $g_k(n)$ is $\frac{1}{2}$.

Remark 1. We now derive explicit expressions for $g_k(n)$ for $k = 2, 3, 4, 5$. From Theorem B (ii) we deduce that

$$\begin{aligned} 3g_2(n) &= \binom{3}{1}(n - g_1(n)) + n^3 + 3n^2 - n \\ &= 3\left(n - \frac{n(n+1)}{2}\right) + n^3 + 3n^2 - n \\ &= n^3 + n^2\ \frac{3}{2} + \frac{n}{2}, \end{aligned}$$

yielding

$$g_2(n) \equiv \sum_{j=1}^{n} j^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

Next,

$$\begin{aligned} 4g_3(n) &= \binom{4}{1}(n - g_1(n)) + \binom{4}{2}(n^2 - g_2(n)) \\ &\quad + n^4 + 4n^3 - n \\ &= 4\left(n - \frac{n(n+1)}{2}\right) \\ &\quad +6\left(n^2 - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\right) \\ &\quad +n^4 + 4n^3 - n \end{aligned}$$

$$\begin{aligned} &= n^4 + n^3\left(4 - \frac{6}{3}\right) + n^2\left(\frac{6}{2} - \frac{4}{2}\right) \\ &\quad +n\left(\frac{4}{2} - \frac{6}{6} - 1\right) \\ &= n^4 + 2n^3 + n^2, \end{aligned}$$

yielding

$$g_3(n) = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}.$$

Next,

$$\begin{aligned}
5g_4(n) &= \binom{5}{1}(n - g_1(n)) + \binom{5}{2}(n^2 - g_2(n)) \\
&\quad + \binom{5}{3}(n^3 - g_3(n)) + n^5 + 5n^4 - n \\
&= 5\left(n - \frac{n(n+1)}{2}\right) \\
&\quad +10\left(n^2 - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\right) \\
&\quad +10\left(n^3 - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right)\right) + n^5 + 5n^4 - n \\
&= n^5 + n^4\left(5 - \frac{10}{4}\right) + n^3\left(\frac{10}{2} - \frac{10}{3}\right) \\
&\quad +n^2\left(-\frac{5}{2} + \frac{10}{2} - \frac{10}{4}\right) + n\left(\frac{5}{2} - \frac{10}{6} = 1\right) \\
&= n^5 + n^4\frac{5}{2} + n^3\frac{10}{6} + n^2 \cdot 0 + n\left(-\frac{1}{6}\right),
\end{aligned}$$

yielding

$$g_4(n) = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

Next,

$$\begin{aligned}
6g_5(n) &= \binom{6}{1}(n - g_1(n)) + \binom{6}{2}(n^2 - g_2(n)) \\
&\quad + \binom{6}{3}(n^3 - g_3(n)) + \binom{6}{4}(n^4 - g_4(n)) \\
&\quad + n^6 + 6n^5 - n \\
&= 6\left(n - \left(\frac{n^2}{2} + \frac{n}{2}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+15\left(n^2-\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\right)\\
&+20\left(n^3-\left(\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}\right)\right)\\
&+15\left(n^4-\left(\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}\right)\right)\\
&+n^6+6n^5-n\\
= \;& n^6+n^5(6-3)+n^4\left(\frac{15}{2}-\frac{20}{4}\right)\\
&+n^3\left(\frac{20}{2}-\frac{15}{3}-\frac{15}{3}\right)\\
&+n^2\left(-\frac{6}{2}+\frac{15}{2}-\frac{20}{4}\right)+n\left(\frac{6}{2}-\frac{15}{6}+\frac{15}{30}-1\right)\\
= \;& n^6+3n^5+n^4\frac{5}{2}+n^3.0-\frac{n^2}{2}+n.0,
\end{aligned}
$$

yielding

$$g_5(n)=\frac{n^6}{6}+\frac{n^5}{2}+n^4\frac{5}{12}-\frac{n^2}{12}.$$

This process can be continued and $g_k(n)$ can be computed recursively for all integers $k \geq 1$. In Section 6, we compute $g_k(n)$ for $k = 1, 2, 3, \ldots, 12$.

