EMBEDDINGS OF 3–MANIFOLDS VIA OPEN BOOKS

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Abstract. In this note, we discuss embeddings of 3–manifolds via open books. First we show that every open book of every closed orientable 3–manifold admits an open book embedding in any open book decompistion of $S^2 \times S^3$ and $S^2 \widetilde{\times} S^3$ with the page a disk bundle over S^2 and monodromy the identity. We then use open book embeddings to reprove that every closed orientable 3–manifold embeds in S^5.

1. Introduction

Let M be a closed oriented smooth manifold and B be a co-dimension 2 oriented smooth submanifold with a trivial normal bundle in M. We say that M has an *open book decomposition*, denoted by $\mathcal{O}b(B, \pi)$, if $M \setminus B$ is a locally trivial fibre bundle over S^1 such that the fibration π in a neighborhood of B looks like the trivial fibration of $(B \times D^2) \setminus (B \times \{0\}) \to S^1$ sending (x, r, θ) to θ, where $x \in B$ and (r, θ) are polar co-oridinates on D^2 and the boundary of each fibre is B which we call the binding. The closure of each fibre is called the page of the open book and the monodromy of the fibration is called the monodromy of the open book. An open book decomposition of M is determined – up to diffeomorphism of M – by the topological type of the page Σ and the isotopy class of the monodromy which is an element of the mapping class group of Σ.

In [Al], J. Alexander proved that every closed oriented 3–manifold admits an open book. Open book decompositions of closed oriented simply connected manifolds were studied by H. Winkelnkemper in [Wi], where he proved the existence of such decompositions on closed oriented simply connected manifolds of dimension n at least 6, provided n is not divisible by 4. He also established that if the dimension $n > 6$ of a closed simply connected manifold is divisible by 4, then it admits an open book decomposition if and only if its signature is zero. Winkelnkemper's results were then extended by J. Lawson [La], F.Quinn [Qu] and I. Tamura [Ta]. Due to their works, the conditions under which a manifold admits an open book decomposition is now well known. These conditions are generally very mild and hence a very large class of manifolds satisfy them. A particular class of manifolds that will be of interest to us consists of odd dimensional closed orientable manifolds. It can be easily deduced from [Wi] that every closed orientable odd dimensional manifold admits an open book decomposition. See also [Qu] for more details.

In recent times, the study of open book decompositions of manifolds has become very prominent due to the connection – discovered by E. Giroux [Gi] – between the open book decompositions and the contact structures. Let M be an odd dimensional smooth manifold. Recall that a contact structure ξ on M is a nowhere integrable co-dimension one distribution. Giroux in his seminal work [Gi] showed that there is a one to one correspondence between the isotopy classes of co-oriented contact structures on a closed oriented 3–manifold M and the open book decompositions of M up to positive stabilizations. By a positive stabilization operation on an open book of M, we mean just the plumbing of a positive Hopf band to the page of the open book. More precisely, it is adding a 1–handle to the page and modifying the monodromy by adding a positive Dehn twist along a curve which goes over the handle exactly once. For more details on this, see [Et]. See also, [Gi] and [Ko] for more regarding Giroux's correspondence.

In this article, we first study open book embeddings of 3–manifolds. We say that a smooth manifold M with a given open book decomposition admits an open book embedding in an open book decompistion of a smooth manifold N, provided there is an embedding of M in N such that –as a submanifold of M– the given

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open book decomposition on M is compatible with the open book decomposition of N. Such a submanifold as an open book is also known as a nested open book of N with the given open book, see [DK]. A more precise definition of an open book embedding is given in Section 3.2.

Open book embeddings were used by A. Mori [Mr] to prove that every closed co-oriented contact 3-manifold open book embeds in S^7. His result was generalized by D. Martinez Torres in [Mr] to show that there is an open book embedding of any contact manifold of dimension $2n+1$ in S^{4n+3}. In addition, the article by J. Etnyre and Y. Lekili [EL] uses open book embeddings to produce contact embeddings. In particular, they establish that there exists a contact structure ξ_{ot} on $S^2 \times S^3$ in which every co-orinentable contact 3–manifold embeds in such a way that the pull-back of ξ_{ot} via this embedding induces the given contact structure on the contact 3–manifold. We would like to mention that it is not known, if every 3–manifold admits an open book embedding in S^5. Our first theorem is regarding open book embeddings of 3–manifolds in $S^3 \times S^2$ and $S^2 \widetilde{\times} S^3$. Before we state this result we would like to mention that in this article, we work in smooth category, i.e., all maps and manifolds if not stated otherwise are smooth. We now state the theorem:

Theorem 1. *Let M be a closed oriented connected 3–dimensional manifold together with an open book decomposition $\mathcal{O}b(B,\pi)$. Then, open book $\mathcal{O}b(B,\pi)$ admits an open book embedding in any open book decomposition associated to $S^3 \times S^2$ with pages a disk bundle over S^2 of even Euler number and monodromy the identity as well as in any open book of $S^3 \widetilde{\times} S^2$ with pages a disk bundle over S^2 of odd Euler number and monodromy the identity.*

