

HOMOTOPICAL HEIGHT

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ABSTRACT. Given a group G and a class of manifolds \mathcal{C} (e.g. symplectic, contact, Kähler etc), it is an old problem to find a manifold $M_G \in \mathcal{C}$ whose fundamental group is G . This article refines it: for a group G and a positive integer r find $M_G \in \mathcal{C}$ such that $\pi_1(M_G) = G$ and $\pi_i(M_G) = 0$ for $1 < i < r$. We thus provide a unified point of view systematizing known and new results in this direction for various different classes of manifolds. The largest r for which such an $M_G \in \mathcal{C}$ can be found is called the homotopical height $ht_{\mathcal{C}}(G)$. Homotopical height provides a dimensional obstruction to finding a $K(G, 1)$ space within the given class \mathcal{C} , leading to a hierarchy of these classes in terms of “softness” or “hardness” à la Gromov. We show that the classes of closed contact, CR, and almost complex manifolds as well as the class of (open) Stein manifolds are soft.

The classes \mathcal{SP} and \mathcal{CA} of closed symplectic and complex manifolds exhibit intermediate “softness” in the sense that every finitely presented group G can be realized as the fundamental group of a manifold in \mathcal{SP} and a manifold in \mathcal{CA} . For these classes, $ht_{\mathcal{C}}(G)$ provides a numerical invariant for finitely presented groups. We give explicit computations of these invariants for some standard finitely presented groups.

We use the notion of homotopical height within the “hard” category of Kähler groups to obtain partial answers to questions of Toledo regarding second cohomology and second group cohomology of Kähler groups. We also modify and generalize a construction due to Dimca, Papadima and Suciu to give a potentially large class of projective groups violating property FP.

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1. INTRODUCTION

Given a group G and one's favorite class of manifolds \mathcal{C} (e.g. symplectic, contact, Kähler etc), it is an old problem to find a manifold $M_G \in \mathcal{C}$ whose fundamental group is G . This article refines it: for a group G and a positive integer r find $M_G \in \mathcal{C}$ such that $\pi_1(M_G) = G$ and $\pi_i(M_G) = 0$ for $1 < i < r$. This paper provides a unified point of view systematizing known and new results in this direction for various different classes of manifolds. The largest r for which such an $M_G \in \mathcal{C}$ can be found shall be called the *homotopical height* $ht_{\mathcal{C}}(G)$ which provides a numerical/dimensional answer to the following basic question:

Question 1.1. Given a finitely presented group G and a class \mathcal{C} of smooth manifolds (e.g. symplectic, contact, Kähler etc), what is the obstruction to constructing a $K(G, 1)$ manifold within the class \mathcal{C} ?

We now formally introduce the notion of *homotopical height*.

Definition 1.2. Let G be a finitely presented group, and let \mathcal{C} be a given class of smooth manifolds of dimension greater than zero. We shall say that the \mathcal{C} -homotopical height of G is $-\infty$ if G is not the fundamental group of some manifold in \mathcal{C} . If $\pi_2(M) \neq 0$ for all $M \in \mathcal{C}$ with $\pi_1(M) = G$, then we shall say that the \mathcal{C} -homotopical height of G is 2. We shall say that the \mathcal{C} -homotopical height of G is greater than or equal to n if there exists a manifold $M \in \mathcal{C}$ such that $\pi_1(M) = G$

and $\pi_i(M) = 0$ for every $1 < i < n$. The \mathcal{C} -homotopical height of G is the maximum value of n such that the \mathcal{C} -homotopical height of G is greater than or equal to n .

We shall denote the \mathcal{C} -homotopical height of G as $ht_{\mathcal{C}}(G)$.

Note that if there exists a $K(G, 1)$ space in \mathcal{C} , then the \mathcal{C} -homotopical height of G is infinite. Thus the \mathcal{C} -homotopical height provides a dimensional obstruction to the construction of a $K(G, 1)$ manifold within the class \mathcal{C} .

In Definition 1.2, the dimension restriction (> 0) on elements of \mathcal{C} is for technical reasons to do with the exceptional nature of the trivial group; this is explained in Section 2.4.

Definition 1.3. A manifold $M \in \mathcal{C}$ is said to **realize** the \mathcal{C} -homotopical height of G if $\pi_1(M) = G$ and $\pi_i(M) = 0$ for all $1 < i < ht_{\mathcal{C}}(G)$. If a manifold $M \in \mathcal{C}$ realizing $ht_{\mathcal{C}}(G)$ has the property that there is no manifold in \mathcal{C} of strictly smaller dimension realizing $ht_{\mathcal{C}}(G)$, then the dimension of M is called the *\mathcal{C} -homotopical dimension*. The \mathcal{C} -homotopical dimension of G will be denoted by $hdim_{\mathcal{C}}(G)$.

Any $K(G, 1)$ manifold in \mathcal{C} realizes the \mathcal{C} -homotopical height (which is ∞). The notions of homotopical height and homotopical dimension provide, as we hope to show in this paper, an organizing principle and language to address a number of existing problems.

Let G be a finitely presented group.

- (1) G is of type FP_{∞} if it admits a $K(G, 1)$ space (in the category of CW complexes) with finitely many cells in each dimension.
- (2) G is of type FP_k if it admits a $K(G, 1)$ space (in the category of CW complexes) with finitely many cells in each dimension less than $(k + 1)$.
- (3) G is of type FP if it admits a finite $K(G, 1)$ space, i.e., a $K(G, 1)$ space with finitely many cells.

(See [Bro82].)

Our investigations lead us to the following gradation of categories \mathcal{C} in degrees of hardness or softness (à la Gromov [Gro86a, Gro10, Gro86b]) viewed through the lens of homotopical height: We shall say that a class \mathcal{C} is of

Type 1: if $ht_{\mathcal{C}}(G) \geq k$ for all finitely presented groups of type FP_k , in particular, $ht_{\mathcal{C}}(G) = \infty$ for all finitely presented groups of type FP_{∞} ;

Type 2: if $ht_{\mathcal{C}}(\{1\}) = \infty$ for the trivial group;

Type 3: if $ht_{\mathcal{C}}(G) \geq 0$ for all finitely presented groups; and

Type 4: if $ht_{\mathcal{C}}(G) = -\infty$ for some finitely presented group of type FP_{∞} .

Clearly classes \mathcal{C} of Type 1 are the softest and those of Type 4 are hardest in the above list. However, as we shall see below, Type 2 and Type 3 are classes that exhibit intermediate behavior of different (and not quite comparable) kinds. Note that a class \mathcal{C} is of Type 3 if and only if every finitely presented group G appears as the fundamental group of some manifold in \mathcal{C} .

1.1. Motivational Problems. There were a number of problems and theorems that led us to the formulation of Definition 1.2 and the consequent “hard/soft” framework. In the next subsection, we shall summarize the partial answers/approaches we obtain in this paper to some of the questions below.

Theorem 1.4 ([Gom95]). *Every finitely presented group is the fundamental group of a closed symplectic manifold.*

In the language of this paper, Theorem 1.4 says that the class \mathcal{SP} of closed symplectic manifolds is of Type 3 (we return to this later).

Theorem 1.5 ([Tau92]). *Every finitely presented group is the fundamental group of a closed complex manifold.*

Theorem 1.5 says that the class \mathcal{CA} of closed complex manifolds is of Type 3. The following question asks if the category \mathcal{CA} of complex manifolds is “soft”:

Question 1.6 ([CCL00, p. 238], [Gro10, Yau93]). *Does every almost complex closed manifold of real dimension greater than four admit a complex structure?*

We adapt Question 1.6 to the context of this paper and ask:

Question 1.7. *For a finitely presented group G of type FP , is $ht_{\mathcal{CA}}(G) = \infty$?*

We note in passing that a group that provides a negative answer to Question 1.7 will also furnish a counterexample to Question 1.6.

The fundamental groups of compact Kähler manifolds (respectively, complex projective manifolds) will be called Kähler groups (respectively, projective groups).

The following problem, due to Kollár, asks for a kind of softness for projective groups. The class \mathcal{P} of complex projective manifolds and the class \mathcal{K} of closed Kähler manifolds belong to Type 4 in the above classification. We think of these classes as “hard”.

Question 1.8 ([Kol95, Section 0.3.1]). *Is a projective group G_1 commensurable to a group G , admitting a $K(G, 1)$ space which is a smooth quasi-projective variety?*

Dimca, Papadima and Suciu [DPS09] have furnished examples of finitely presented groups giving a negative answer to Question 1.8. However, these examples are not of type FP_k for some k and hence do not satisfy $ht_{\mathcal{SM}}(G) = \infty$ even for the class \mathcal{SM} of closed smooth manifolds. We therefore modify Question 1.8 as follows:

Question 1.9.

- (1) Let G_1 be a projective group of type FP_∞ . Is G_1 commensurable to a group G , admitting a $K(G, 1)$ space which is a smooth quasi-projective variety?
- (2) In particular, if G is a torsion-free projective group of type FP , does it admit a $K(G, 1)$ space which is a quasi-projective variety?
- (3) If G_1 is a projective group of type FP_k , is G_1 commensurable to a group G with $ht_{\mathcal{QP}}(G) \geq k$, where \mathcal{QP} denotes the category of smooth quasiprojective varieties.

The examples in [DPS09] do not provide counterexamples to Question 1.9(3). In the form Question 1.9(3), Kollár’s question applies to every finitely presented projective group (take $k = 2$).

The next set of questions specifically pertain to the hard (i.e., Type 4) class \mathcal{K} of Kähler manifolds. Our framework of “homotopical height” gives an approach and partial positive answers to these. An outstanding conjecture on Kähler groups, due to Carlson and Toledo, states the following:

Conjecture 1.10 ([Kol95]). *Let G be an infinite Kähler group. Then there exists a finite index subgroup G_1 of G such that the second Betti number $b_2(G_1)$ is positive.*

We were led to the following question thanks to Domingo Toledo.

Question 1.11. Let G be a Kähler group such that $H^2(G, \mathbb{Z}G) \neq 0$. Is G the fundamental group of a compact Riemann surface?

1.2. Outline of the Paper and Results. Let \mathcal{C} be a subclass of the class of (not necessarily closed) smooth manifolds of *positive dimension*. The classes we shall be most interested here are as follows, arranged roughly in decreasing order of “softness”.

- (1) The class \mathcal{SM} of closed smooth manifolds.
- (2) The class \mathcal{ACR} of closed almost CR manifolds.
- (3) The class \mathcal{AC} of closed almost complex manifolds.
- (4) The class \mathcal{CT} of closed contact manifolds.
- (5) The class \mathcal{CR} of closed CR manifolds.
- (6) The class \mathcal{St} of (open) Stein manifolds.
- (7) The class \mathcal{SP} of closed symplectic manifolds.
- (8) The class \mathcal{CA} of closed complex manifolds.
- (9) The class \mathcal{AF} of smooth affine varieties.
- (10) The class \mathcal{QP} of smooth quasi-projective varieties.
- (11) The class \mathcal{K} of compact Kähler manifolds.
- (12) The class \mathcal{P} of smooth complex projective manifolds.
- (13) The subclass \mathcal{HC} of \mathcal{P} consisting of smooth complex projective manifolds with holomorphically convex universal cover.
- (14) The subclass \mathcal{S} of \mathcal{P} consisting of smooth complex projective manifolds with Stein universal cover.

Other natural classes to which Definition 1.2 can and do apply will be defined as we go along. This paper has roughly three thematic parts:

- 1) Section 3 deals with the “soft” categories: $\mathcal{CT}, \mathcal{ACR}, \mathcal{AC}, \mathcal{CR}, \mathcal{St}$.
- 2) Section 4 deals primarily with the “intermediate” categories: $\mathcal{SP}, \mathcal{CA}$.
- 3) The later sections deal primarily with the “hard” categories \mathcal{K} and \mathcal{P} .

The *techniques* used in these three types of categories are different but not unrelated. We give below a representative selection of results from each of these three categories.

Soft Categories:

The main theorem here on the “soft category” proves softness (Type 1) for the classes $\mathcal{CT}, \mathcal{ACR}, \mathcal{AC}, \mathcal{CR}, \mathcal{St}$ (see Theorems 2.1, 3.3, 3.5, 3.6).

Theorem 1.12. *The classes $\mathcal{SM}, \mathcal{CT}, \mathcal{AC}, \mathcal{CR}, \mathcal{St}$ are each of Type 1. In particular, $ht_{\mathcal{C}}(G) = \infty$ for all groups G of type FP_{∞} and $\mathcal{C} = \mathcal{SM}, \mathcal{CT}, \mathcal{AC}, \mathcal{CR}$ or \mathcal{St} .*

Softness of the class of closed almost complex manifolds \mathcal{AC} provides a possible approach towards a *negative* answer to Question 1.6. A group providing a negative answer to Question 1.7 or to the more general Question 4.1 below will immediately furnish an almost complex manifold which does not admit a complex structure.

Intermediate Categories:

The classes \mathcal{SP} and \mathcal{CA} exhibit intermediate “softness” in the sense that every finitely presented group G can be realized as the fundamental group of a manifold in \mathcal{SP} and also a manifold in \mathcal{CA} (by Theorem 1.4 and Theorem 1.5 respectively). For these classes, $ht_{\mathcal{C}}(G)$ provides a numerical invariant for finitely presented groups

G . In Section 4, we give explicit computations of these invariants for some standard finitely presented groups.

The proof of the following theorem is in Propositions 4.4, 4.5, 4.6, 4.9.

Theorem 1.13. *We have $ht_{\mathcal{CA}}(G) = \infty$ for*

- (a) *any finitely generated abelian group G ,*
- (b) *for any finite group G , and*
- (c) *the (integral) Heisenberg group H^{2m+1} .*

Let G be a finitely presented group such that $b_2(G) = 0$. Then $ht_{\mathcal{SP}}(G) = 2$. In particular, all finite groups and finitely generated free groups have $ht_{\mathcal{SP}}(G) = 2$.