4. A Matrix Method

Now that we know that a_{kr}, $r = 1, 2, \ldots (k+1)$ are determined uniquely, here is a matrix inversion method to find them. Let

$$\psi_k(\ell)=\sum_{r=1}^{\ell} r^k, \ell = 1, 2, \ldots, (k+1). \qquad (14)$$

Then, since

$$\sum_{r=1}^{(k+1)} a_{kr}\ell^r = S_{k,\ell} = \psi_k(\ell), \quad \ell = 1, 2, \ldots, (k+1), \quad (15)$$

it follows that

$$A_k \begin{pmatrix} a_{k1} \\ a_{k2} \\ . \\ a_{k(k+1)} \end{pmatrix} = \begin{pmatrix} \psi_k(1) \\ \psi_k(2) \\ . \\ \psi_k(k+1) \end{pmatrix}, \qquad (16)$$

where A_k is the $(k+1) \times (k+1)$ matrix with entries in the ℓth row given by

$$(\ell, \ell^2, \ldots, \ell^{(k+1)}), \quad \ell = 1, 2, \ldots, (k+1).$$

It can be checked that

$$A_k \begin{pmatrix} x_1 \\ . \\ . \\ x_{k+1} \end{pmatrix} = \begin{pmatrix} 0 \\ . \\ . \\ 0 \end{pmatrix}$$

implies that $x_1 = 0,\ x_2 = 0, \ldots, x_{n+1} = 0$.

This implies that A_k is invertible and hence, for any integer $k \geq 1$,

$$\begin{pmatrix} a_{k1} \\ a_{k2} \\ . \\ a_{k(k+1)} \end{pmatrix} = A_k^{-1} \begin{pmatrix} \psi_k(1) \\ \psi_k(2) \\ . \\ \psi_{k(k+1)} \end{pmatrix}. \qquad (17)$$

5. An Example

We now give an example where the function $g_k(n)$ arises in a counting situation.

Let S_n be the set $\{0, 1, 2, \ldots, n\}$ of integers. How many closed intervals $[i, j]$ are there where $i, j \in S_n$ and $i < j$? There are exactly n intervals of length one each. These are $[i, i+1]$ for $i = 0, 1, 2, (n-1)$. There are exactly $(n-1)$ intervals of length two each. These are $[i, i+2], \quad i = 0, 1, \ldots, (n-2)$. And in general there are exactly $(n-k+1)$ intervals of length k each. These are

$[i, i+k]$, $\quad i = 0, 1, \ldots, (n-k)$. Thus the total number of closed intervals $[i, j]$ with i, j in S_n and $i < j$ is $n + (n-1) + \ldots + (n-k+1) + \ldots + 1$. This is precisely $g_1(n)$.

Next, consider the set $S_{n,2} \equiv \{(i, j)\ i, j \in S_n\}$. How many squares are there with all vertices in $S_{n,2}$ and sides parallel to the axes? There are exactly n^2 such squares whose side length is one. These are with vertices $\{(i, j), (i+1, j), (i, j+1), (i+1, j+1)\}$ with $0 \leq i \leq n-1,\ 0 \leq j \leq n-1$. Next, for any integer $\ell,\ 1 \leq \ell \leq n$ there are exactly $(n-\ell+1)^2$ squares whose side length is ℓ with vertices all in $S_{n,2}$. These are the ones with vertices $\{(i, j), (i+\ell, j), (i, j+\ell), (i+\ell, j+\ell)\}$ with $0 \leq i \leq n-\ell,\ 0 \leq j \leq n-\ell$. Thus, the total number of squares with all vertices in $S_{n,2}$ is $\sum_{\ell=1}^{n}(n-\ell+1)^2 = \sum_{j=1}^{n} j^2$ which is precisely $g_2(n)$.