Using methods used in establishing the Theorem 1, we establish the following:

Theorem 2. *Every closed orientable 3–manifold admits a smooth embedding in S^5.*

This theorem was first discovered by M. Hirsch in [Hi]. Embeddings of manifolds in Euclidean spaces have a long history starting from the seminal work of H.Whitney [Wh] establishing that every closed n–manifold admits an embedding in $\mathbb{R}^{2n}$. In fact, a general result of A. Heafliger and M.Hirsch [HH] implies that every odd dimensional closed orientable manifold embeds in $\mathbb{R}^{2n-1}$. There are other proofs of Theorem 2. See, for example, the article [HLM] for a proof using what is now known as braided embeddings and also the article [Ka] for embeddings of closed orientable 3–manifolds in S^5 using surgery description of 3-manifolds and Kirby calculus. We refer to [Wa] and [Ro] for embeddings of non-orientable 3–manifolds in $\mathbb{R}^5$.

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2. Preliminaries

In this section, we quickly review notions necessary for this article pertaining mapping class groups and open book decompositions.

2.1. Mapping class group.

Let us begin by recalling the definition of a mapping class group as in [FM].

Definition 3 (Mapping class group)**.** *Let Σ be an orientable manifold. By the mapping class group of Σ, we mean the group of orientation preserving self diffeomorphisms of Σ upto isotopy. In case, Σ has a non-empty boundary $\partial\Sigma$, then we always assume that diffeomorphisms and the isotopies are the identity in a collar neighborhood of the boundary.*

We denote the mapping class group of a surface Σ by $\mathcal{MCG}(\Sigma)$. In case, Σ has a non-empty boundary and we want to emphasis this fact, we will denote the mapping class group by $\mathcal{MCG}(\Sigma, \partial\Sigma)$. In this article, unless specified otherwise, we say that two diffeomorphisms f and g of a manifold $(M, \partial M)$ are equal provided they represent the same element in $\mathcal{MCG}(M, \partial M)$.

Lickorish in [Li] showed that every element of the mapping class group of an orientable surface is a product of Dehn twists.

Recall that, by definition, a Dehn twist along the circle $S^1 \times \{1\}$ in $S^1 \times [0,2]$ is a self diffeomorphism of $S^1 \times [0,2]$ given by $(e^{i\theta}, t)$ going to $(e^{i(\theta \pm \pi t)}, t)$. Note that the Dehn twist fixes both the boundary components of $S^1 \times [0,2]$. Hence, given an embedded circle c in an orientable surface Σ, we can define the Dehn twist of an annular neighborhood $N(c) = S^1 \times [0,2]$ of $c = S^1 \times \{1\}$ in Σ which is the identity when restricted to the boundary of this annular neighborhood. Clearly, we can extend this diffeomorphism by the identity in the complement of the annular neighborhood to produce a self diffeomorphism of Σ. This diffeomorphism is called a Dehn twist along the embedded cirlcle c in the surface Σ. For more details, refer [FM].

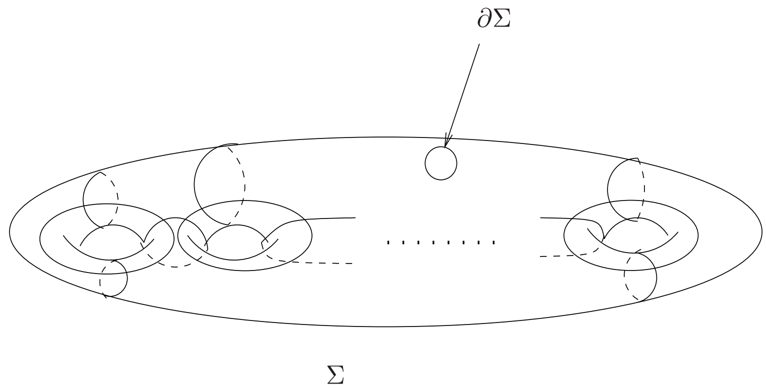

FIGURE 1. Figure depicts genus g compact orientable surface Σ with one boundary component. The embedded curves on the surface represents the standard Lickorish generators corresponding to the presentation of the mapping class group of Σ as given in [Jo].

In fact, it was later established by Lickorish that every element of the mapping class group of a closed orientable surface of genus g is a product of Dehn twists along curves depicted in the Figure 1. For an orientable surface with connected boundary, it was established by D.L. Johnson [Jo] that the same generators that generate the mapping class group of the closed surface obtained by attaching a disk along the boundary of the surface are sufficient to generate the mapping class group. In case, the boundary of the surface has more than one connected components, then we need – additionally – Dehn twists along certain simple closed curves which separate boundary components. See [FM, p. 133] for a picture depicting generators in this case.

Let $\mathcal{C}$ be the finite collection of simple closed curve embedded on a surface Σ as depicted in [FM, p. 133]. We know that Dehn twists along the curves in $\mathcal{C}$ generate the mapping class group of Σ. Any curve $c \in \mathcal{C}$ will be refered as a *Lickorish curve* and Dehn twists along these Lickorish curves as *Lickorish generators*.