We also provide some weak positive evidence towards Conjecture 1.10 (see Proposition 4.12):

Proposition 1.14. *Let G be an infinite finitely presented group (e.g. an infinite Kähler group) with $ht_{\mathcal{SP}}(G) > 2$. Then $b_2(G)$ is positive.*

It follows that Kähler symplectically aspherical groups of class \mathcal{A} in the terminology of [IKRT04] verify Conjecture 1.10.

Hard Categories:

We turn to the harder categories $\mathcal{K}, \mathcal{P}, \mathcal{HC}, \mathcal{S}$ in Sections 5, 6 and 7 where we discuss restrictions on homotopical height resulting from:

- a) Lefschetz Hyperplane Theorem (Section 5.1),
- b) Formality (Section 5.2), and
- c) Group Cohomology (Section 6).

In Proposition 5.3, we obtain the following exact computation of homotopical dimension in the class \mathcal{S} :

Proposition 1.15. *If $ht_{\mathcal{S}}(G) = n < \infty$, then there exists a smooth complex projective manifold M of complex dimension exactly n realizing it. Further, no smooth projective manifold M of complex dimension less than n realizes $ht_{\mathcal{S}}(G)$. Hence $hdim_{\mathcal{S}}(G) = 2n$.*

Furthermore, the universal cover \widetilde{M} of the above complex projective manifold M is homotopy equivalent to a wedge of n -spheres.

We also obtain partial positive answers to Question 1.11 (the details are in Propositions 6.6, 6.8, 6.7).

Theorem 1.16. *If $ht_{\mathcal{S}}(G) > 2$, then either $H^2(G, \mathbb{Z}G) = 0$ or G is the fundamental group of a compact Riemann surface.*

Suppose $ht_{\mathcal{S}}(G) = 2$, and M is a smooth projective surface realizing the homotopical height of G . If in addition the action of $\pi_1(M) = G$ on $\pi_2(M)$ is trivial, then either $H^2(G, \mathbb{Z}G) = 0$ or G is virtually the fundamental group of a compact Riemann surface.

If $ht_{\mathcal{S}}(G) > n$, then either $H^n(G, \mathbb{Z}G) = 0$ or G is a $\text{PD}(n)$ group, in which case $H^n(G, \mathbb{Z}G) = \mathbb{Z}$.

In Section 6.3, we note a simple but curious connection between Kollár's Question 1.8 and Carlson-Toledo's Conjecture 1.10.

Section 7 is devoted to examining, generalizing and topologically characterizing certain counterexamples to Kollár's Question 1.8 discovered by Dimca, Papadima

and Suciu. In particular we prove the following Theorem (see Theorem 7.9) which provides (via Corollary 7.10) a generalization of the counterexamples constructed in [DPS09]. Our approach replaces the constructions using characteristic varieties in [DPS09] by some group cohomology computations.

Theorem 1.17. *Let $f : M \rightarrow S$ be an irrational topological Lefschetz fibration that is not a Kodaira fibration, with $\dim M = 2n + 2$, $n \geq 2$. Let K be the finite critical set of f . Further suppose that \widetilde{M} is contractible. Let F denote the regular fiber and $N = \pi_1(F)$. Then*

- a) $\pi_k(F) = 0$ for $1 < k < n$,
- b) $\pi_n(F)$ is a free $\mathbb{Z}N$ -module, with generators in one-to-one correspondence with $K \times \pi_1(S)$,
- c) \widetilde{F} is homotopy equivalent to a wedge of n -spheres,
- d) N cannot be of type FP; in particular, there does not exist a quasiprojective $K(N, 1)$ space.

The examples in [DPS09] come from complex analytic maps $f : M \rightarrow S$ between a product $M = S_1 \times \cdots \times S_r$ of Riemann surfaces and a Riemann surface S of genus one. The next Theorem (see Theorem 7.15) gives a purely topological characterization of homotopy classes of maps $f : M \rightarrow S$ that come from irrational Lefschetz fibrations (that are not Kodaira fibrations).

Theorem 1.18. *Let M be a product $S_1 \times \cdots \times S_r$ of closed, orientable surfaces S_i of positive genus with $r \geq 2$. Fix $(x_1, \dots, x_r) \in S_1 \times \cdots \times S_r$ and let $j_k : S_k \rightarrow S_1 \times \cdots \times S_r$ be given by $j_k(y_k) = (x_1, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_r)$.*

If there exists a product complex structure on M (coming from complex structures on S_k) and an irrational Lefschetz fibration $f : M \rightarrow S$ which is not a Kodaira fibration, then

- a) S has genus one, and
- b) for all k , we have $\deg((f \circ j_k)) > [\pi_1(N) : f_*(\pi_1(M))]$.

Conversely, suppose that S is a closed orientable surface of positive genus and there exists a continuous map $f : M \rightarrow S$ such that for all k ,

$$\deg((f \circ j_k)) > [\pi_1(N) : f_*(\pi_1(M))].$$

Then S has genus one and there exist complex structures on S_k, S such that $f : M \rightarrow S$ is homotopic to a Lefschetz fibration (for M equipped with the product complex structure). Further, there exist complex analytic maps $\phi_k : S_k \rightarrow S$ such that $\phi(z_1, \dots, z_k) = m(\phi_1(z_1), \dots, \phi_r(z_r))$, where $m(., .)$ denotes addition on S .

Finally, in Section 8, we end with a zoo of examples, counterexamples and questions.

2. PRELIMINARIES

2.1. Preliminaries on CW complexes and smooth manifolds. A well-known theorem going back to Dehn [Sti93], Markov [Mar58] and Boone-Haken-Poenaru [BHP68] states that every finitely presented group appears as the fundamental group of a smooth closed 4-manifold. The main ideas of their proof can be generalized as follows.

Theorem 2.1. *The inequality $ht_{\mathcal{SM}}(G) \geq k$ holds for all finitely presented groups G of type FP_k . In particular, $ht_{\mathcal{SM}}(G) = \infty$ for all finitely presented groups G of type FP_∞ .*

Proof. Let X be a $K(G, 1)$ space with finitely many cells in each dimension less than $(k + 1)$, and let X_k denote its k -skeleton. Embed X_k simplicially in a simplex Δ^N of large enough dimension N . Embed Δ^N simplicially in \mathbb{R}^N where the latter is equipped with a standard simplicial structure. Then X_k is a simplicial subcomplex of \mathbb{R}^N and hence by [Dol95, p. 95, Ex. 3], the subspace X_k is a strong deformation retract of a neighborhood U_k in \mathbb{R}^N .

The argument in [BHP68] (see also [Sti93, Ch 9.4]) ensures that the boundary of U_k in \mathbb{R}^N may be taken to be a closed $(N - 1)$ manifold and hence the closure $W_k = \overline{U_k}$ is a manifold with boundary. Let $\partial W_k = V_k$. Since W_k has the same homotopy type as X_k , it follows that $\pi_i(W_k) = 0$ for $1 < i < k$.

We now embed \mathbb{R}^N in \mathbb{R}^{N+k+1} and consider a regular neighborhood Z_k of W_k in \mathbb{R}^{N+k+1} . Then

$$Y_k = \partial Z_k = (W_k \times S^k) \cup (D^{k+1} \times V_k),$$

where S^k and D^{k+1} denote the k -sphere and the $(k + 1)$ -disk respectively. Observe that $\pi_i(W_k \times S^k) = 0$ for $1 < i < k$, as $\pi_i(W_k) = 0$ for $1 < i < k$. Finally, consider the relative homotopy exact sequence of the pair $(Y_k, W_k \times S^k)$:

$$\longrightarrow \pi_i(W_k \times S^k) \longrightarrow \pi_i(Y_k) \longrightarrow \pi_i(Y_k, W_k \times S^k) \longrightarrow \pi_{i-1}(W_k \times S^k).$$

From it we have $\pi_i(Y_k) = \pi_i(Y_k, W_k \times S^k)$ for $2 < i < k$. Further,

$$\pi_1(W_k \times S^k) = \pi_1(X_k) = \pi_1(Y_k)$$

as in [BHP68] (see also [Sti93, Ch 9.4]), where the isomorphisms are natural and given via inclusions into Z_k . Hence $\pi_i(Y_k) = \pi_i(Y_k, W_k \times S^k)$ for $2 \leq i < k$.

Now, $\pi_i(Y_k, W_k \times S^k) = \pi_i((D^{k+1} \times V_k)/\sim)$, where \sim identifies the boundary $S^k \times V_k = \partial(D^{k+1} \times V_k)$ to a point. Hence $\pi_i(Y_k, W_k \times S^k) = 0$ for $i < k$ and so $\pi_i(Y_k) = 0$ for $2 \leq i < k$. Since $G = \pi_1(X_k) = \pi_1(Y_k)$, the result follows. \square

Let STT denote the class of smooth closed manifolds with stably trivial tangent bundle. The proof of Theorem 2.1 gives us more. Observe that Y_k is a codimension one closed submanifold of \mathbb{R}^{N+k+1} . So the normal bundle to Y_k in \mathbb{R}^{N+k+1} is a trivial line bundle. Hence the tangent bundle to Y_k is stably trivial. This gives us the following corollary of Theorem 2.1 which will be useful later:

Corollary 2.2. *The inequality $ht_{STT}(G) \geq k$ holds for all finitely presented groups G of type FP_k . In particular, $ht_{STT}(G) = \infty$ for all finitely presented groups G of type FP_∞ .*

2.2. Preliminaries on Duality and Poincaré Duality groups. We refer the reader to [BE73, Bro82] for details about duality and Poincaré Duality (PD) groups.

Lemma 2.3. *Let G be a one-ended group. Let M be a closed orientable 4-manifold such that $\pi_1(M) = G$ and $\pi_2(M) = 0$. Then M is a $K(G, 1)$ space and G is a $PD(4)$ group.*

Proof. Let \widetilde{M} be the universal cover of M . Since $\pi_2(M) = \pi_2(\widetilde{M})$, it follows that $\pi_2(\widetilde{M}) = 0$. As $\dim H^1(G, \mathbb{Z}G) + 1$ coincides with the number of ends of M , and

G is one-ended, it follows that $H^1(M, \mathbb{Z}G) = 0$. Hence $H_c^1(\widetilde{M}) = 0$. By Poincaré Duality, $H_3(\widetilde{M}) = 0$. Therefore, $\pi_3(\widetilde{M}) = 0$ by Hurewicz' Theorem.

Also, since G is one-ended, it is infinite. Hence \widetilde{M} is non-compact, and $H_4(\widetilde{M}) = 0$. Again, by Hurewicz' Theorem, $\pi_4(\widetilde{M}) = 0$. Further, since $H_i(\widetilde{M}) = 0$ for all $i > 4$, we conclude that $\pi_i(\widetilde{M}) = 0$ for all $i \geq 1$. Consequently, \widetilde{M} is contractible and the result follows. \square

We shall also need the following theorem due to Kleiner (see also [Bow98]).

Theorem 2.4 ([Kle]). *Suppose G is a group which is FP_2 over \mathbb{Z} . If $H^2(G, \mathbb{Z}G)$ has a 1-dimensional G -invariant submodule, then G is virtually a surface group. Hence H is a $\text{PD}(2)$ group over \mathbb{Q} if and only if H is virtually a surface group.*

2.3. Preliminaries on Kähler Groups. We begin with some known facts and simple observations.

We recall a well-known theorem of Gromov.

Theorem 2.5 ([Gro89]). *Any infinite Kähler group is one-ended.*

A more general result is proved in [DG05].

Lemma 2.3 and Theorem 2.5 together yield the following:

Corollary 2.6. *Let M be a smooth complex projective surface with $\pi_2(M) = 0$. Then M is a $K(\pi_1(M), 1)$ space, and $\pi_1(M)$ is a $\text{PD}(4)$ group.*

A similar statement holds in every dimension. Towards this, we first recall a classical theorem of Andreotti, Frankel and Narasimhan [AF59, AN62, Nar67].

Theorem 2.7. *Let M be a (not necessarily smooth) complex projective variety of complex dimension n such that the universal cover \widetilde{M} is Stein. Then $H_i(\widetilde{M}, \mathbb{Z}) = 0$ for $i > n$. In fact \widetilde{M} is homotopy equivalent to a CW-complex of real dimension n . Also, the second homotopy group $\pi_2(M)$ is free abelian.*

If X is any Stein manifold embedded in \mathbb{C}^N for some N , then for a dense set of points $x \in \mathbb{C}^N$, the square of the distance function $z \mapsto d^2(x, z)$ is Morse and strongly plurisubharmonic on X .

Proposition 2.8. *Let M be a smooth complex projective variety of complex dimension n whose universal cover \widetilde{M} is Stein. Also suppose that $\pi_i(M) = 0$ for all $1 < i \leq n$. Then M is a $K(\pi_1(M), 1)$ space, and $\pi_1(M)$ is a $\text{PD}(2n)$ group. In particular, $\pi_i(M) = 0$ for all $i > 1$.*

Proof. Since $\pi_i(M) = \pi_i(\widetilde{M})$ for $i > 1$, it follows that $\pi_i(\widetilde{M}) = 0$ for $1 < i \leq n$. From Theorem 2.7 and repeated application of Hurewicz' Theorem, it follows that $\pi_i(\widetilde{M}) = 0$ for all i , and the proposition follows. \square

Remark 2.9. For the conclusion in Proposition 2.8 we do not quite need to assume that universal cover \widetilde{M} is Stein. It suffices to assume that $\pi_i(M) = 0$ for $1 < i < 2n - 1$ in view of the argument in the proof of Lemma 2.3.

2.4. The trivial group. We illustrate the notion of homotopical height with the example of the trivial group, and justify the dimensional restriction in Definition 1.2.