Next, consider the set $S_{n,3} \equiv \{(i, j, k), i, j, k \in S_n\}$. How many cubes are there with all vertices in $S_{n,3}$ and faces parallel to the coordinate planes? There are exactly n^3 such cubes whose side length is one. These are the ones with vertices $\{(i, j, k), (i+1, j, k), (i, j+1, k), (i, j, k+1), (i+1, j+1, k), (i, j+1, k+1), (i+1, j, k+1), (i+1, j+1, k+1)\}$ with $0 \leq i \leq n-1, 0 \leq j \leq n-1, 0 \leq k \leq n-1$. Similarly, there are $(n-\ell+1)^3$ cubes of side length ℓ with vertices in $S_{n,3}$. Thus, the total number of cubes with vertices in $S_{n,3}$ is $\sum_{\ell=1}^{n}(n-\ell+1)^3 = \sum_{j=1}^{n} j^3$ which is precisely $g_3(n)$. Next, for any integer $k > 3$ consider the set

$$S_{n,k} \equiv \{(i_1, i_2, \ldots, i_k)\ \ i_j \in S_n, j = 1, 2, \ldots, k\}.$$

How many k-dimensional cubes are there with all vertices in $S_{n,k}$ and faces parallel to the coordinate hyperplanes? Arguing as before, the number of such cubes with side length ℓ and vertices in $S_{n,k}$ is $(n-\ell+1)^k$. Thus, the total number of k-dimensional cubes with all vertices in $S_{n,k}$ is $\sum_{\ell=1}^{n}(n-\ell+1)^k = \sum_{j=1}^{n} j^k$ which is precisely $g_k(n)$.

6. Determination of a_{kj} for $j = (k+1), k, \ldots, (k-10)$

Returning to (7)–(9) we note that

$$a_{k(k+1)} = \frac{1}{(k+1)}, \quad a_{kk} = \frac{k}{2}, \quad a_{k(k-1)} = \frac{k}{12}.$$

Next, for $k \geq 3$

$$\begin{aligned} a_{k(k-2)} &\\ &= \frac{\left(\binom{k}{3} - a_{k(k+1)}\binom{k+1}{4} - a_{kk}\binom{k}{3} - a_{k(k-1)}\binom{k-1}{2}\right)}{\binom{k-2}{1}} \\ &= k(k-1)\left(\frac{1}{3!}\,\frac{1}{2} - \frac{1}{4!} - \frac{1}{24}\right) = 0. \end{aligned}$$

For $k \geq 4$,

$$\begin{aligned} a_{k(k-3)} &= \frac{\left(\binom{k}{4} - a_{k_1(k+1)}\binom{k+1}{5} - a_{rk}\binom{k}{4} - a_{k_1(k-1)}\binom{k-1}{3}\right)}{\binom{k-3}{1}} \\ &= k(k-1)(k-2)\left(\frac{1}{2}\,\frac{1}{4!} - \frac{1}{5!} - \frac{1}{12}\frac{1}{3!}\right) \\ &= -\frac{k(k-1)(k-2)}{720}. \end{aligned}$$

For $k \geq 5$,

$$\begin{aligned} a_{k(k-4)} &= \left(\frac{\binom{k}{5} - a_{k(k+1)}\binom{k+1}{6} - a_{kk}\binom{k}{5} - a_{k(k-1)}\binom{k-1}{4}}{\binom{k-4}{1}}\right) \\ &\qquad \times \left(\frac{-a_{k_1(k-3)}\binom{k-3}{2}}{\binom{k-4}{1}}\right) \\ &= k(k-1)(k-2)(k-3) \\ &\quad \times \left(\frac{1}{2}\,\frac{1}{5!} - \frac{1}{6!} - \frac{1}{12}\,\frac{1}{4!} + \frac{1}{1440}\right) = 0. \end{aligned}$$

For $k \geq 6$,

$$\begin{aligned} a_{k(k-5)} &= \left(\binom{k}{6} - a_{k(k+1)}\binom{k+1}{7} - a_{kk}\binom{k}{6}\right. \\ &- \ a_{k(k-1)}\binom{k-1}{5} \\ &\left. -a_{k_1(k-3)}\binom{k-3}{3}\right)\Big/\binom{k-5}{1} \\ &= k(k-1)(k-2)(k-3)(k-4) \\ &\quad \times \left(\frac{1}{2}\ \frac{1}{6!} - \frac{1}{7!} - \frac{1}{12}\ \frac{1}{5!} + \frac{1}{720}\ \frac{1}{3!}\right) \\ &= \frac{k(k-1)(k-2)(k-3)(k-4)}{6!}\frac{1}{7\times 6}. \end{aligned}$$