2.2. **Open books.**

Let us review few results related to open book decompositions of manifolds. We first recall the following:

Definition 4 (Open book decomposition)**.** *An open book decomposition of a closed oriented manifold M consists of a co-dimension 2 oriented submanifold B with a trivial normal bundle in M and a locally trivial fibration $\pi : M \setminus B \to S^1$ such that $\pi^{-1}(\theta)$ is an interior of a co-dimension 1 submanifold N_θ and $\partial N_\theta = B$, for all θ. The submanifold B is called the binding and N_θ is called a page of the open book. We denote the open book decomposition of M by $(M, \mathcal{O}b(B, \pi))$ or sometimes simply by $\mathcal{O}b(B, \pi)$.*

Next, we discuss the notion of an *abstract open book decomposition*. To begin with, let us recall the following:

Definition 5 (Mapping torus)**.** *Let Σ be a manifold with non-empty boundary $\partial\Sigma$. Let ϕ be an element of the mapping class group of Σ. By the mapping torus $\mathcal{MT}(\Sigma, \phi)$, we mean*

$$\Sigma \times [0,1]/\sim$$

where $\sim$ is the equivalence relation identifying $(x, 0)$ with $(\phi(x), 1)$.

We are now in a position to define an abstract open book decomposition.

Definition 6. *Let Σ and ϕ as in the previous definition. An abstract open book decomposition of M is pair (Σ, ϕ) such that M is diffeomorphic to*

$$\mathcal{MT}(\Sigma, \phi) \cup_{id} \partial\Sigma \times D^2$$

where id denotes the identity mapping of $\partial\Sigma \times S^1$.

Note that the mapping class ϕ determines M uniquely up to diffeomorphism. The map ϕ is called the *monodromy* of the open book. The manifold obtained by identifying the boundary of $\mathcal{M}T(\Sigma, \phi)$ with the boundary of $\partial\Sigma \times D^2$ as described in the Definition 6 will be denoted by $\mathcal{A}ob(\Sigma, \phi)$. A manifold M together with a given abstract open book decomposition will be denoted by $(M, \mathcal{A}ob(\Sigma, \phi))$.

One can easily see that an abstract open book decomposition of M gives an open book decomposition of M up to diffeomorphism and vice versa. Hence, we will not generally distinguish between open books and abstract open books.

Remark 7.
(1) *Notice that S^n admits an open book decomposition with pages D^{n-1} and the monodromy the mapping class Id of the identity map of D^{n-1}. We call this open book the* trivial open book*. For more details regarding open books, refer the lecture notes* [Et] *and* [Gi]*[chpt-4.4.2].*
(2) *$S^3 \times S^2$ admits an open book decomposition with pages the unit disk bundle of T^*S^2 and monodromy the mapping class of the identity map of the unit disk bundle of T^*S^2. We call this open book decomposition of $S^3 \times S^2$ the standard open book decomposition of $S^3 \times S^2$.*

3. Open book embeddings of 3-manifolds in $S^3 \times S^2$ and $S^2 \widetilde{\times} S^3$

In this section, we produce open book embeddings of closed oriented 3-manifolds in any open books of $S^3 \times S^2$ and $S^2 \widetilde{\times} S^3$ with pages any disk bundle and monodromy the identity. We begin by reviewing quickly some well known results about embedded Hopf band in S^3. We can view S^3 as the unit sphere in $\mathbb{C}^2$. The Hopf links $H^{\pm}$ are the pre-images of 0 under the maps $(z_1, z_2) \to z_1 z_2$ and $(z_1, z_2) \to z_1 \bar{z_2}$, respectively restricted to the unit sphere S^3 of $\mathbb{C}^2$. A Hopf annulus is a Seifert surface for a Hopf link and a positive/negative Hopf band in S^3 is an embedded annulus with the boundary $H^{\pm}$. See, for example, [Et] for more details.

3.1. Hopf band in S^3 and the mapping class group of an annulus.

To begin with, we go through the proofs of the following well known results. These are also proved in [HY].

Lemma 8. *Let A be an annulus and let ϕ be an element of the mapping class group $\mathcal{MCG}(A)$ of A. Then, there exists an embedding f of A in S^3 that satisfies the following:*

(1) *$f(A)$ is a Hopf band in S^3.*
(2) *There exists a diffeomorphism of Ψ_1 of S^3, isotopic to the identity via an isotopy Ψ_t such that $f^{-1} \circ \Psi_1 \circ f = \phi$.*
(3) *The isotopy Ψ_t fixes the boundary of A pointwise for all t.*