Proposition 2.10. *For the trivial group $G = \{1\}$, we have $ht_{\mathcal{SP}}(G) = ht_{\mathcal{K}}(G) = ht_{\mathcal{P}}(G) = ht_{\mathcal{HC}}(G) = ht_{\mathcal{S}}(G) = 2$, and $ht_{\mathcal{CT}}(G) = ht_{\mathcal{CR}}(G) = ht_{\mathcal{St}}(G) = ht_{\mathcal{AF}}(G) = ht_{\mathcal{CA}}(G) = ht_{\mathcal{AC}}(G) = \infty$.*

Proof. Let M be a simply connected closed symplectic manifold. Then, by the Hurewicz' Theorem, $H_2(M) = \pi_2(M)$. Since M is symplectic, $H_2(M) \neq 0$ and it follows that $ht_{\mathcal{SP}}(G) = 2$. As \mathcal{K} is a subclass of \mathcal{SP} , and the 2-sphere S^2 belongs to the class, it follows that $ht_{\mathcal{SP}}(G) = ht_{\mathcal{K}}(G) = 2$. Also, since we are considering only manifolds of positive dimension, the classes $\mathcal{P}, \mathcal{HC}, \mathcal{S}$ are all subclasses of \mathcal{K} and so $ht_{\mathcal{P}}(G) = ht_{\mathcal{HC}}(G) = ht_{\mathcal{S}}(G) = 2$.

Next, since the sphere of dimension $(2n + 1)$ admits both a contact and CR structure, it follows that $ht_{\mathcal{CT}}(G) = ht_{\mathcal{CR}}(G) = \infty$.

Since the affine plane is Stein, we have $ht_{\mathcal{St}}(G) = ht_{\mathcal{AF}}(G) = \infty$.

Finally, the Calabi–Eckmann manifolds are complex with underlying real manifold $S^{2m+1} \times S^{2n+1}$. Therefore, we have $ht_{\mathcal{CA}}(G) = \infty$. Complex manifolds are, of course, almost complex, implying that $ht_{\mathcal{AC}}(G) = \infty$. \square

3. SOFT CLASSES OF MANIFOLDS

3.1. Open almost complex manifolds. We first show that the category ACO of open almost complex manifolds is soft of Type 1.

Theorem 3.1. *The inequality $ht_{ACO}(G) \geq k$ holds for all finitely presented groups G of type FP_k . In particular, $ht_{ACO}(G) = \infty$ for all finitely presented groups G of type FP_∞ .*

Proof. By Corollary 2.2, we have $ht_{STT}(G) \geq k$ for all finitely presented groups G of type FP_k . In particular, $ht_{STT}(G) = \infty$ for all finitely presented groups G of type FP_∞ . Hence for a finitely presented group G of type FP_k , there exists a manifold M such that

- (a) $\pi_1(M) = G$,
- (b) $\pi_i(M) = 0$ if $1 < i < k$, and
- (c) the tangent bundle TM of M is stably trivial.

Let E_m denote the trivial vector bundle of rank m over M . Then there exists an integer N such that the vector bundle $W_M = TM \oplus E_m$ over M is trivial for all $m \geq N$. Taking $m \geq N$ to be odd or even according as the dimension of M is odd or even, we see that the open manifold $M \times \mathbb{R}^m$ is even dimensional and has a trivial tangent bundle. So, $T(M \times \mathbb{R}^m) = (M \times \mathbb{R}^m) \times \mathbb{R}^{2k} = (M \times \mathbb{R}^m) \times \mathbb{C}^k$. Hence $M \times \mathbb{R}^m$ is almost complex. The theorem follows. \square

Remark 3.2. Note that in the proof of Theorem 3.1, m (the dimension of the factor \mathbb{R}^m) can be taken to be arbitrarily large. This will be useful in the proof of Theorem 3.5 below.

3.2. Almost CR and almost complex manifolds. Following [Bla02], we use the terminology *almost CR* for manifolds whose tangent bundles have a codimension one distribution equipped with a complex structure J . If, moreover, this distribution is formally integrable, we say that the manifold is CR. Sometimes, in the literature, the former is called CR and the latter integrable CR. We explicate this to avoid confusion. We denote the class of closed almost CR manifolds by ACR .

We observe that Theorem 3.1 immediately furnishes softness (Type 1) for closed almost CR manifolds and almost complex manifolds. Theorem 3.3 below is a

stronger version of a Theorem of Kotschick [Kot92] that every finitely presented group occurs as the fundamental group of an almost contact manifold.

Theorem 3.3. *The inequality $ht_{ACR}(G) \geq k$ holds for all finitely presented groups G of type FP_k . In particular, $ht_{ACR}(G) = \infty$ for all finitely presented groups G of type FP_∞ .*

The inequality $ht_{AC}(G) \geq k$ holds for all finitely presented groups G of type FP_k . In particular, $ht_{AC}(G) = \infty$ for all finitely presented groups G of type FP_∞ .

Proof. Given a finitely presented group G of type FP_k , by Theorem 3.1 there exists a manifold M and a natural number n such that

- (a) $\pi_1(M) = G$,
- (b) $\pi_i(M) = 0$ for all $1 < i < k$, and
- (c) $M \times \mathbb{R}^m$ carries an almost complex structure $(M \times \mathbb{R}^m, J)$ for all $m \geq n$ such that $m + \dim M$ is even.

The smooth real hypersurface $H = M \times S^{m-1}$ of the almost complex manifold $(M \times \mathbb{R}^m, J)$ has a complex tangent subspace $TH \cap J(TH) \subset TH$ forming a (not necessarily integrable) complex tangent distribution. Clearly, $TH \cap J(TH)$ is of real codimension one in TH . Hence J restricted to TH gives an almost CR-structure on H . If $m > k$, then

- (a) $\pi_1(H) = G$, and
- (b) $\pi_i(H) = 0$ for $1 < i < k$.

The first statement of the theorem, which is for closed almost CR manifolds, follows.

To prove the second statement, consider $W = H \times S^{2m+1}$. Both H and S^{2m+1} admit almost CR structures; let $\mathcal{H} \subset TH$ and $\mathcal{D} \subset TS^{2m+1}$ be distributions giving almost CR structures on H and S^{2m+1} respectively. Let ξ and η denote the unit vector fields on H and S^{2m+1} normal to \mathcal{H} and \mathcal{D} respectively. Let J_H and J_S denote the almost complex structures on \mathcal{H} and \mathcal{D} respectively.

We define an almost complex structure J on TW as follows. First, identify TW naturally with the direct sum $\mathcal{H} \oplus \mathcal{D} \oplus \mathbb{R}\xi \oplus \mathbb{R}\eta$. Now define, for every $p \in W$, a linear homomorphism $J_p : T_pW \rightarrow T_pW$ by

- (1) $J_p(v) = J_H(v)$ for $v \in \mathcal{H}$,
- (2) $J_p(v) = J_D(v)$ for $v \in \mathcal{D}$,
- (3) $J_p(\xi_p) = \eta_p$, and
- (4) $J_p(\eta_p) = -\xi_p$.

Since J_p is clearly a smoothly varying family of automorphisms satisfying $J_p^2 = -\text{Id}$, it is an almost complex structure on W . Further,

- (a) $\pi_1(W) = G$, and
- (b) $\pi_i(W) = 0$ for $1 < i < k$.

The second statement of the theorem now follows. □

3.3. Stein, CR and Contact manifolds. We shall require the following celebrated theorem of Eliashberg to move from the category of open almost complex manifolds to the category of Stein manifolds.

Theorem 3.4 ([Eli90]). *Let X be a real $2n$ -dimensional smooth manifold, with $n > 2$, such that X carries an almost complex structure J and a proper Morse function ϕ all of whose critical points are of index at most n . Then J is homotopic*

to a complex structure J' on X with respect to which ϕ is plurisubharmonic. In particular, (X, J') is Stein.

We now have the tools ready to show that the category St of Stein manifolds is soft of type 1.

Theorem 3.5. *The inequality $ht_{St}(G) \geq k$ holds for all finitely presented groups G of type FP_k . In particular, $ht_{St}(G) = \infty$ for all finitely presented groups G of type FP_∞ .*

Proof. By Theorem 3.1, we have $ht_{ACO}(G) \geq k$ for all finitely presented groups G of type FP_k . In particular, $ht_{ACO}(G) = \infty$ for all finitely presented groups G of type FP_∞ . By Remark 3.2, for a finitely presented group G of type FP_k , there exists an open almost complex manifold $(M \times \mathbb{R}^m, J)$ with M compact such that

- (a) $\pi_1(M \times \mathbb{R}^m) = \pi_1(M) = G$,
- (b) $\pi_i(M \times \mathbb{R}^m) = \pi_i(M) = 0$ if $1 < i < k$, and
- (c) $m \geq \dim M$.

Since $m \geq \dim M$, there exists a proper Morse function ϕ on $M \times \mathbb{R}^m$ all of whose critical points are of index at most $\frac{1}{2} \dim(M \times \mathbb{R}^m)$. Hence by Theorem 3.4, the almost complex structure J is homotopic to a complex structure J' such that $(M \times \mathbb{R}^m, J')$ is Stein. The theorem follows. \square

As an immediate corollary of Theorem 3.5, we obtain softness for the class of closed integrable CR manifolds as well as closed contact manifolds.

Theorem 3.6. *The inequality $ht_{CR}(G) \geq k$ holds for all finitely presented groups G of type FP_k . In particular, $ht_{CR}(G) = \infty$ for all finitely presented groups G of type FP_∞ .*

The inequality $ht_{CT}(G) \geq k$ holds for all finitely presented groups G of type FP_k . In particular, $ht_{CT}(G) = \infty$ for all finitely presented groups G of type FP_∞ .

Proof. Let G be of type FP_k . Then by Theorem 3.5, there exists a Stein manifold M such that

- (1) $\pi_1(M) = G$, and
- (2) $\pi_i(M) = 0$ for all $1 < i < k$.

By the property of being Stein, there exists a natural number N such that M can be embedded in \mathbb{C}^N . We assume that M comes equipped with such an embedding. The second part of Theorem 2.7 says that the square of the distance function $z \mapsto d^2(x, z)$ is Morse and strongly plurisubharmonic for a dense set of points $x \in \mathbb{C}^N$. Without loss of generality, let the origin 0 be such a point. Then the intersection with the sphere $S(0, R) \cap M =: M_R$ is a strongly pseudo-convex integrable CR manifold for all R that are regular values of the function $z \mapsto d^2(0, z)$.

Since G is of type FP_k , the above Morse function has finitely many critical values of index less than k . Suppose all critical values of index less than k are contained in the ball of radius R_0 about 0 . Then for all $R > R_0$, the strongly pseudo-convex integrable CR manifold M_R satisfies the following two conditions:

- (1) $\pi_1(M_R) = G$, and
- (2) $\pi_i(M_R) = 0$ for all $1 < i < k$.

Therefore, the softness of closed integrable CR manifolds (the first statement of the theorem) follows.

Since strongly pseudo-convex integrable closed CR manifolds are contact [Bla02, p. 72-73], softness of closed contact manifolds (the second statement of the theorem) follows. \square

A'Campo and Kotschick, [AK94], showed that an arbitrary finitely presented group can be realized as the fundamental group of a closed contact manifold of dimension five. Consequently, closed contact manifolds are of Type 3 in our classification. Theorem 3.6 upgrades \mathcal{CT} from Type 3 to Type 1 in the softness scale.

4. COMPLEX AND SYMPLECTIC MANIFOLDS

The classes \mathcal{SP} and \mathcal{CA} of closed symplectic manifolds and closed complex manifolds respectively, exhibit intermediate behavior in terms of softness and hardness. By Theorem 1.4 and Theorem 1.5, both \mathcal{SP} and \mathcal{CA} are of Type 3.

4.1. Complex Manifolds. The class \mathcal{CA} is also of Type 2. However, we do not know whether the class \mathcal{CA} can be upgraded to Type 1. We rephrase and elaborate Question 1.7, which essentially asks if this can be done.

Question 4.1. Is $ht_{\mathcal{CA}}(G) \geq k$ for every finitely presented group G of type FP_k ? For a finitely presented group G of type FP_∞ , is $ht_{\mathcal{CA}}(G) = \infty$?

Remark 4.2. Theorem 3.3 shows that the answer to Question 4.1 is “Yes” if the class \mathcal{CA} is replaced by the class \mathcal{AC} of closed almost complex manifolds. Hence a group for which the answer to Question 4.1 is negative will produce an almost complex manifold which does not admit a complex structure.

Proposition 2.10 answers Question 4.1 positively for the trivial group. Note that Taubes’ twistor construction in [Tau92] furnishes manifolds whose second homotopy group is non-zero and hence we cannot hope to answer Question 4.1 positively using his techniques. We give a number of other examples of G below for which the answer of Question 4.1 is positive.

We shall need the following theorem due to Morimoto. Let M be an almost contact manifold. The almost contact structure induces a natural almost complex structure on $M \times \mathbb{R}$. If this almost complex structure is integrable, then the almost contact manifold M is said to be *normal*. In particular, a Sasakian manifold admits a normal almost contact structure.

Theorem 4.3 ([Mor63]). *The Cartesian product of two normal almost contact manifolds admits a complex structure.*

Lemma 4.4. *The height $ht_{\mathcal{CA}}(G) = \infty$ if G is any of the following groups:*

- (a) *finitely generated abelian groups, and*
- (a) *fundamental group of any spherical space form.*

Proof. Groups in (a): By taking Cartesian products of manifolds, it reduces to showing that $ht_{\mathcal{CA}}(G) = \infty$ for every cyclic group G . Now, if $G = \mathbb{Z}$, then the Hopf manifolds M_n homeomorphic to $S^1 \times S^{2n+1}$ have $\pi_i(M_n) = 0$ for $1 < i \leq 2n$. Hence $ht_{\mathcal{CA}}(\mathbb{Z}) = \infty$.