For $k \geq 7$,

$$\begin{aligned} a_{k(k-6)} &= \left(\binom{k}{7} - a_{k(k+1)}\binom{k+1}{8} - a_{kk}\binom{k}{7}\right. \\ &- \ a_{k(k-1)}\binom{k-1}{6} - a_{k(k-3)}\binom{k-3}{4} \\ &\left. - \ a_{k(k-5)}\binom{k-5}{2}\right)\Big/\binom{k-6}{1} \\ &= k(k-1)\dots(k-5) \\ &\quad \times \left(\frac{1}{2}\ \frac{1}{7!} - \frac{1}{8!} - \frac{1}{12}\ \frac{1}{6!} + \frac{1}{720}\ \frac{1}{4!} - \frac{1}{7!6}\ \frac{1}{2}\right) \\ &= k(k-1)\dots(k-5)\ 0 = 0. \end{aligned}$$

For $k \geq 8$,

$$\begin{aligned} a_{k(k-7)} &= \left(\binom{k}{8} - a_{k(k+1)}\binom{k+1}{9}\right. \\ &- \ a_{kk}\binom{k}{8} - a_{k(k-1)}\binom{k-1}{7} \end{aligned}$$

$$
\begin{aligned}
&- \quad a_{k(k-3)}\binom{k-3}{5} \\
&- \quad a_{k(k-5)}\binom{k-5}{3}\Bigg)\Bigg/\binom{k-7}{1}
\end{aligned}
$$

$$
\begin{aligned}
&= \quad k(k-1)\ldots(k-6)\times \\
&\quad \Bigg(\frac{1}{2}\ \frac{1}{8!}-\frac{1}{9!}-\frac{1}{12}\ \frac{1}{7!}+\frac{1}{720}\ \frac{1}{5!} \\
&\quad -\frac{1}{42\times 61}\ \frac{1}{3!}\Bigg) \\
&= \quad \frac{-3k(k-1)\ldots(k-6)}{7!\ 6!}.
\end{aligned}
$$

For $k \geq 9$,

$$
\begin{aligned}
a_{k(k-8)} \quad &= \quad \Bigg(\binom{k}{9} - a_{k(k+1)}\binom{k+1}{10} - a_{kk}\binom{k}{9} \\
&- \quad a_{k(k-1)}\binom{k-1}{8} - a_{k(k-3)}\binom{k-3}{6} \\
&- \quad a_{k(k-5)}\binom{k-5}{4} - a_{k(k-7)}\binom{k-7}{2}\Bigg)\Bigg/\binom{k-5}{1} \\
&= \quad k(k-1)\ldots(k-7)\Bigg(\frac{1}{2}\ \frac{1}{9!}-\frac{1}{10!}-\frac{1}{12}\ \frac{1}{8!} \\
&\quad +\frac{1}{6!}\ \frac{1}{6!}-\frac{1}{4!}\ \frac{1}{6!}\ \frac{1}{42}+\frac{3}{2}\ \frac{1}{7!6!}\Bigg) \\
&= \quad \frac{k(k-1)\ldots(k-7)}{10!}(5-1-7-\frac{1}{2}+7 \\
&\quad -5+\frac{3}{2}=0.
\end{aligned}
$$

For $k \geq 10$,

$$
a_{k_1(k-9)} \quad = \quad \Bigg(\binom{k}{10} - a_{k(k+1)}\binom{k+1}{11}
$$

$$
\begin{aligned}
&- \quad a_{kk}\binom{k}{10} - a_{k(k-1)}\binom{k-1}{9} \\
&- \quad a_{k(k-3)}\binom{k-3}{7} - a_{k(k-1)}\binom{k-5}{5} \\
&- \quad a_{k(k-7)}\binom{k-7}{3}\Bigg)\Bigg/\binom{k-9}{1} \\
&= \quad k(k-1)\dots(k-8)\Bigg(\frac{1}{2}\ \frac{1}{10!} - \frac{1}{11!} - \frac{1}{12}\ \frac{1}{9!} + \\
&\quad \frac{1}{720}\ \frac{1}{7!} - \frac{1}{6!}\ \ 42\ \ \frac{1}{5!} + \frac{3}{7!6!3!}\Bigg) \\
&= \quad \frac{k(k-1)\dots(k-8)\ 10}{12!}.
\end{aligned}
$$