Proof. We know that S^3 admits an open book decomposition with pages a Hopf band and the monodromy the Dehn twist around its center circle. This, in particular, implies that there exists a flow Φ_t on S^3 whose time 1 map Φ_1 maps a Hopf annulus – say $\mathcal{A}$ – to itself and Φ_1 restricted to $\mathcal{A}$ is a Dehn twist along the center circle on $\mathcal{A}$. We consider an embedding f of A in S^3 such that $f(A) = \mathcal{A}$. The lemma is now a straight forward consequence of the fact that every element of the mapping class group of an annulus is just a power of the Dehn twist along its center circle. □

Note that $S^3 \times [0,1]$ can be regarded as a collar of ∂D^4 in D^4 with $\partial D^4 = S^3 \times 1$. Since Ψ constructed in the Lemma 8 is isotopic to the identity, we have the following:

Corollary 9. *There exists a proper embedding f of an annulus A in $(D^4, \partial D^4)$ which satisfies the property that for every element $\phi \in \mathcal{MCG}(A)$, there exists a diffeomorphism Γ_1 of $(D^4, \partial D^4)$ isotopic to the identity such that $\phi = f^{-1} \circ \Gamma_1 \circ f$.*

Proof. First, we consider a proper embedding of A in $S^3 \times [0,1]$ as follows: We smoothly push a Hopf annulus, say $\mathcal{A}$ from $\partial D^4 = S^3 \times \{1\}$ to the level $S^3 \times \{0\}$ keeping the boundary of the Hopf annulus fixed such that $S^3 \times \{t\} \cap \mathcal{A}$ is a Hopf link for each $t \in (0,1]$. We consider the proper embedding f of A such that image of f is the pushed Hopf annulus $\mathcal{A}$.
Now, let Ψ_t be the isotopy of S^3 such that Ψ_1 realizes the given element of $\mathcal{MCG}(A)$. Using the isotopy Ψ_t, we construct a diffeomorphism Γ_1 of $S^3 \times [-1,1]$ that satisfies the following:

(1) Γ_1 is isotopic to the identity via a family of diffeomorphisms Γ_t.
(2) Γ_1 restricted to $S^3 \times \{0\}$ is Ψ_1.

This diffeomorphism is defined as follows:

$$\Gamma_1(x,t) = \begin{cases} \Psi_{1-t}(x) & \text{if } t \geq 0 \\ \Psi_{t+1}(x) & \text{if } t \leq 0 \end{cases}$$

Since $S^3 \times [-1,1]$ can be regarded as a collar of ∂D^4 in $(D^4, \partial D^4)$, we are through as Γ_1 clearly can be extended smoothly to a diffeomorphism of $(D^4, \partial D^4)$ by the identity in the complement of the collar. □

3.2. **Open book embeddings.**

In this section, we review the notion of open book embeddings. More concretely, we will make precise the notion of abstract open book embeddings. However, as earlier remarked, since open book decomposition and abstract open book decompositions are closely related, we will often not distinguish between abstract open book embeddings and open book embeddings.

Definition 10 (Open book embedding). *Let M^k be a manifold with open book decomposition $\mathcal{O}b(B_1, \pi_1)$ and N^l be another manifold with open book decomposition $\mathcal{O}b(B_2, \pi_2)$. We say an embedding $f : M \hookrightarrow N$ is an open book embedding of $(M, \mathcal{O}b(B_1, \pi_1))$ in $(N, \mathcal{O}b(B_1, \pi_2))$ provided f embeds B_1 in B_2 and the following diagram commutes:*

$$\begin{array}{ccc} M \setminus B_1 & \xrightarrow{f} & N \setminus B_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ S^1 & \xrightarrow{id} & S^1 \end{array}$$

Just as an abstract open book is defined, we can define an abstract open book embedding as follows:

Definition 11 (Abstract open book embeddings). *Let $M = \mathcal{A}ob\,(\Sigma_1, \phi_1)$ and $N = \mathcal{A}ob\,(\Sigma_2, \phi_2)$ be two abstract open books. We say that there exists an abstract open book embedding of M in N provided there exists a proper embedding f of Σ_1 in Σ_2 such that $\phi_2 = f^{-1} \circ \phi_1 \circ f$.*

It is clear from the definition that an abstract open book embedding produces an embedding for the associated open book and vice versa.

Remark 12. *The open book embeddings defined above are also known as spun embeddings in the litreture.*

There are some obvious examples of open book embeddings.

Examples 13. (1) *Each sphere S^n embeds in S^{n+k} with $k > 0$ via obvious inclusion such that the trivial open book of S^{n+k} restricts to the trivial open book of S^n.*
(2) *Notice that since we can embed $S^{n-1} \times I$ in D^{n+1} properly, $S^1 \times S^n$ admits an open book embedding in S^{n+2}.*

Now, we have an easy consequence using the Corollory 9.

Proposition 14. *Any closed oriented 3–manifold with an open book decomposition having pages an annulus A and the monodromy any mapping class ϕ of the annulus admits an open book embedding in the trivial open book of S^5.*

Proof. The corollory 9 implies that the abstract open book $\mathcal{A}ob\,(A, \phi)$ asociated to M abstract open book embeds in the abstract open book $\mathcal{A}ob\,\left(D^4, Id\right)$ associated to S^5. Hence, the result follows. □

3.3. The proof of the Theorem 1.

In this subsection, we establish the Theorem 1. Recall that we need to show that every closed oriented 3–dimensional manifold with a given open book decomposition open book embeds in any open book $S^3 \times S^2$ and $S^2 \widetilde{\times} S^3$ having pages a disk bundle over S^2 and its monodromy the identity.