For any positive integer p , let L_p^{2n+1} denotes the $(2n + 1)$ -dimensional Lens space with fundamental group $\mathbb{Z}/p\mathbb{Z}$. Then $W_n = L_p^{2n+1} \times S^{2n+1}$ admits a complex structure by Theorem 4.3, and also $\pi_i(W_n) = 0$ for $1 < i \leq 2n$. Hence $ht_{\mathcal{CA}}(\mathbb{Z}_p) = \infty$.

Groups in (b): Let $M = S^N/G$, where G acts freely by isometries on the round sphere S^N of odd dimension N . Take m times iterated join $S^N * S^N * \cdots * S^N$. The group G acts on it freely using the diagonal action, with G acting on each factor as before. Therefore, we obtain a free action by isometries of G on the round sphere S^{mN+m-1} . Define $M_m := S^{mN+m-1}/G$, and let $W_m := M_m \times S^{2m+1}$. Then, by Theorem 4.3 again, W_m admits a complex structure, and $\pi_i(W_m) = 0$ for $1 < i \leq 2m$. Hence $ht_{\mathcal{CA}}(G) = \infty$. For a classification of fundamental groups of spherical space forms, see [Wol11, Ch. 7]. \square

We shall now rework an argument going back to Oliver [Oli79, p. 546–7] to provide a “very inefficient” model space for arbitrary finite groups. The models in Lemma 4.4 are considerably more efficient. In fact we shall be using the argument for finite cyclic groups in the proof of Lemma 4.4 implicitly in the proof of Proposition 4.5 below.

Proposition 4.5. *For any finite group G ,*

$$ht_{\mathcal{CA}}(G) = \infty.$$

Proof. Let G be any finite group. Let H be a subgroup of G and M a complex manifold on which H acts freely by holomorphic automorphisms. Denote the index $\#G/H$ by N . We identify the product complex manifold M^N with the space of all maps from G/H to M . Then M^N can be naturally identified with the space of all H -equivariant maps from G to M . The action of H on M^N is the diagonal one. This action of H on M^N naturally extends to a G -action on M^N using the left-translation action of G on G/H . Clearly this action of G on M^N is also by holomorphic automorphisms.

We denote by C_n the Calabi–Eckmann manifold with underlying manifold $S^{2n+1} \times S^{2n+1}$. We observe that for any m , the cyclic group $\mathbb{Z}/m\mathbb{Z}$ acts freely by complex automorphisms on C_n (see [Mor63] for instance). Hence for any non-trivial element $g \in G$, the cyclic group $\langle g \rangle \subset G$ acts freely by complex automorphisms on C_n .

Let $i_g = [G : \langle g \rangle]$ denote the index of $\langle g \rangle$ in G . We have seen in the first part of the proof that there is an action of G on the Cartesian product $C_n^{i_g}$ by holomorphic automorphisms, such that the action of the subgroup $\langle g \rangle$ is free. Hence the diagonal action of G on the product

$$W_n := \prod_{g \in G, g \neq 1} C_n^{i_g}$$

is free and by holomorphic automorphisms. Let $V_n = W_n/G$ denote the quotient manifold. Then $\pi_1(V_n) = G$, and $\pi_i(V_n) = 0$ for $1 < i \leq 2n$.

Hence $ht_{\mathcal{CA}}(G) = \infty$ for any finite group G . \square

Proposition 4.6. *The height $ht_{\mathcal{CA}}(G) = \infty$ if G is one of the following:*

- (a) *the (integral) Heisenberg group H^{2m+1} , and*
- (b) *the fundamental group of an aspherical closed Sasakian manifold, or, more generally, an aspherical manifold equipped with an almost normal contact structure.*

Proof. Groups in (a): In this case G is the fundamental group of a non-trivial circle bundle over an even dimensional torus. Hence $G = \pi_1(M)$, where M admits a normal almost contact structure. By Theorem 4.3, the manifold $V_n := M \times S^{2n+1}$

admits a complex structure. We have $\pi_1(V_n) = G$ and $\pi_i(V_n) = 0$ for $1 < i \leq 2n$. Hence $ht_{\mathcal{CA}}(G) = \infty$.

Groups in (b): The argument is same as in part (a), namely take products with odd dimensional spheres. \square

Remark 4.7. If *NACS* (respectively, *SAS*) denotes the category of closed manifolds equipped with an almost normal contact structure (respectively, closed Sasakian manifolds), then the above argument shows that

$$ht_{SAS}(G) \leq ht_{NACS}(G) \leq ht_{\mathcal{CA}}(G)$$

for all G .

The next question gives two natural test cases for Question 4.1.

Question 4.8. Is $ht_{\mathcal{CA}}(G) = \infty$ for the solvable groups or the fundamental groups of closed real hyperbolic manifolds of dimension greater than two?

4.2. Symplectic Manifolds. All the results in this subsection apply to the more general class of c -symplectic (cohomologically symplectic) manifolds, i.e., closed manifolds M^{2n} with $\omega \in H^2(M, \mathbb{R})$ such that $\omega^n \neq 0$. We, nevertheless, prefer to retain the notation \mathcal{SP} and talk about symplectic manifolds to prevent further proliferation of categories.

Theorem 2.10 shows that $ht_{\mathcal{SP}}(\{1\}) = 2$. This phenomenon is far more general as shown by the following proposition.

Proposition 4.9. *Let G be a finitely presented group such that $b_2(G) = 0$. Then $ht_{\mathcal{SP}}(G) = 2$. Similarly, if $b_{2k}(G) = 0$, then $ht_{\mathcal{SP}}(G) \leq 2k$.*

Proof. Suppose it is not true. Then there exists a closed symplectic manifold M such that $\pi_1(M) = G$ and $\pi_2(M) = 0$. By attaching cells of dimension greater than three to M , we obtain a $K(G, 1)$ space which will be denoted by X . Then $b_2(M) = b_2(X) = b_2(G) = 0$. This contradicts the assumption that M is symplectic.

Similarly, if $b_{2k}(G) = 0$, and M is a closed symplectic manifold such that $\pi_1(M) = G$ and $\pi_i(M) = 0$ for $1 < i \leq 2k$, then we can construct a $K(G, 1)$ space X by attaching cells of dimension greater than $(2k + 1)$ to M . In that case, $b_{2k}(M) = b_{2k}(X) = b_{2k}(G) = 0$, contradicting the fact that $b_{2k}(M) > 0$ (ensured by the existence of the symplectic structure). \square

It follows that all finite groups and the infinite cyclic group $G = \mathbb{Z}$ have $ht_{\mathcal{SP}}(\mathbb{Z}) = 2$, and, more generally, $b_{2k}(\mathbb{Z}^{2k-1}) \leq 2k$.

Recall that the \mathcal{C} -homotopical dimension $hdim_{\mathcal{C}}(G)$ is the smallest number among dimensions of manifolds realizing $ht_{\mathcal{C}}(G)$. Lemma 2.3 furnishes the following lower bounds on $hdim_{\mathcal{SP}}$.

Proposition 4.10. *If a one-ended finitely presented group G has $ht_{\mathcal{SP}}(G) > 2$, then*

- (a) *either $hdim_{\mathcal{SP}}(G) = 2$, and G is the fundamental group of a 2-manifold of positive genus,*
- (b) *or $hdim_{\mathcal{SP}}(G) = 4$ and G is a $PD(4)$ group,*
- (c) *or $hdim_{\mathcal{SP}}(G) \geq 6$.*

Proof. Suppose that $hdim_{\mathcal{SP}}(G) < 6$. Then there exists a symplectic manifold M of dimension 2 or 4 such that $\pi_1(M) = G$ and $\pi_2(M) = 0$. If M is 2-dimensional, then we land in case (a). If M is 4-dimensional, then Lemma 2.3 says that G is a PD(4) group. \square

Fundamental groups of closed orientable surfaces of positive genus are called *surface groups*.

The next observation shows that $hdim$ imposes restrictions on homotopical height. This might be an approach to Question 1.11.

Proposition 4.11. *Assume that G is not a surface group, and $H^2(G, \mathbb{Z}G) \neq 0$. Also, assume that $hdim_{\mathcal{SP}}(G) = 4$. Then, $ht_{\mathcal{SP}}(G) = 2$.*

Proof. Suppose that $ht_{\mathcal{SP}}(G) > 2$. Consequently, there exists a symplectic 4-manifold M with $\pi_1(M) = G$ and $\pi_2(M) = 0$. Therefore, $\pi_2(\widetilde{M}) = 0$, where \widetilde{M} is the universal cover of M . By the Hurewicz' Theorem, $H_2(\widetilde{M}) = 0$, and hence by Poincaré duality, $H_c^2(\widetilde{M}) = 0$. Therefore $H^2(M, \mathbb{Z}G) = 0$. Since \widetilde{M} is 2-connected, and has a finite 3-skeleton modulo G , we have $H^2(M, \mathbb{Z}G) = H^2(G, \mathbb{Z}G)$. Hence $H^2(G, \mathbb{Z}G) = 0$, yielding a contradiction. \square

The next proposition provides some weak evidence towards Conjecture 1.10.

Proposition 4.12. *Let G be an infinite finitely presented group with $ht_{\mathcal{SP}}(G) > 2$. Then $b_2(G)$ is positive.*

Proof. As usual, first suppose that $ht_{\mathcal{SP}}(G)$ is finite. Let M be a closed smooth symplectic manifold realizing $ht_{\mathcal{SP}}(G)$. Then $\pi_2(M) = 0$. Hence a $K(G, 1)$ space X may be constructed from M by adding cells in dimensions greater than three to kill higher homotopy groups. Therefore, $b_2(G) = b_2(X) = b_2(M)$; the last equality holds because M and X differ only in cells of dimension four or more. Since M is symplectic, we have $b_2(M) > 0$, and the proposition follows in this case.

Next, assume that $ht_{\mathcal{SP}}(G) = \infty$. Then there exists $M \in \mathcal{SP}$ with $\pi_1(M) = G$ and $\pi_2(M) = 0$. The same argument now furnishes the conclusion that $b_2(G)$ is positive. \square

Remark 4.13. We note that Proposition 4.12 offers the following approach to Conjecture 1.10. Given an infinite Kähler group G and a compact Kähler manifold M with fundamental group G , can one construct a new symplectic (or c -symplectic) manifold M_1 from M by attaching cells in dimensions greater than three such that $\pi_2(M_1) = 0$? This construction can be generalized a bit by demanding that we are required to attach cells only to a finite-sheeted cover of M , converting the latter into a manifold with vanishing π_2 .

For groups G admitting such constructions, Proposition 4.12 shows that virtually $b_2(G)$ is positive. The advantage of the approach is that it takes the problem into the domain of symplectic topology, where surgery techniques are much more plentifully available. In particular, Kähler symplectically aspherical groups of class \mathcal{A} in the terminology of [IKRT04] verify Conjecture 1.10.

Cohomological dimension also imposes restrictions on homotopical height for straightforward reasons as follows.

Proposition 4.14. *If $cd(G) = 2n$, then either $ht_{\mathcal{SP}}(G) = \infty$ with G being a PD($2n$) group, or $ht_{\mathcal{SP}}(G) \leq 2n + 2$. If $cd(G) = 2n + 1$, then $ht_{\mathcal{SP}}(G) \leq 2n + 2$.*

Proof. Suppose that G is not a PD(2n) group that is realizable as the fundamental group of a symplectic $K(G, 1)$ manifold. Also, assume that $cd(G) = 2n$ or $2n+1$. If $ht_{\mathcal{SP}}(G) > 2n+2$, then there exists a symplectic manifold M such that $\pi_1(M) = G$ and $\pi_i(M) = 0$ for $1 < i \leq 2n+2$. By attaching cells of dimension greater than $2n+3$ we obtain a $K(G, 1)$ space X . Then

$$b_{2n+2}(G, \mathbb{Z}) = b_{2n+2}(X, \mathbb{Z}) = b_{2n+2}(M, \mathbb{Z}) > 0,$$

contradicting the assumption that $cd(G) = 2n$ or $2n+1$. \square

We shall give *lower bounds* on $ht_{\mathcal{SP}}(G)$ for certain groups including \mathbb{Z}^N (for N odd) later on (see Proposition 8.2) using some techniques from the hard category of projective manifolds, in particular the Lefschetz hyperplane theorem.

4.3. Affine and Quasi-projective varieties. Let \mathcal{AF} and \mathcal{QP} denote the classes of smooth affine and smooth quasiprojective varieties respectively. Homotopical height does not distinguish between these categories:

Lemma 4.15. *For G a finitely presented group, we have $ht_{\mathcal{AF}}(G) = ht_{\mathcal{QP}}(G)$.*

Proof. Jouanolou showed that for any quasi-projective variety M , there exists an affine variety W mapping surjectively onto M with contractible fibers [Jou73]. In particular, this W is homotopy equivalent to M . It follows that $ht_{\mathcal{AF}}(G) \geq ht_{\mathcal{QP}}(G)$.

Since $\mathcal{AF} \subset \mathcal{QP}$, we have $ht_{\mathcal{QP}}(G) \geq ht_{\mathcal{AF}}(G)$. \square

5. BASIC RESTRICTIONS ON HOMOTOPICAL HEIGHT OF KÄHLER GROUPS

This and the next two sections deal with the “hard” categories of Kähler (\mathcal{K}) and complex projective (\mathcal{P}) manifolds. We shall also deal with the subclasses \mathcal{HC} and \mathcal{S} of \mathcal{P} consisting of smooth projective varieties whose universal covers are, respectively, holomorphically convex and Stein. This section deals with some basic restrictions on homotopical height coming from the Lefschetz hyperplane theorem and formality.