For $k \geq 11$,

$$
\begin{aligned}
a_{k(k-10)} \quad &= \quad \Bigg(\binom{k}{11} - a_{k(k+1)}\binom{k+1}{12} - a_{kk}\binom{k}{11} \\
&- \quad a_{k(k-1)}\binom{k-1}{10} - a_{k(k-3)}\binom{k-3}{8} \\
&- \quad a_{k(k-5)}\binom{k-5}{6} - a_{k(k-7)}\binom{k-7}{4} \\
&- \quad a_{k(k-9)}\binom{k-9}{2}\Bigg)\Bigg/\binom{k-10}{1} \\
&= \quad \frac{k(k-1)\dots(k-9)}{12!}\Bigg(\frac{12}{2} - 1 - 11 \\
&+ \quad \frac{120 \times 11 \times 10 \times 9}{720} \\
&\quad -\frac{12 \times 11 \times 10 \times 9 \times 8}{6 \times 6!} \\
&\quad + 3\frac{12 \times 11 \times 10 \times 9 \times 8}{6!}\frac{1}{41} - \frac{10}{2}\Bigg) \\
&= \quad \frac{k(k-1)\dots(k-9)}{12!}\ 0 = 0.
\end{aligned}
$$

Summarizing the above we have:

$$\begin{array}{ll}
\text{for } k \geq 1 & a_{k(k+1)} = \frac{1}{(k+1)} \; a_{kk} = \frac{1}{2} \\
\text{for } k \geq 2 & a_{k(k-1)} = \frac{k}{12} \\
\text{for } k \geq 3 & a_{k(k-2)} = 0 \\
\text{for } k \geq 4 & a_{k(k-3)} = -\frac{k(k-1)(k-2)}{720} \\
\text{for } k \geq 5 & a_{k(k-4)} = 0 \\
\text{for } k \geq 6 & a_{k(k-5)} = \frac{k(k-1)(k-2)(k-3)(k-4)}{6!} \; \frac{1}{7\times 6} \\
\text{for } k \geq 7 & a_{k(k-6)} = 0 \\
\text{for } k \geq 8 & a_{k(k-7)} = -3\frac{k(k-1)\ldots(k-6)}{7!6!} \\
\text{for } k \geq 9 & a_{k(k-8)} = 0 \\
\text{for } k \geq 10 & a_{k(k-9)} = \frac{k(k-1)\ldots(k-8)10}{12!} \\
\text{for } k \geq 11 & a_{k(k-10)} = 0.
\end{array}$$

This, in turn, yields for integers $n \geq 1$,

$$\begin{aligned}
S_{1,n} &= \frac{n^2}{2} + \frac{n}{2} \\
S_{2,n} &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \\
S_{3,n} &= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \\
S_{4,n} &= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \\
S_{5,n} &= \frac{n^6}{6} + \frac{n^5}{2} + \frac{n^4 5}{12} - \frac{n^2}{12} \\
S_{6,n} &= \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42} \\
S_{7,n} &= \frac{n^8}{8} + \frac{n^7}{2} + n^6\frac{7}{12} - n^4\frac{7}{24} + \frac{n^2}{12} \\
S_{8,n} &= \frac{n^9}{9} + \frac{n^8}{2} + n^7\frac{2}{3} - n^5\frac{7}{15} + n^3\frac{2}{9} - \frac{n}{30} \\
S_{9,n} &= \frac{n^{10}}{10} + \frac{n^9}{2} + n^8\frac{3}{4} - n^6\frac{7}{10} + \frac{n^4}{2} - \frac{3}{20}n^2 \\
S_{10,n} &= \frac{n^{11}}{11} + \frac{n^{10}}{2} + n^9\frac{5}{6} - n^7 + n^5 - \frac{n^3}{3} + \frac{5}{66}n
\end{aligned}$$