We begin by introducing few terminologies. We refer to [GS] for more details regarding these. We know that when we add a 2–handle to a 4–ball B^4 along an unknot on the boundary with framing $m, m \in \mathbb{Z}$ we produce a disk bundle with Euler number m. Let us denote this disk bundle by $\mathcal{D}E(m)$.

Next, we establish a lemma. The techniques used in the proof of this lemma is adopted from techniques developed by Hirose and Yasuhara in [HY] to establish *flexible* embeddings of closed surfaces in certain 4–manifolds. Hirose and Yasuhara called an embedding f of a surface Σ in a 4–manifold M flexible provided for every element ϕ of the mapping class group of Σ, there exists a diffeomorphism Ψ of M, isotopic to the identity, which maps $f(\Sigma)$ to itself and $f^{-1} \circ \Psi|_{f(\Sigma)} \circ f = \phi$.

Lemma 15. *Let $(\Sigma, \partial\Sigma)$ be a surface with non-empty boundary. There exists an embedding f of Σ in a disk bundle $\mathcal{D}E(m)$, for any $n \in \mathbb{Z}$, which satisfies the following:*
(1) *The embedding is proper.*
(2) *Given any diffeomorphism ϕ of $(\Sigma, \partial\Sigma)$, there exists a family Ψ_t of diffeomorphisms of $\mathcal{D}E(m)$ with $\Psi_0 = id$ such that Ψ_1 maps Σ to itself and satisfies the property that $f^{-1} \circ \Psi_1 \circ f$ is isotopic to the given diffeomorphism ϕ of $(\Sigma, \partial\Sigma)$.*

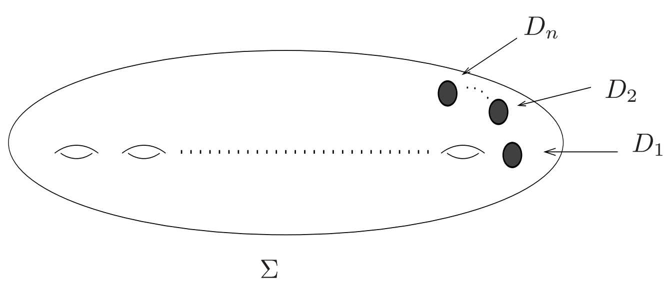

Figure 2. Embedding of Σ together with disks $D_1, \cdots, D_n$

Proof. We know that $\mathcal{D}E(m)$ is obtained by attaching a 2-handle to B^4 along an unknot with its framing m. This implies that we can regard it as a union of B^4 with $D^2 \times D^2$. We first describe an embedding of $(\Sigma, \partial\Sigma)$ in $S^3 = \partial B^4$ that we will need in order to establish the Lemma. Let us assume that $\partial\Sigma$ has $n \in \mathbb{N}$ boundary components. Let us denote by Σ the closed surface obtained from $(\Sigma, \partial\Sigma)$ after attaching disks to each boundary component of $\partial\Sigma$. First, embed Σ in S^3 such that it bounds the standard unknotted handle-body as shown in the Figure 1.

Now, observe that by removing the disks $D_1, D_2, \cdots, D_n$ as shown in Figure 2, we get an embedding of $(\Sigma, \partial\Sigma)$ in S^3 such that each boundary component of $(\Sigma, \partial\Sigma)$ is the boundary of D_i for some i.

Next, we attach a band with one full-twist around a properly embedded arc in the disk D_1 to the surface Σ as shown in Figure 3. This produces an embedded surface S with $(n+1)$ boundary components in S^3. Notice that out of these $n+1$ boundary components, $n-1$ boundary components correspond to boundaries of the disks $D_i, i = 2, \cdots, n$. The remaining two boundary components form a Hopf link as depicted in Figure 3. We denote these boundary components by H_1 and H_2. We use this embedding of the surface S with $n+1$ boundary components to properly embed the surface $(\Sigma, \partial\Sigma)$ in $\mathcal{D}E(m)$ in the following way:

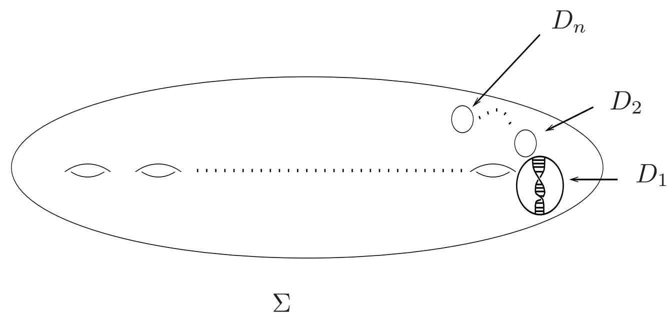

FIGURE 3. Embedding of the surface S with $n+1$ boundary components which contains a Hopf band H as a subsurface