5.1. Lefschetz Hyperplane Theorem.

Lemma 5.1. *If $-\infty < ht_{\mathcal{P}}(G) < \infty$, then there exists a smooth complex projective variety M realizing it such that the complex dimension of M is at most $ht_{\mathcal{P}}(G)$.*

Proof. Let N be a smooth complex projective manifold realizing $ht_{\mathcal{P}}(G) = n$. Assume that the complex dimension of N is strictly bigger than n (otherwise there is nothing to prove). Let N_1 be a hyperplane section of N . By the Lefschetz hyperplane theorem, we have $\pi_i(N_1) = \pi_i(N)$ for all $i < \dim_{\mathbb{C}} N_1$. In particular, $\pi_1(N_1) = G$ and $\pi_i(N_1) = 0$ for $1 < i < n$. We can repeat this argument replacing N by N_1 and iterating till the dimension of the hyperplane section reduces to n . \square

Remark 5.2. The same argument works for the categories \mathcal{HC} and \mathcal{S} as hyperplane sections preserve both the categories \mathcal{HC} and \mathcal{S} (see Proposition 2.1 of [BM12] for instance). Thus if $ht_{\mathcal{HC}}(G) = n$ (respectively, $ht_{\mathcal{S}}(G) = n$), then there exists a complex projective manifold $M \in \mathcal{HC}$ (respectively, $M \in \mathcal{S}$) of complex dimension at most n realizing it.

For the category \mathcal{S} we have a stronger result.

Proposition 5.3. *Assume that $-\infty < ht_{\mathcal{S}}(G) = n < \infty$. Then there exists a complex projective manifold M of complex dimension exactly n realizing it. Further, no smooth projective manifold M of complex dimension less than n realizes $ht_{\mathcal{S}}(G)$. Hence $hdim_{\mathcal{S}}(G) = 2n$.*

Also, the universal cover \widetilde{M} of the above complex projective manifold M is homotopy equivalent to a wedge of n -spheres.

Proof. By Lemma 5.1 and Remark 5.2, there exists a smooth complex projective manifold M of dimension at most n realizing $ht_{\mathcal{S}}(G)$. The first assertion in the proposition follows from the second assertion of Lemma 5.1.

To prove the second assertion, suppose that there exists a complex projective manifold M of dimension m less than n realizing $ht_{\mathcal{S}}(G)$. Then by the definition of $ht_{\mathcal{S}}(G)$, we have $\pi_i(M) = 0$ for $2 \leq i \leq m$. Hence $\pi_i(\widetilde{M}) = 0$ for $1 \leq i \leq m$. But $H_i(\widetilde{M}) = 0$ for $i > m$ by Theorem 2.7. Hence $\pi_i(\widetilde{M}) = 0$ for all i implying that \widetilde{M} is a $K(G, 1)$ space. But this contradicts the hypothesis that $ht_{\mathcal{S}}(G) = n < \infty$. Therefore, the dimension of a complex projective manifold realizing $ht_{\mathcal{S}}(G)$ is at least n .

Combining the first two assertions we have $hdim_{\mathcal{S}}(G) = 2n$.

The last assertion now follows from the last statement of Theorem 2.7, which says that \widetilde{M} is homotopy equivalent to an n -complex K . Since $\pi_i(K) = 0$ for $i < n$, the n -complex K must be homotopy equivalent to a wedge of n -spheres. \square

5.1.1. Lefschetz Hyperplane Theorem for Symplectic Manifolds. The following fundamental Theorem of Donaldson [Don96] generalizes the Lefschetz Hyperplane Theorem to the context of symplectic manifolds.

Theorem 5.4 ([Don96, Theorem 1; Proposition 39]). *Let (V, ω) be a compact symplectic manifold of dimension $2n$, and suppose that the de Rham cohomology class $[\frac{\omega}{2\pi}] \in H^2(V, \mathbb{R})$ lies in the integral lattice $H^2(V, \mathbb{Z})/Torsion$. Let $h \in H^2(V, \mathbb{Z})$ be a lift of $[\frac{\omega}{2\pi}]$ to an integral class. Then for sufficiently large integers k the Poincaré dual of kh in $H_{2n-2}(V, \mathbb{Z})$ can be realized by a symplectic submanifold $W \subset V$.*

Let $L \rightarrow V$ be a complex line bundle over V with compatible almost complex structure, and with first Chern class $c_1(L) = [\frac{\omega}{2\pi}]$. Let W_k be the zero-set of a section s of $L^{\otimes k}$. When k is sufficiently large, the general W_k is a submanifold and the inclusion $i : W_k \rightarrow V$ induces an isomorphism on homotopy groups π_p for $p < n - 2$ and a surjection on π_{n-1} .

As an immediate corollary, we have the following. The proof of it is an exact replica of the proof of Lemma 5.1 and we omit it.

Corollary 5.5. *If $ht_{\mathcal{SP}}(G) < \infty$, then there exists a symplectic manifold M realizing it such that the dimension of M is at most $2ht_{\mathcal{SP}}(G)$; in other words, $hdim_{\mathcal{SP}}(G) \leq 2ht_{\mathcal{SP}}(G)$.*

5.2. Restrictions from formality. We refer to [PS09] for a survey of formality of spaces and algebras relevant to our context. Recall that a commutative differential graded algebra (cdga for short) (A, d_A) is formal if it is weakly equivalent to its cohomology algebra, endowed with the zero differential. Also a group G is formal when the Eilenberg–MacLane space $K(G, 1)$ is formal.

Definition 5.6 ([Mac10]). A $\text{cdga } (A, d_A)$ is q -stage formal if there is a zig-zag of homomorphisms connecting (A, d_A) to $(H^*(A, d_A), 0)$, with each one of these homomorphisms inducing an isomorphism in cohomology up to degree q , and a monomorphism in degree $q + 1$.

A group G is q -stage formal when the Eilenberg–MacLane space $K(G, 1)$ is q -stage formal.

The next theorem is a special case of a Theorem of Macinic.

Theorem 5.7 ([Mac10]). *Suppose that M is a formal space but $\pi_1(M)$ is not q -stage formal for some $q \geq 2$. Then $\pi_i(M) \neq 0$ for some $i \in [2, q]$.*

The following bound on $ht_{\mathcal{K}}(G)$ is an immediate consequence.

Proposition 5.8. *Let G be a Kähler group. Let q_0 be the smallest value of q such that G is not q -stage formal. Then $ht_{\mathcal{K}}(G) \leq q_0$.*

Proof. Let M be any compact Kähler manifold with fundamental group G . A fundamental result of Deligne, Griffiths, Morgan and Sullivan asserts that M is a formal space (cf. [ABCKT, Chapter 3]). Hence using Theorem 5.7 we have $\pi_i(M) \neq 0$ for some $i \in [2, q_0]$. So $ht_{\mathcal{K}}(G) \leq q_0$ by Definition 1.2. \square

6. CONSEQUENCES IN GROUP COHOMOLOGY

Gromov [Gro89] showed that $H^1(G, \mathbb{Z}G) = 0$ (Theorem 2.5) for any infinite Kähler group G . In this section we start looking at higher group cohomologies. The guiding questions are Question 1.11 and Conjecture 1.10.

6.1. Cohomology with local coefficients. We first consider the Leray–Serre cohomology spectral sequence of the classifying maps $M := \widetilde{M}/G \rightarrow K(G, 1)$ for the principal G -bundle $\widetilde{M} \rightarrow M$ (cf. [Hu59, p. 286], [Dye76, Theorem 2.2], [Kli10, Proposition 1], [BM12, Proposition 3.5]).

Proposition 6.1. *Let M be a smooth complex projective variety with Stein universal cover \widetilde{M} . Assume that $\pi_i(M) = 0$ for all $1 < i < n$. Also, assume that $G := \pi_1(M)$ is torsion-free. Let R be any left $\mathbb{Z}G$ -module. Then*

- (1) $H^p(G, R) = H^p(M, R)$ for $0 < p < n$, and
- (2) *there is an exact sequence of G -modules,*

$$\begin{aligned} 0 &\rightarrow H^n(G, R) \rightarrow H^n(M, R) \rightarrow H^n(\widetilde{M}, R)^G \rightarrow H^{n+1}(G, R) \\ &\rightarrow H^{n+1}(M, R) \rightarrow H^1(G, H^n(\widetilde{M}, R)) \rightarrow H^{n+2}(G, R) \rightarrow H^{n+2}(M, R) \\ &\rightarrow H^2(G, H^n(\widetilde{M}, R)) \rightarrow H^{n+3}(G, R) \rightarrow \dots \rightarrow H^{n-1}(G, H^n(\widetilde{M}, R)) \\ &\rightarrow H^{2n}(G, R) \rightarrow H^{2n}(M, R) \rightarrow H^n(G, H^n(\widetilde{M}, R)) \rightarrow H^{2n+1}(G, R) \rightarrow 0. \end{aligned}$$

- (3) $H^{p+n+1}(G, R) = H^p(G, H^n(\widetilde{M}, R))$ for all $p \geq n + 1$.

Proof. Let $\widetilde{M} \rightarrow M$ be a principal G -bundle. Take a $K(G, 1)$ space K ; its universal cover \widetilde{K} is contractible. Let $f : M \rightarrow K$ be a classifying map. Let

$$\widetilde{g} : \widetilde{M} \times \widetilde{K} \rightarrow \widetilde{K}$$

be the natural projection. The group G acts on \tilde{K} through deck transformations, and it acts on $\tilde{Y} \times \tilde{K}$ via the diagonal action. Since \tilde{g} is equivariant with respect to these actions, it induces a map

$$g : W := (\tilde{M} \times \tilde{K})/G \longrightarrow \tilde{K}/G = K.$$

The fibers of g are homotopy equivalent to \tilde{Y} (see [Hu59, pp. 285–286] for more details).

Note that $H^i(\tilde{M}, R) = 0$ for $i > n$ by Theorem 2.7. Hence $H^i(\tilde{M}, R) = 0$ for $i \neq 0, n$.

The Leray–Serre cohomology spectral sequence for the above fibration with local coefficients R gives

$$H^p(K, (H^q(\tilde{M}, R))) \implies H^{p+q}(M, R),$$

and hence

$$H^p(G, (H^q(\tilde{M}, R))) \implies H^{p+q}(M, R)$$

since K is a $K(G, 1)$ space.

As $H^i(\tilde{M}, R) = 0$, $i \neq 0, n$, it follows that $E_2^{p,0} = H^p(G, R)$ and $E_2^{p,n} = H^p(G, H^n(\tilde{M}, R))$ are the only (possibly) non-zero $E_2^{p,q}$ terms. Since $E_2^{p,i} = 0$, for $0 < i < n$, the differentials $d_2 \cdots, d_n = 0$. Also, the differentials d_i are zero for $i > n + 1$. Thus d_{n+1} is the only (possibly) non-zero differential. Hence

$$E_{n+1} = \cdots = E_2 \quad \text{and} \quad E_{n+2} = E_{n+3} = \cdots = E_\infty,$$

and also

- $E_\infty^{0,0} = H^0(G, H^0(\tilde{M}, R)) = H^0(G, R) = R^G$ (see [Bro82, p. 58] for instance),
- $E_\infty^{p,0} = H^p(G, H^0(\tilde{M}, R)) = H^p(G, R)$, for $p = 1, \dots, n$,
- $E_\infty^{p,q} = H^p(G, H^q(\tilde{M}, R)) = H^p(G, 0) = 0$, for $q \neq 0, n$.

Further, we have the following exact sequences:

- $0 \longrightarrow E_\infty^{0,n} \longrightarrow H^n(\tilde{M}, R)^G \xrightarrow{d_{n+1}} H^{n+1}(G, R) \longrightarrow 0$,
- $0 \longrightarrow H^{p-n-1}(G, H^n(\tilde{M}, R)) \xrightarrow{d_{n+1}} H^p(G, R) \longrightarrow E_\infty^{p,0} \longrightarrow 0$, for all $p \geq n + 1$, and
- $0 \longrightarrow E_\infty^{p,n} \longrightarrow H^p(G, H^n(\tilde{M}, R)) \xrightarrow{d_{n+1}} H^{p+n+1}(G, R) \longrightarrow 0$, for all $p \geq 1$.

The above descriptions of the $E_\infty^{p,q}$ terms can be assembled to produce the following exact sequences for the fibration:

$$0 \longrightarrow H^p(G, R) \longrightarrow H^p(M, R) \longrightarrow 0$$

(assembling $E_\infty^{0,p}$ and $E_\infty^{p,0}$), for $0 < p < n$,

$$0 \longrightarrow H^n(G, R) \longrightarrow H^n(M, R) \longrightarrow (H^n(\tilde{M}, R))^G \xrightarrow{d_{n+1}} H^{n+1}(G, R)$$

(assembling $E_\infty^{0,n}$ and $E_\infty^{n,0}$), and

$$H^{p-n-1}(G, H^n(\tilde{M}, R)) \xrightarrow{d_{n+1}} H^p(G, R) \longrightarrow H^p(M, R) \longrightarrow H^{p-n}(G, H^n(\tilde{M}, R)) \xrightarrow{d_{n+1}} H^{p+1}(G, R)$$

(assembling $E_\infty^{0,p}$ and $E_\infty^{p,0}$), for all $p \geq n + 1$.

Since $H^p(M, R) = 0$ for all $p > 2n$, we immediately get from the above exact sequence that

$$H^{p+1}(G, R) = H^{p-n}(G, H^n(\widetilde{M}, R))$$

for all $p \geq 2n + 1$. \square

Remark 6.2. Note that in the proof of Proposition 6.1, the Stein property of \widetilde{M} is only used to conclude that the homology of \widetilde{M} is concentrated in the middle dimension. Thus, Proposition 6.1 and its conclusions remain valid when M is only a closed orientable manifold of dimension $2n$ such that $H_i(\widetilde{M}) = 0$ for $i \neq 0, n$.