$$S_{11,n} = \frac{n^{12}}{12} + \frac{n^{11}}{2} + n^{10}\frac{11}{12} - n^8\frac{11}{8} + n^6\frac{11}{6} - n^4\frac{33}{24} + n^2\frac{5}{12}$$

$$S_{12,n} = \frac{n^{13}}{13} + \frac{n^{12}}{2} + n^{11} - n^9\frac{11}{6} + n^7\frac{22}{7} - n^5\frac{33}{10} + \frac{5}{3}n^3 - \frac{691}{2730}n.$$

A natural conjecture from our calculations is that for any $k \geq (2r+1)$, k, r positive integers, $a_{k(k-2r)} = 0$. It may be noted that we have verified it for $r = 1, 2, \ldots, 5$ and $k \geq (2r+1)$.

7. Concluding Remarks

7.1 *Asymptotic Behavior of* $S_{k,n}$ *For Large* n

It follows from

$$S_{k,n} = \sum_{j=1}^{n} j^k \equiv \sum_{j=1}^{k+1} a_{k_j} n^j$$

that

$$\begin{aligned} \frac{1}{n^{k+1}} \sum_{j=1}^{n} j^k &= a_{k(k+1)} + \frac{\sum_{j=1}^{k} a_{kj} n^j}{n^{k+1}} \\ &= \frac{1}{(k+1)} + \sum_{j=1}^{k} \frac{a_{kj} n^j}{n^{k+1}} \\ \Rightarrow \lim_{n\to\infty} \sum_{j=1}^{n} \frac{j^k}{n^{k+1}} &= \frac{1}{(k+1)}. \end{aligned}$$

This also follows from the Riemann sum approximation

$$\sum_{j=1}^{n} \frac{j^k}{n^{k+1}} = \frac{1}{n} \sum_{j=1}^{n} f_k\left(\frac{j}{n}\right), \text{ where } f_k(x) = x^k,\ 0 \leq x \leq 1$$

which converges to

$$\to \int_0^1 f_k(x)dx = \frac{1}{(k+1)}.$$

Similarly, the second order behavior is given by

$$\left(\frac{1}{n^{k+1}}\sum_{j=1}^{n} j^k - \frac{1}{(k+1)}\right) = \frac{1}{2}\frac{1}{n} + \sum_{j=1}^{k-1}\frac{a_{kj}n^j}{n^{k+1}}$$
$$\Rightarrow n\left(\frac{1}{n^{k+1}}\sum_{j=1}^{n} j^k - \frac{1}{(k+1)}\right) \to \frac{1}{2}.$$

And so on.

7.2 *No Induction Involved*

For the cases $k = 1, 2, 3$, the proof in elementary texts involves merely the verification of the formula using the principle of induction. Our proofs of the polynomial formula in Theorem A involves no induction but are from first principles.

7.3 *Other Treatments*

There are many papers on this subject. Some of them are listed in the Suggested Reading.

Suggested Reading

[1] Walter Eugene Clark, *Translation of Aryabhatiya of Aryabhata*, The University of Chicago Press, verse 22, pp.37, 1930.

[2] B Sury, Bernoulli numbers and the Riemann Zeta function, *Resonance*, No.7, pp.54–62, 2003.

[3] A F Beardon, Sums of powers of integers, *Am. Math. Monthly*, Vol. 103, pp.201–213, 1996.

[4] A W F Edwards, Sums of powers of integers: A Little of the History, *The Mathematical Gazette*, Vol.66, No.435, 1982.

[5] H K Krishnapriyan, Eulerian polynomials and Faulhaber's result on sums of powers of integers, *The College Mathematics Journal*, Vol.26, pp.118–123, 1995.

[6] Robert Owens, Sums of powers of integers, *Mathematics Magazine*, Vol.65, pp.38–40, 1992.

[7] S Shirali, On sums of powers of integers, *Resonance*, Indian Academy of Sciences, July 2007.

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