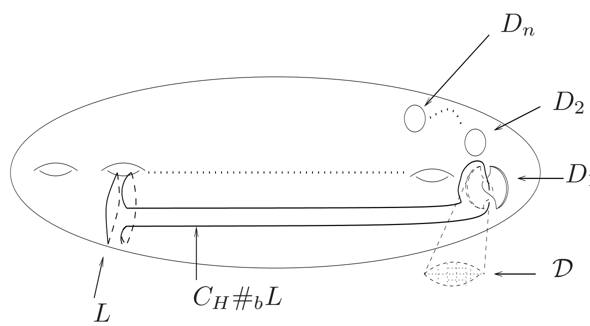

FIGURE 4. Embedding of the surface S with $n+1$ boundary components. The boundary component with dashed line bounds a properly embedded disk in $\mathcal{D}E(m)$

Observe that by construction, S admits an embedding of a Hopf band H with the boundary components H_1 and H_2 as shown in Figure 3. Now, consider S^3 being embedded as $S^3 \times \{\frac{1}{2}\}$ in $S^3 \times [0,1]$, where we regard $S^3 \times [0,1]$ as a collar of ∂B^4. We now observe that we can attach a 2-handle along one of the boundary components of the Hopf band in such a way that we obtain $\mathcal{D}E(m)$ from B^4. More precisely, consider one of the boundary components – say H_1 – of the Hopf band and consider the cylinder $H_1 \times [\frac{1}{2}, 1]$ and assume that $H_1 \times \{1\}$ is the unknot along which the 2–handle with framing n is attached. In Figure 4, the boundary component H_1 is denoted by a dashed circle. Thus, H_1 bounds a disk D in $\mathcal{D}E(m)$. We attach this disk to the surface S to get a new embedding – say $\widetilde{f}$ – of $(\Sigma, \partial\Sigma)$ in $\mathcal{D}E(m)$ with its n boundary components. Let us denote these boundary components by $\partial D_1, \partial D_2, \cdots, \partial D_n$ as shown in Figure 4.

Consider n cylinders $\partial D_i \times [\frac{1}{2}, 1]$ for $i = 1$ to n. Using these cylinders, we now modify the embedding $\widetilde{f}$ to get a proper embedding f of $(\Sigma, \partial\Sigma)$ in $\mathcal{D}E(m)$. This we do by considering the union $\widetilde{f}(\Sigma) \cup \partial D_1 \times [\frac{1}{2}, 1] \cup \cdots \cup \partial D_n \times [\frac{1}{2}, 1]$.

The embedding described above then clearly gives a proper embedding of $(\Sigma, \partial\Sigma)$ in $\mathcal{D}E(m)$. We perturb this embedding – if necessary – to make it into a smooth and proper embedding. By slight abuse of notation, let us again denote this embedding of $(\Sigma, \partial\Sigma)$ by f.

We now observe that the embedding f satisfies the property that any simple closed curve C and its ambient band connected sum with the center curve C_H (depicted by dark cure in the Figure 4) of the Hopf band H, are ambiently isotopic. This is because, C_H is isotopic the boundary component of H_1 which bounds the disk D. Hence, C_H can be shrunk to a point in the interior of $f(\Sigma)$. This implies that we can isotope C to $C\#C_H$ using the disk D.

Note that the regular neighborhood of the curve $C\#_b C_H$ is a Hopf annulus. We claim that there is an isotopy – say Φ_t – of $\mathcal{D}E(m)$ which is fixed near the boundary of $\mathcal{D}E(m)$ and which induces a Dehn twist along $C\#_b C_H$. In fact, the isotopy can be assumed to be the identity when restricted to the 2–handle as well. This can be done as follows:

To begin with, recall that the whole surface Σ except the 2-disk D coming for the attached 2–handle is still embedded in B^4. In fact, we would like to point out that everything except the cylinders $\partial D_i \times [\frac{1}{2}, 1]$ are still embedded in the level $S^3 \times \{\frac{1}{2}\}$ of the collar $S^3 \times [0, 1]$ of ∂B^4. In particular, a fixed neighborhood $\mathcal{N}(C\#_b C_H)$ is contained in $S^3 \times \{\frac{1}{2}\}$.

In order to get the isotopy Ψ_t as claimed we first describe how to produce an isotopy Φ_t of $\mathcal{D}E(m)$ which induces the Dehn twist along $C\#_b C_H$ on Σ.