Corollary 6.3. *If $ht_S(G) = n$, then $cd(G) \geq n$.*

Proof. Choose $R = \mathbb{Z}G$ in Proposition 6.1. Then $H^p(G, \mathbb{Z}G) = H^p(M, \mathbb{Z}G)$ for $p < n$. Now $H^p(M, \mathbb{Z}G) = H_c^p(\widetilde{M})$ and by Poincaré Duality $H_c^p(\widetilde{M}) = H_{2n-p}(\widetilde{M})$. Further, by Theorem 2.7, we have $H_{2n-p}(\widetilde{M}) = 0$ if $p < n$. Hence $H^p(G, \mathbb{Z}G) = 0$ for $p < n$. The corollary follows. \square

Now by Proposition 5.3, if $ht_S(G) = n$ is finite, then there exists a smooth complex projective variety M of complex dimension n realizing $ht_S(G)$ with Stein universal cover \widetilde{M} .

Corollary 6.4. *Suppose $ht_S(G) = n$ is finite and M realizes $ht_S(G)$. Then*

- (1) $H^p(G, \mathbb{Z}G) = 0$ for $0 < p < n$,
- (2) *there is an exact sequence of G -modules,*

$$0 \longrightarrow H^n(G, \mathbb{Z}G) \longrightarrow H^n(M, \mathbb{Z}G) \longrightarrow (H^n(\widetilde{M}, \mathbb{Z}G))^G \longrightarrow H^{n+1}(G, \mathbb{Z}G) \longrightarrow 0,$$

(3)

$$H^i(G, H^n(\widetilde{M}, \mathbb{Z}G)) = H^{n+i}(G, \mathbb{Z}G) \quad 1 \leq i \leq n-2,$$

- (4) *there is an exact sequence of G -modules,*

$$\begin{aligned} 0 \longrightarrow H^{n-1}(G, H^n(\widetilde{M}, \mathbb{Z}G)) &\longrightarrow H^{2n}(G, \mathbb{Z}G) \longrightarrow \mathbb{Z} \\ &\longrightarrow H^n(G, H^n(\widetilde{M}, \mathbb{Z}G)) \longrightarrow H^{2n+1}(G, \mathbb{Z}G) \longrightarrow 0, \end{aligned}$$

- (5) $H^{p+n+1}(G, \mathbb{Z}G) = H^p(G, H^n(\widetilde{M}, \mathbb{Z}G))$ for all $p \geq n+1$.

Proof. This follows from Proposition 6.1 putting $R = \mathbb{Z}G$ and using the following two facts:

- (1) $H^{n+i}(M, \mathbb{Z}G) = H_c^{n+i}(\widetilde{M}) = H_{n-i}(\widetilde{M}) = \pi_{n-i}(\widetilde{M}) = 0$ for $0 < i < n$.
- (2) $H^p(M, \mathbb{Z}G) = H_c^p(\widetilde{M}) = H_{2n-p}(\widetilde{M}) = 0$ $0 < p < n$, where the last equality is from Theorem 2.7
- (3) $H^{2n}(M, \mathbb{Z}G) = H_c^{2n}(\widetilde{M}) = \mathbb{Z}$

\square

Remark 6.5. Note that the proof of Corollary 6.4 does not use the complex analytic (Stein) properties of the universal cover \widetilde{M} . It uses only the following topological facts:

- a) M is a closed orientable $2n$ -dimensional manifold,
- b) \widetilde{M} is homotopy equivalent to a wedge of n -spheres.

Thus the conclusions of Corollary 6.4 hold whenever $G = \pi_1(M)$ with M satisfying properties (a), (b).

6.2. Higher Group Cohomologies.

Proposition 6.6. *If $ht_S(G) > 2$, then either $H^2(G, \mathbb{Z}G) = 0$ or G is the fundamental group of a Riemann surface.*

Proof. We first deal with the case where $ht_S(G) = n > 2$ is finite. Let M be a projective variety of (complex) dimension m realizing the homotopical height. By Corollary 2.6, we have $m > 2$.

Then by Proposition 6.1, $H^2(G, \mathbb{Z}G) = H^2(M, \mathbb{Z}G)$ and the latter group is isomorphic to $H_c^2(\widetilde{M})$. By Poincaré Duality and Theorem 2.7, $H_c^2(\widetilde{M}) = H_{2m-2}(\widetilde{M}) = 0$ and so $H^2(G, \mathbb{Z}G) = 0$.

Next suppose that $ht_S(G)$ is infinite. Then one of the following two holds:

- (a) For all $m > 2$, there exists $n \geq m$ and a projective variety $M \in \mathcal{S}$ with $\pi_1(M) = G$ such that $\pi_i(M) = 0$ for all $1 < i < n$, but $\pi_n(M) \neq 0$. The same proof as above goes through in this case to show that $H^2(G, \mathbb{Z}G) = 0$.
- (b) There exists a complex projective variety $M \in \mathcal{S}$ of (complex) dimension m such that M is a $K(G, 1)$ space. Then from Proposition 2.8, this G is a PD(2m) group. If the complex dimension of M is greater than one, then again $H^2(G, \mathbb{Z}G) = 0$.

The only case left is that the complex dimension of M is equal to one, when M is a Riemann surface and G is the fundamental group of a Riemann surface. \square

The following slightly weaker statement holds in all dimensions.

Proposition 6.7. *If $ht_S(G) > n$, then either $H^n(G, \mathbb{Z}G) = 0$ or G is a PD(n) group, in which case $H^n(G, \mathbb{Z}G) = \mathbb{Z}$.*

Proof. As before, first suppose that $ht_S(G)$ is finite. Let M be a projective variety of complex dimension m realizing the homotopical height. From Proposition 2.8 we have $m > n$.

Again, as in the proof of Proposition 6.6, we have

$$H^n(G, \mathbb{Z}G) = H^n(M, \mathbb{Z}G) = H_c^n(\widetilde{M}) = H_{2m-n}(\widetilde{M}) = 0$$

and so $H^n(G, \mathbb{Z}G) = 0$.

If on the other hand, $ht_S(G)$ is infinite, then, one of the following two holds:

- (a) For all $m > n$, there exists $k \geq m$ and a projective variety $M \in \mathcal{S}$ with $\pi_1(M) = G$ such that $\pi_i(M) = 0$ for all $1 < i < k$, and $\pi_k(M) \neq 0$. The same proof as above goes through in this case to show that $H^n(G, \mathbb{Z}G) = 0$.
- (b) There exists a complex projective variety $M \in \mathcal{S}$ of (complex) dimension m such that M is a $K(G, 1)$ space. Then, again from Proposition 2.8, this G is a PD(2m) group. If $n \neq 2m$, then again $H^n(G, \mathbb{Z}G) = 0$. Otherwise, $n = 2m$, G is a PD(n) group, and $H^n(G, \mathbb{Z}G) = \mathbb{Z}$.

This completes the proof. \square

We next investigate $H^2(G, \mathbb{Z}G)$ when $ht_S(G) = 2$.

Proposition 6.8. *Suppose that $ht_S(G) = 2$, and M is a smooth complex projective surface realizing the homotopical height of G . Assume that the action of $G = \pi_1(M)$ on $\pi_2(M)$ is the trivial one. Then either $H^2(G, \mathbb{Z}G) = 0$ or G is virtually the fundamental group of a Riemann surface.*

Proof. We use Proposition 6.1 with $n = 2$.

From Poincaré Duality and the Hurewicz' Theorem,

$$H^2(M, \mathbb{Z}G) = H_c^2(\widetilde{M}) = H_2(\widetilde{M}) = \pi_2(M)$$

and the latter is a trivial $\mathbb{Z}G$ module by hypothesis. By Theorem 2.7, the group $\pi_2(M)$ is free abelian.

The long exact sequence of Proposition 6.1 shows that $H^2(G, \mathbb{Z}G)$ is a $\mathbb{Z}G$ -submodule of $H^2(M, \mathbb{Z}G)$. Hence $H^2(G, \mathbb{Z}G)$ is a trivial $\mathbb{Z}G$ module.

Then either $H^2(G, \mathbb{Z}G) = 0$ or it contains a copy of the trivial $\mathbb{Z}G$ -module \mathbb{Z} . The latter case forces G to be virtually the fundamental group of a Riemann surface by Kleiner's Theorem 2.4. \square

6.3. Groups with a quasi-projective $K(G, 1)$ space. We shall denote the class of smooth complex quasiprojective varieties as \mathcal{QP} . We recall Kollár's question 1.8 below and observe in this subsection a simple but curious connection with Carlson-Toledo's Conjecture 1.10.

Question 6.9 ([Kol95]). If a torsion-free group $G = \pi_1(M)$ for some $M \in \mathcal{P}$, does there exist a $K(G, 1)$ space in \mathcal{QP} ?

The only known set of counter-examples to Question 6.9 arise from some recent examples due to Dimca, Papadima and Suciu [DPS09] (see Theorem 7.9 below for generalizations). We believe therefore that the class of groups admitting a quasiprojective $K(G, 1)$ space deserves special attention. We denote the class of quasiprojective $K(G, 1)$ spaces as \mathcal{QPK} .

Proposition 6.10. *Suppose $M \in \mathcal{QPK}$ contains a positive dimensional projective subvariety. Then $b_2(\pi_1(M)) > 0$.*

Proof. Let $N \subset M$ denote a positive dimensional projective subvariety of M . Let

$$M \subset \overline{M} \subset \mathbb{C}\mathbb{P}^N$$

be the closure of an embedding of M . Restrict the usual Fubini–Study Kähler class ω on $\mathbb{C}\mathbb{P}^N$ to \overline{M} and thence to N . Let $Z \subset N$ be an algebraic curve in N (obtained by taking enough hyperplane sections of N , say). Then

$$(6.1) \quad \int_Z \omega \neq 0.$$

Let $i_0 : M \hookrightarrow \overline{M}$, $i : N \hookrightarrow M$ and $j : Z \hookrightarrow N$ be the inclusions. From (6.1) it follows that

$$(i_0 \circ i \circ j)^* : H^2(\overline{M}, \mathbb{C}) \longrightarrow H^2(Z, \mathbb{C})$$

is non-zero. Therefore, $(i \circ j)^* : H^2(M, \mathbb{C}) \longrightarrow H^2(Z, \mathbb{C})$ is non-zero. Hence $b_2(M)$ is nonzero. Since M is a $K(\pi_1(M), 1)$ space, this implies that $b_2(\pi_1(M)) > 0$. \square

Remark 6.11. All known examples of projective groups G having quasiprojective $K(G, 1)$'s are constructed in a way that the quasi-projective $K(G, 1)$ contains a projective subvariety with G as its fundamental group (see [ABCKT, Ch. 8] for instance). They thus satisfy the hypothesis of Proposition 6.10

Corollary 6.12. *Suppose $M \in \mathcal{QPK}$ admits a nonconstant holomorphic map ϕ from a compact complex analytic manifold N . Then $b_2(\pi_1(M)) > 0$.*

Proof. Consider $M \subset \overline{M} \subset \mathbb{C}\mathbb{P}^N$ as in the proof of Proposition 6.10. Then, by Chow's Theorem, the image $\phi(N) \subset M \subset \mathbb{C}\mathbb{P}^N$ is algebraic. Since ϕ is non-constant, $\phi(N)$ is positive dimensional, and hence Proposition 6.10 completes the proof. \square

Remark 6.13. If G is an infinite Kähler group admitting a quasiprojective $K(G, 1)$ space M , in order to show that $b_2(G) > 0$, according to Corollary 6.12 it suffices to show that M admits a nonconstant holomorphic map from some compact complex analytic manifold.

Recall Jouanolou's trick [Jou73] (cf. Lemma 4.15), where he showed that for any quasi-projective variety M , there exists an affine variety W mapping surjectively onto M with contractible fibers. Thus, if a group G admits a quasi-projective $K(G, 1)$ space, it also admits an affine $K(G, 1)$ space. Since affine varieties do not admit any nonconstant holomorphic maps from any compact complex analytic manifold, the restriction on M in Corollary 6.12 is nontrivial.

Instead of demanding that M admits a nonconstant holomorphic map from a compact complex analytic manifold, we could also demand that M receives a symplectic map ϕ from a symplectic manifold of positive dimension.

Proposition 6.14. *Suppose $M \in \mathcal{QPK}$ equipped with some symplectic structure receives a symplectic map ϕ from a compact symplectic manifold N of positive dimension. Then $b_2(\pi_1(M)) > 0$.*

Proof. We have a compact symplectic manifold (N, ω) of positive dimension, a symplectic form η on M , and a map $\phi : N \rightarrow M$ such that $\phi^*\eta = \omega$. Since the cohomology class $0 \neq [\omega] \in H^2(N, \mathbb{R})$, it follows that $[\eta] \in H^2(M, \mathbb{R})$ is nonzero, and hence $b_2(\pi_1(M)) = b_2(M)$ is nonzero. \square

Remark 6.15. Examples of fundamental groups G of smooth projective varieties admitting a quasiprojective $K(G, 1)$ space are constructed by Toledo in [Tol90] and in Theorem 8.8 of [ABCKT] using work of Goresky–MacPherson [GM88, Section 2]. These examples satisfy the hypotheses in both Proposition 6.10 and Proposition 6.14.

7. FINITENESS PROPERTIES

In this section we compute explicitly the \mathcal{S} -homotopical dimension of certain projective groups discovered by Dimca, Papadima and Suciu [DPS09] (see also [Su13]). We then show how to generalize and topologically characterize them. Our techniques replace the constructions using characteristic varieties in [DPS09] by the group cohomology computations developed in Section 6.

7.1. Projective groups violating FP_r . The examples in [DPS09] are inspired by certain examples due to Bestvina and Brady [BB97], but adapted to the context of Kähler groups.

Theorem 7.1 ([DPS09, Theorem A]). *For every $r \geq 3$, there is an $(r - 1)$ -dimensional smooth complex projective variety H_r with fundamental group G_r satisfying the following conditions:*

- (1) *The homotopy group $\pi_i(H_r)$ vanish for all $2 \leq i \leq r - 2$, while $\pi_{r-1}(H) \neq 0$.*
- (2) *The universal cover \widetilde{H}_r of H_r is a Stein manifold.*

- (3) The group G_r is of type FP_{r-1} , but not of type FP_r .
- (4) The group G_r is not commensurable (up to finite kernels) to any group having a classifying space of finite type.
- (5) The homotopy group $\pi_{r-1}(H_r)$ is an infinitely generated free $\mathbb{Z}G_r$ -module.