This is done as follows: Push the neighborhood $\mathcal{N}(C\#_b C_H)$ slightly towards $S^3 \times \{0\}$ in the collar in such a way that at a fixed level between 0 and $\frac{1}{2}$ the intersection of this pushed neighborhood is a Hopf annulus and this Hopf annulus contains the pushed curve $C\#_b C_H$ as its center curve. Let us denote this level by $S^3 \times \{s_0\}$. We now perform an isotopy to induce a Dehn twist along the pushed $C\#_b C_H$ in such a way that this isotopy is supported in a small neighborhood of $S^3 \times \{s_0\}$ not intersecting $S^3 \times \{\frac{1}{2}\}$. After performing this isotopy, we further isotope the pushed neighborhood $\mathcal{N}(C\#C_H)$ back to its original place in $S^3 \times \{\frac{1}{2}\}$. Clearly, the effect of successive compositions of these isotopies is an isotopy Φ_t which induces the Dehn twist along $C\#C_H$ on Σ.

We are almost done. We now recall that the mapping class group of $(\Sigma, \partial\Sigma)$ is generated by Dehn twists along Lickorish curves as described in the Figure 1 for an orientable surface with one boundary component and as described in [FM, p. 133] for an orientable surface with more than one boundary components. Since on each Lickorish curve it is possible to perform a Dehn twist via an ambient isotopy of $\mathcal{D}E(m)$, we get the isotopy Ψ_t with the required properties.

□

Remark 16. *Notice that the Lemma 15 above shows that in $\mathcal{D}E(m)$, there exists a proper flexible embedding of a the surface Σ.*

Proof of Theorem 1. Consider the abstract open book $\mathcal{A}ob\,(\mathcal{D}E(m), Id)$. Recall from [Ko] that if m is even, then $\mathcal{A}ob\,(\mathcal{D}E(m), Id)$ represents the manifold $S^3 \times S^2$ and if m is odd then it represents $S^2 \widetilde{\times} S^3$.

Observe that the Lemma 15 implies that there is an abstract open book embedding of $\mathcal{A}ob(\Sigma, \phi)$ in $\mathcal{A}ob(\mathcal{D}E(m), Id)$, for any n, where $\mathcal{A}ob(\Sigma, \phi)$ is an abstract open book associated to the open book $\mathcal{O}b(B, \pi)$ of M. Hence, $(M, \mathcal{O}b(B, \pi))$ admits an open book embedding in any open book decomposition associated to $S^3 \times S^2$ and $S^2 \widetilde{\times} S^3$ with page a disk bundle over S^2 and monodromy the identity.

□

4. Embeddings of 3–manifolds in S^5

In this section, we use techniques developed to establish the Lemma 15 to reprove the theorem that every closed orientable 3–manifold embeds in S^5.

Proof of Theorem 2. In order to produce an embedding of a closed orientable 3–manifold M in S^5, we first notice that it is sufficient to embed M in $S^3 \times \mathbb{R}^2$. Hence, in what follows, we show how to embed M in $S^3 \times \mathbb{R}^2$.

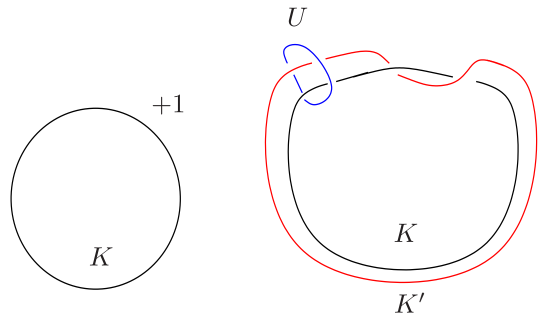

FIGURE 5. In the left side of the above figure, we depict the Kirby diagram of $\mathcal{D}E(1)$. The unknot K with framing $+1$ is the attaching circle for the 2-handle of $\mathcal{D}E(1)$. While in the right side, we depict the unknot K together with the unknot K' which is the boundary of a slightly pushed copy of the core of the attaching handle. The blue knot is the unknot U linking both K and K' once. The knot K' is assumed to be on the boundary of the attaching region which is a solid torus around K.

To begin with, we fix and review some notations. We parametrize a collar of ∂B^4 by $S^3 \times [0,1]$ such that $\partial B^4 = S^3 \times \{1\}$. The unknot K which is the attaching circle of the 2-handle is then contained in $S^3 \times \{1\}$. This is depicted in the left of the Figure 5 by black circle with framing $+1$. Let us denote the zero section of the bundle $\mathcal{D}E(1)$ by $\mathcal{S}$. We can regard $\mathcal{S}$ as the sphere obtained by considering the union of attaching disk of the 2-handle with $K \times [0,1]$ and the obvious disk $K \times \{0\}$ bounds in $S^3 \times \{0\}$. Next, we denote by $\mathcal{N}(K)$ a tubular neighborhood of K which is the attaching region of the 2–handle $H_2 = D^2 \times D^2$. If p is a point on the boundary of D^2, then the disk $D^2 \times \{p\}$ embedded in the 2–handle H_2 intersects the boundary $S^3 \times \{1\}$ in a curve K' which links K once. This is depicted by the red curve in the Figure 5. Notice that K' lies on the boundary of the attaching region $\mathcal{N}(K)$.

We now describe how to embed a surface with one boundary component which is disjoint from the zero section $\mathcal{S}$ of $\mathcal{D}E(1)$ and is flexible in $\mathcal{D}E(1)$. Consider an unknot U which links the attaching region $\mathcal{N}(K)$ as depicted in the right of Figure 5. Consider the two circles $U \times \{\frac{1}{2}\}$ and $K' \times \{\frac{1}{2}\}$ in the sphere $S^3 \times \{\frac{1}{2}\}$. Notice that the complement of $\mathcal{N}(K) \times \{\frac{1}{2}\}$ in $S^3 \times \{\frac{1}{2}\}$ is a solid torus $S^1 \times D^2$. The circle $U \times \{\frac{1}{2}\}$ is the center circle $S^1 \times \{0\}$ of this solid torus while $K' \times \{\frac{1}{2}\}$ is a curve going once around the longitude and once around the meridian of the solid torus. This implies circles $U \times \{\frac{1}{2}\}$ and $K' \times \{\frac{1}{2}\}$ bound a Hopf annulus in $S^3 \times \{\frac{1}{2}\}$ which is disjoint for $K \times \{\frac{1}{2}\}$ as it lies inside the solid torus $S^1 \times D^2$. Let us call this Hopf annulus $\mathcal{A}$. Now, observe that the boundary component of the annulus $\mathcal{A}$ corresponding to $K' \times \{\frac{1}{2}\}$ bounds a disk – say $\mathcal{D}$ – in $\mathcal{D}E(1)$ by construction. By attaching the annulus $U \times [\frac{1}{2}, 1]$ to $\mathcal{D}$ along its boundary $U \times \{\frac{1}{2}\}$, we produce a properly embedded disk in $\mathcal{D}E(1)$.

Next, let $\widetilde{\Sigma}$ be a standardly embedded handle-body contained in the solid torus $S^1 \times D^2$ which is disjoint from the Hopf annulus $\mathcal{A}$. Let Σ be the boundary of this handle-body. We can perform an ambient connected sum of Σ with $\mathcal{A}$ in $S^3 \times \{\frac{1}{2}\}$ such that the surface obtained after the ambient connected sum is still contained in the complement of $K \times \{\frac{1}{2}\}$ in $S^3 \times \{\frac{1}{2}\}$. Notice that since $\mathcal{A}$ is an annulus in a properly embedded disk described in the previous paragraph, this connected sum operation produces a properly embedded surface with one boundary component. By a slight abuse of notation, let us continue to denote this surface by Σ.

Observe that an argument similar to the one used in the proof of the Lemma 15 implies that Σ is a properly embedded flexible surface in $\mathcal{D}E(1)$. This is because the embedded surface Σ admits an embedding of a Hopf annlus such that one of the boundary component of this Hopf annlus bounds a disk in the surface by the construction. Hence, we can isotope every generator of the mapping class group of Σ in such a way that it admits a neighborhood which is a Hopf annulus embedded in $S^3 \times \{\frac{1}{2}\}$. Furthermore, notice that Σ does not intersect the zero section $\mathcal{S}$ of the bundle $\mathcal{D}E(1)$ as it does not intersect the core disk of the 2–handle as well as the annulus $K \times [\frac{1}{2}, 1]$.

Now, let M be any closed orientable 3–manifold. Observe that it was established in [My] that we can regard M as $\mathcal{A}ob(\Sigma, \phi)$ for some orientable surface Σ with one boundary component. Since Σ admits a flexible embedding in $\mathcal{D}E(1)$, there exists an open book embedding of M in $S^2 \widetilde{\times} S^3 = \mathcal{A}ob(\mathcal{D}E(1), Id)$.

We now notice that since Σ does not intersect the zero section $\mathcal{S}$, the mapping torus $\mathcal{MT}(\Sigma, \phi)$ associated to the abstract open book $M = \mathcal{A}ob(\Sigma, \phi)$ is in fact, properly embedded in a manifold diffeomorphic to $S^1 \times S^3 \times (0, 1]$. This follows from the fact that the complement of the zero section in $\mathcal{D}E(1)$ is $S^3 \times (0, 1]$, see, for example, [GS, p. 119].

Next, consider the disjoint union of $S^1 \times S^3 \times (0, 1]$ and $S^3 \times D^2$. Consider the quotient manifold obtained by identifying the boundary $S^1 \times S^3 \times \{1\}$ of $S^1 \times S^3 \times (0, 1]$ with the boundary $S^3 \times S^1$ of $S^3 \times D^2$ by the identity. Notice that the resulting quotien manifold is diffeomorphic to $S^3 \times D^2$.

Notice that since the mapping torus $\mathcal{MT}(\Sigma, \phi)$ is properly embedded in $S^1 \times S^3 \times (0, 1]$, we clearly get that the embedding of M in $S^2 \widetilde{\times} S^3$ obtained via the open book embedding of M in $S^2 \widetilde{\times} S^3 = \mathcal{A}ob(\mathcal{D}E(1), Id)$ is contained in a manifold diffeomorphic to $S^3 \times \mathbb{R}^2$ as required. This completes our argument. $\square$

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