Proposition 5.3 shows that the above Conditions 1, 2 are satisfied by a large class of manifolds: Take any G that appears as the fundamental group of a manifold in \mathcal{S} . Then any manifold $M \in \mathcal{S}$ realizing $ht_{\mathcal{S}}(G)$ satisfies Conditions 1, 2. Further, any hyperplane section of such an M also satisfies Conditions 1, 2. We now observe that the groups in Theorem 7.1 have precisely identifiable \mathcal{C} -homotopical height.

Proposition 7.2. *With the notation of Theorem 7.1, we have*

$$ht_{\mathcal{K}}(G_r) = ht_{\mathcal{P}}(G_r) = ht_{\mathcal{HC}}(G_r) = ht_{\mathcal{S}}(G_r) = r - 1.$$

Also, $b_2(G_r) > 0$.

Proof. By Condition 1 of Theorem 7.1, we have $ht_{\mathcal{K}}(G_r) \geq r - 1$. If $ht_{\mathcal{K}}(G_r) \geq r$, then there exists a smooth complex projective variety M such that $\pi_i(M)$ vanish for $2 \leq i \leq r - 1$. By attaching cells of dimension greater than r to M , we can construct a $K(G, 1)$ space with finite r -skeleton (equal to the r -skeleton of M). This contradicts Condition 3 of Theorem 7.1 which says that G is not of type FP_r .

Since H_r is Stein and we have

$$ht_{\mathcal{K}}(G_r) \geq ht_{\mathcal{P}}(G_r) \geq ht_{\mathcal{HC}}(G_r) \geq ht_{\mathcal{S}}(G_r)$$

(a general fact) the first claim follows.

For $r > 3$, we have $b_2(G_r) = b_2(H_r) > 0$ as H_r is Kähler. It remains to show that $b_2(G_3) > 0$.

The authors of [DPS09] construct a surjective holomorphic map of the product W_r of r Riemann surfaces to an abelian variety V_r of dimension r . The abelian variety V_r is then mapped to an elliptic curve E and the generic smooth fiber of the composition from W_r to E is H_r with fundamental group G_r . It follows that there is a surjective map from G_3 to $\mathbb{Z}^{6-2} = \mathbb{Z}^4$. Therefore, $b_1(G_3) > 0$. Hence the Albanese variety of H_3 is non-trivial and so (see [KKM11] for instance) $b_2(G_3) > 0$. \square

Remark 7.3. Proposition 7.2 shows that the examples in [DPS09] (Theorem 7.1) **do not** provide counterexamples to Question 1.9 (3). The authors of [DPS09] observe that $cd(G_3)$ is strictly greater than $ht_{\mathcal{K}}(G_3)$ ($= 2$) as G_3 is not of type FP_3 .

We denote the free $\mathbb{Z}G_r$ module $\pi_{r-1}(H_r)$ by $\mathbb{Z}G_r^\alpha$ for α a countably infinite indexing set. Then $H^{r-1}(\widetilde{M}, \mathbb{Z}G_r) = Hom_{\mathbb{Z}}(\pi_{r-1}(H_r), \mathbb{Z}G_r)$ and we denote this $\mathbb{Z}G$ module by $\mathbb{Z}G_r^{\alpha*}$. Also, $(Hom_{\mathbb{Z}}(\pi_{r-1}(H_r), \mathbb{Z}G_r))^{G_r} = Hom_{\mathbb{Z}G_r}(\pi_{r-1}(H_r), \mathbb{Z}G_r)$, which we denote as $\mathbb{Z}G_r^{\alpha 0}$. Combining Condition 5 of Theorem 7.1 with Proposition 7.2 and Corollary 6.4, we obtain the following information regarding the cohomology groups $H^n(G, \mathbb{Z}G_r)$.

Proposition 7.4. *With the notation of Theorem 7.1, the following hold:*

- (1) $H^p(G, \mathbb{Z}G_r) = 0$ for $0 < p < r - 1$.
- (2) There is an exact sequence of G -modules,

$$0 \longrightarrow H^{r-1}(G, \mathbb{Z}G_r) \longrightarrow \mathbb{Z}G_r^\alpha \longrightarrow \mathbb{Z}G_r^{\alpha 0} \longrightarrow H^r(G, \mathbb{Z}G_r) \longrightarrow 0.$$

- (3) $H^i(G, \mathbb{Z}G_r^{\alpha*}) = H^{r-1+i}(G, \mathbb{Z}G_r)$ if $1 \leq i \leq r-3$.
(4) There is an exact sequence of G -modules,

$$\begin{aligned} 0 \longrightarrow H^{r-2}(G_r, \mathbb{Z}G_r^{\alpha*}) &\longrightarrow H^{2(r-1)}(G, \mathbb{Z}G_r) \longrightarrow \mathbb{Z} \\ &\longrightarrow H^{r-1}(G, \mathbb{Z}G_r^{\alpha*}) \longrightarrow H^{2r-1}(G, \mathbb{Z}G_r) \longrightarrow 0. \end{aligned}$$

- (5) $H^{p+r}(G, \mathbb{Z}G_r) = H^p(G, \mathbb{Z}G_r^{\alpha*})$ for all $p \geq r$.

7.2. Topological Lefschetz Fibrations and FP_r . The construction in [DPS09] can be generalized in different directions. We shall use the purely topological structure underlying the (complex) Morse theory argument (à la Lefschetz [La81, DPS09]) to prove that there exists M with $G = \pi_1(M)$ satisfying the hypotheses of Corollary 6.4 or more generally those of Remark 6.5 (Conditions (a), (b)).

Then we shall use Theorem 7.7 below from group cohomology to conclude that G violates property FP . Since quasiprojective varieties have the homotopy type of finite CW complexes, it follows immediately that such a G cannot have a quasiprojective $K(G, 1)$ space. Thus we have a large source of counterexamples to Kollár's Question 1.8. The aim of this subsection is thus to replace the arguments in [DPS09] involving resonance and characteristic varieties by group cohomology computations and thus generalize those examples to a purely topological context.

We refer the reader to [Don96, Don99] for details of the theory of topological and symplectic Lefschetz fibrations and adapt the definition from [Don99, Gom04] below.

Definition 7.5. A *topological Lefschetz fibration* on a smooth, closed, oriented $2n$ -manifold M consists of the following data:

- (1) a closed orientable 2-manifold S ,
- (2) a finite set of points $K = \{b_i\} \subset M$ called the critical set,
- (3) a smooth map $f : M \rightarrow S$ whose differential df is surjective outside K ,
- (4) for each critical point x of f , there are orientation preserving coordinate charts about x and $f(x)$ (into \mathbb{C}^n and \mathbb{C} , respectively) in which f is given by $f(z_1, \dots, z_n) = \sum_{i=1}^n z_i^2$, and
- (5) f is injective on the critical set $K \subset X$.

A topological Lefschetz fibration with base S a surface of positive genus will be called an *irrational* topological Lefschetz fibration. A Lefschetz fibration without singular fibers is called a *Kodaira fibration*.

If S is an open orientable manifold instead of being closed, and K is only discrete rather than finite, the fibration will be called an *open topological Lefschetz fibration*.

Note that the definition of a topological Lefschetz fibration ensures that $|f|^2$ is a Morse function near singular fibers and a regular fibration away from the critical values. It turns out that the structure of a topological Lefschetz fibration is all that one needs for the usual Lefschetz hyperplane theorem argument (cf. [La81]) to go through and for the proof of Corollary 5.4 of [DPS09] to work out in a topological context (thus replacing a hypothesis involving characteristic varieties in [DPS09]).

Proposition 7.6. *Let $f : M \rightarrow S$ be a (possibly irrational) topological Lefschetz fibration with $\dim M = 2n+2$, $n \geq 2$. Let K be the finite critical set of f . Suppose that \widetilde{M} is $(n+1)$ -connected. Let F denote the regular fiber and $N = \pi_1(F)$. Then*

- a) $\pi_k(F) = 0$ for $1 < k < n$,
- b) $\pi_n(F)$ is a free $\mathbb{Z}N$ -module, with generators in one-to-one correspondence with

$K \times \pi_1(S)$, and

c) if, furthermore, \widetilde{M} is contractible, then \widetilde{F} is homotopy equivalent to a wedge of n -spheres.

Proof. The proof below is based on ideas in [La81, DPS09], particularly the proofs of Lefschetz-type theorems using Morse theory.

Let M_N denote the cover of M corresponding to the (image of the) subgroup N ($= \pi_1(F)$) of $\pi_1(M)$ (we shall observe below that N injects into $\pi_1(M)$). Then f induces a (possibly open) topological Lefschetz fibration $f_N : M_N \rightarrow \widetilde{S}$ with regular fiber F , where \widetilde{S} is the universal cover of S .

It follows (cf. [La81, Don96, DPS09]) that M_N can be obtained from F (up to homotopy) by attaching exactly one $n + 1$ cell \mathbb{D}_i^{n+1} for every critical value v_i of f_N along a vanishing cycle of the Lefschetz fibration $f_N : M_N \rightarrow \widetilde{S}$. This is because f is assumed to be injective on the critical set K . Since $n \geq 2$, we have $\pi_1(M_N) = N$. Also, $\pi_k(M_N, F) = 0$ for $k \leq n$.

Since \widetilde{M} is $(n + 1)$ -connected, we have $\pi_k(M_N) = 0$, for $1 < k \leq n + 1$. The homotopy exact sequence of the pair (M_N, F) gives the following:

$$\cdots \rightarrow \pi_k(M_N) \rightarrow \pi_k(M_N, F) \rightarrow \pi_{k-1}(F) \rightarrow \pi_{k-1}(M_N) \rightarrow \cdots$$

and hence $\pi_k(M_N, F) = \pi_{k-1}(F)$ for $2 < k \leq n + 1$. In particular, $\pi_{n+1}(M_N, F) = \pi_n(F)$. Since $\pi_k(M_N, F) = 0$ for $k \leq n$, we have $\pi_{k-1}(F) = 0$ for $2 < k \leq n$, which is statement (a) of the proposition.

Further, $\pi_{n+1}(M_N, F)$ is generated as a free $\pi_1(M_N)$ -module precisely by the $(n + 1)$ cells \mathbb{D}_i^{n+1} attached to vanishing cycles of the Lefschetz fibration. If

$$\phi : M_N \rightarrow M$$

denotes the covering map, then the critical set of f_N is $\phi^{-1}(K)$, which in turn can be naturally identified with $K \times \pi_1(S)$. Statement (b) of the Proposition follows.

To prove part (c), first observe that the inclusion of the regular fiber F in M_N lifts to an inclusion of \widetilde{F} into \widetilde{M} , because $\pi_1(M_N) = N$. Hence \widetilde{M} can be obtained from \widetilde{F} up to homotopy by attaching the lifts of the $(n + 1)$ cells \mathbb{D}_i^{n+1} to \widetilde{M} . So $\pi_n(\widetilde{F})$ is a free abelian group with generators in one-to-one correspondence with the indexing set $I = K \times \pi_1(M)$. Let V_I denote the wedge of a collection of n -spheres indexed by I . Let $H : V_I \rightarrow \widetilde{F}$ be a map inducing an isomorphism of π_n 's. Since the contractible space \widetilde{M} can be obtained from \widetilde{F} up to homotopy by attaching the lifts of the $n + 1$ cells \mathbb{D}_i^{n+1} , it follows from the Mayer Vietoris homology exact sequence that H induces an isomorphism of homology groups $H_i(\widetilde{F}) = H_i(V_I)$. By simple connectivity of \widetilde{F} and V_I , and from the Hurewicz' Theorem it follows that \widetilde{F} and V_I are homotopy equivalent. \square

We recall now a classical theorem of Bieri.

Theorem 7.7 ([Bie76]). *Let $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ be a short exact sequence of groups, with N of type (FP) and Q of finite cohomology dimension. Then G is a duality (respectively, PD) group if and only if N and Q are duality (respectively, PD) groups. Further $cd(G) = cd(N) + cd(Q)$.*

As an immediate corollary of it we have:

Corollary 7.8. *Let $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ be a short exact sequence of groups, with both G, Q PD groups. Further suppose that N is not a PD group of finite cohomology dimension $cd(G) - cd(Q)$. Then N is not of type FP. Hence, N cannot have a $K(G, 1)$ space homotopy equivalent to a finite CW complex. In particular, N cannot have a quasiprojective $K(G, 1)$ space.*

We combine Proposition 7.6 with Corollary 7.8 to obtain the following Theorem (and its Corollary 7.10) providing a source of counterexamples to Kollár's Question 1.8 generalizing those in [DPS09].

Theorem 7.9. *Let $f : M \rightarrow S$ be an irrational topological Lefschetz fibration that is not a Kodaira fibration, with $\dim M = 2n + 2$, $n \geq 2$. Let K be the finite critical set of f . Further suppose that \widetilde{M} is contractible. Let F denote the regular fiber and $N = \pi_1(F)$. Then*

- a) $\pi_k(F) = 0$ for $1 < k < n$,
- b) $\pi_n(F)$ is a free $\mathbb{Z}N$ -module, with generators in one-to-one correspondence with $K \times \pi_1(S)$,
- c) \widetilde{F} is homotopy equivalent to a wedge of n -spheres, and
- d) N cannot be of type FP; in particular, there does not exist a quasiprojective $K(N, 1)$ space.

Proof. Statements (a), (b) and (c) follow directly from Proposition 7.6.

Proof of (d): It is at this stage that we replace the arguments in [DPS09] involving resonance and characteristic varieties by group cohomology computations.

Note that $\pi_1(M)$ and $\pi_1(S)$ are PD groups of dimension $(2n + 2)$ and 2 respectively. To show that N cannot be of type FP, it suffices (by Corollary 7.8) to show that N cannot be a PD($2n$) group. By Proposition 6.1 (and Remark 6.2), there is an exact sequence of G -modules,

$$0 \rightarrow H^n(N, \mathbb{Z}N) \rightarrow H^n(F, \mathbb{Z}N) \rightarrow H^n(\widetilde{F}, \mathbb{Z}N)^N \rightarrow H^{n+1}(N, \mathbb{Z}N) \rightarrow \dots$$

If N is a PD($2n$) group, then we have $H^n(N, \mathbb{Z}N) = H^{n+1}(N, \mathbb{Z}N) = 0$ because $n \geq 2$. Hence $H^n(F, \mathbb{Z}N) = H^n(\widetilde{F}, \mathbb{Z}N)^N$.

Now, by Poincaré Duality and the Hurewicz' Theorem, we have,

$$H^n(F, \mathbb{Z}N) = H_c^n(\widetilde{F}) = H_n(\widetilde{F}) = \pi_n(\widetilde{F}) = \bigoplus_I \mathbb{Z},$$

where $\bigoplus_I \mathbb{Z}$ denotes direct sum of a collection of copies of \mathbb{Z} indexed by I . Also,

$$\begin{aligned} H^n(\widetilde{F}, \mathbb{Z}N)^N &= (Hom_{\mathbb{Z}}(H_n(\widetilde{F}), \mathbb{Z}N))^N = (Hom_{\mathbb{Z}}(\pi_n(\widetilde{F}), \mathbb{Z}N))^N = \\ &= Hom_{\mathbb{Z}N}(\pi_n(\widetilde{F}), \mathbb{Z}N) = Hom_{\mathbb{Z}N} \left(\bigoplus_I \mathbb{Z}, \mathbb{Z}N \right) = \prod_I \mathbb{Z}N, \end{aligned}$$

where $\prod_I \mathbb{Z}N$ denotes direct product of a collection of copies of $\mathbb{Z}N$ indexed by I . Hence

$$\bigoplus_I \mathbb{Z} = \prod_I \mathbb{Z}N.$$

Note that as an abelian group $\mathbb{Z}N$ is a direct sum of (a non-zero number of) copies of \mathbb{Z} . Since $f : M \rightarrow S$ is an irrational topological Lefschetz fibration (this is exactly the place where we use irrationality of f) the set I is countably infinite. Therefore, $\bigoplus_I \mathbb{Z}$ is countable and $\prod_I \mathbb{Z}N$ is uncountable and the two cannot be equal. This contradiction shows that N cannot be a PD($2n$) group and hence

cannot be of type FP. In particular, there does not exist a quasiprojective $K(N, 1)$ space. \square

Corollary 7.10. *Let M be a smooth projective variety of complex dimension $n + 1$ with $n \geq 2$, such that \widetilde{M} is Stein and $\pi_{n+1}(M) = 0$. Suppose also that M realizes $ht_S(\pi_1(M))$. Let $f : M \rightarrow S$ be an irrational Lefschetz fibration that is not a Kodaira fibration, and let F denote the regular fiber of the fibration. Denote $N = \pi_1(F)$. Then*

- a) $\pi_k(F) = 0$ for $1 < k < n$,
- b) $\pi_n(F)$ is a free $\mathbb{Z}N$ -module, with generators in one-to-one correspondence with $K \times \pi_1(S)$,
- c) \widetilde{F} is homotopy equivalent to a wedge of n -spheres,
- d) N cannot be of type FP; in particular, there does not exist a quasiprojective $K(N, 1)$ space, and
- e) \widetilde{F} is Stein.

Proof. Since M realizes $ht_S(\pi_1(M))$, we have $\pi_k(M) = 0$ for $1 < k < n + 1$. By hypothesis $\pi_{n+1}(M) = 0$. Hence by Theorem 2.7 and the Hurewicz' Theorem, \widetilde{M} is contractible. Theorem 7.6 now applies directly to prove (a)–(d).

Since $\pi_1(F)$ injects into $\pi_1(M)$, it follows that \widetilde{F} is a complex analytic submanifold of the Stein manifold \widetilde{M} . Hence \widetilde{F} is Stein, proving (e). \square

7.2.1. Products of Riemann surfaces. We modify the notation of Corollary 7.10 slightly: let M be a smooth projective variety with contractible universal cover. We now turn to the specific examples of M constructed in [DPS09], viz. M is a product $S_1 \times \cdots \times S_r$ of $r > 2$ Riemann surfaces, S is a torus, and f is a branched cover restricted to every factor S_i . Dimca, Papadima and Suciu construct explicit such examples in [DPS09]. We provide below a criterion at the level of fundamental groups that furnish (in a sense) all examples arising this way. The main new ingredient is the following theorem of Edmonds [Edm79]:

Theorem 7.11 ([Edm79]). *A map $f : M \rightarrow N$ of closed, orientable surfaces is homotopic to a branched covering if and only if*

$$f_* : \pi_1(M) \rightarrow \pi_1(N)$$

is injective or

$$\deg(f) > [\pi_1(N) : f_*(\pi_1(M))].$$

If $f_ : \pi_1(M) \rightarrow \pi_1(N)$ is injective, then f is homotopic to a covering map. If $\deg(f) > [\pi_1(N) : f_*(\pi_1(M))]$, then f is homotopic to a branched cover map but not to a covering map.*

We collect together a few elementary facts.

Lemma 7.12. *Let M be a product $S_1 \times \cdots \times S_r$ of closed, orientable surfaces S_i of positive genus with $r \geq 2$, and let S be another closed, orientable surface.*

Fix $(x_1, \dots, x_r) \in S_1 \times \cdots \times S_r$ and let $j_k : S_k \rightarrow S_1 \times \cdots \times S_r$ be given by $j_k(y_k) = (x_1, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_r)$. Let $f : M \rightarrow S$ be a map such that

- a) $f_*(\pi_1(M)) (\subset \pi_1(S))$ is not cyclic, and
- b) $f_* \circ j_{k*}$ is not the trivial homomorphism for any k .

Then S must be a torus.

Proof. By hypothesis (b), for all $k = 1, \dots, r$, there exists $a_k \in \pi_1(S_k)$ such that $f_* \circ j_{k*}(a_k)$ is non-trivial. Hence the images $f_* \circ j_{l*}(\pi_1(S_l))$ commute with $f_* \circ j_{k*}(a_k)$ for all $l \neq k$.

By hypothesis (a) $f_*(\pi_1(M)) \neq \langle f_* \circ j_{k*}(a_k) \rangle$, the (non-trivial) cyclic group generated by $f_* \circ j_{k*}(a_k)$. Since $\pi_1(S)$ is torsion-free, this implies that $\pi_1(S)$ contains a free abelian group of rank greater than one. Hence S must be a torus. \square

Lemma 7.13. *Let M be a product $S_1 \times \dots \times S_r$ of Riemann surfaces of positive genus equipped with the product complex structure. Let S be another Riemann surface of positive genus. Fix $(x_1, \dots, x_r) \in S_1 \times \dots \times S_r$ and let $j_k : S_k \rightarrow S_1 \times \dots \times S_r$ be as in Lemma 7.12. Let*

$$f : M \rightarrow S$$

be a map such that $f \circ j_k$ is non-constant for all k . Then S has genus one and $f \circ j_k$ is a branched cover for all k .

Proof. Since any non-constant complex analytic map between Riemann surfaces is a branched cover, $f \circ j_k$ is a branched cover for all k .

Further, any branched cover has positive degree, and hence $[\pi_1(S) : f_* \circ j_{k*}(\pi_1(S_k))]$ is finite for all k . In particular, the hypotheses (a) and (b) of Lemma 7.12 are satisfied. Therefore, the genus of S is one. \square

We shall say that a complex analytic map $f : M \rightarrow S$ between a product $M = S_1 \times \dots \times S_r$ of Riemann surfaces and a Riemann surface S is *completely non-degenerate* if $f \circ j_k$ is a branched cover for all k . The next proposition characterizes homotopy classes of completely non-degenerate maps and shows that the examples constructed in [DPS09] account for all irrational Lefschetz fibrations of a product of Riemann surfaces.

Proposition 7.14. *Let M be a product $S_1 \times \dots \times S_r$ of closed, orientable surfaces S_i of positive genus with $r \geq 2$, and let S be another closed, orientable surface of positive genus. Fix $(x_1, \dots, x_r) \in S_1 \times \dots \times S_r$ and let $j_k : S_k \rightarrow S_1 \times \dots \times S_r$ be as in Lemma 7.12. If a (continuous) map $f : M \rightarrow S$ is homotopic to a completely non-degenerate complex analytic map (for some complex structures on S_k and S) then:*

- a) S has genus one, and
- b) for all k , the homomorphism $(f \circ j_k)_* : \pi_1(S_k) \rightarrow \pi_1(S)$ is either injective or

$$\deg((f \circ j_k)) > [\pi_1(N) : f_*(\pi_1(M))].$$

Conversely, if a continuous map $f : M \rightarrow S$ satisfies (b), then S has genus one and there exist complex structures on S_k, S such that $f : M \rightarrow S$ is homotopic to a completely non-degenerate complex analytic map ϕ for the product complex structure on M . Further, there exist complex analytic maps $\phi_k : S_k \rightarrow S$ such that $\phi(z_1, \dots, z_k) = m(\phi_1(z_1), \dots, \phi_r(z_r))$, where $m(.,.)$ denotes addition on S .

Proof. Suppose f is homotopic to a completely non-degenerate complex analytic map. Then by Lemma 7.13, the surface S has genus one. Also by hypothesis, for all k , the map $f \circ j_k$ is homotopic to a branched cover. Hence by Theorem 7.11, the homomorphism $(f \circ j_k)_* : \pi_1(S_k) \rightarrow \pi_1(S)$ is either injective or $\deg((f \circ j_k)) > [\pi_1(N) : f_*(\pi_1(M))]$.

Conversely, suppose that for all k , the homomorphism $(f \circ j_k)_* : \pi_1(S_k) \rightarrow \pi_1(S)$ is either injective or $\deg((f \circ j_k)) > [\pi_1(N) : f_*(\pi_1(M))]$. Then again by

Theorem 7.11, for all k , the map $f \circ j_k$ is homotopic to a branched cover map ϕ_k . Hence by Lemma 7.12, the surface S has genus one. Equip S with some complex structure and pull back the complex structure to S_k by ϕ_k . Equip M with the resulting product complex structure J . Let $m(.,.)$ denote addition on S . Define $\phi : (M, J) \rightarrow S$ by $\phi(z_1, \dots, z_r) = m(\phi_1(z_1), \dots, \phi_r(z_r))$. It is easy to check that ϕ and f induce the same map at the level of fundamental groups (as $(f \circ j_k)_* = (\phi \circ j_k)_*$ for all k). Since M and S are Eilenberg-MacLane spaces, ϕ and f are homotopic. \square

We finally come to a topological characterization of irrational Lefschetz fibrations of a product of Riemann surfaces. We continue using j_k as in Proposition 7.14.

Theorem 7.15. *Let M be a product $S_1 \times \dots \times S_r$ of closed, orientable surfaces S_i of positive genus with $r \geq 2$. If there exists a product complex structure on M (coming from complex structures on S_k) and an irrational Lefschetz fibration $f : M \rightarrow S$ which is not a Kodaira fibration, then*

- a) S has genus one, and
- b) $\deg((f \circ j_k)) > [\pi_1(N) : f_*(\pi_1(M))]$ for all k .

Conversely, suppose that S is a closed orientable surface of positive genus and there exists a continuous map $f : M \rightarrow S$ such that $\deg((f \circ j_k)) > [\pi_1(N) : f_(\pi_1(M))]$ for all k . Then S has genus one and there exist complex structures on S_k, S such that $f : M \rightarrow S$ is homotopic to a Lefschetz fibration (for M equipped with the product complex structure). Further, there exist complex analytic maps $\phi_k : S_k \rightarrow S$ such that $\phi(z_1, \dots, z_k) = m(\phi_1(z_1), \dots, \phi_r(z_r))$, where $m(.,.)$ denotes addition on S .*

Proof. For the forward direction, observe by Proposition 7.14, that it suffices to rule out the possibility that $(f \circ j_k)_* : \pi_1(S_k) \rightarrow \pi_1(S)$ is injective for some k . If not, then $f \circ j_k$ is homotopic to a unramified covering map. It follows that f is a local submersion at all points, and hence f is homotopic to a Kodaira fibration. But a Lefschetz fibration $f : M \rightarrow S$ which is not a Kodaira fibration cannot be homotopic to a Kodaira fibration. It follows that $(f \circ j_k)_* : \pi_1(S_k) \rightarrow \pi_1(S)$ cannot be injective for any k .

We now prove the converse direction. By Proposition 7.14 the surface S has genus one. Also, for all k , the map $f \circ j_k$ is homotopic to a branched cover map but not a covering map. We can further homotope $f \circ j_k$ so that each branch point b_{kn} has degree exactly 2. Let ϕ_k denote the resulting branched cover maps. Now (as in the proof of the converse direction of Proposition 7.14), equip S with some complex structure and, for all k , pull back the complex structure to S_k by ϕ_k . Equip M with the resulting product complex structure and define $\phi : M \rightarrow S$ by $\phi(z_1, \dots, z_r) = m(\phi_1(z_1), \dots, \phi_r(z_r))$. Note that any tuple of the form $(b_{1i_1}, \dots, b_{ri_r})$ is a non-degenerate critical point of ϕ . Also ϕ is homotopic to f . It follows that $f : M \rightarrow S$ is homotopic to a Lefschetz fibration. \square

8. EXAMPLES, COUNTEREXAMPLES AND QUESTIONS

We shall give examples of groups having arbitrarily large, but finite homotopical height. Several of these groups G admit a closed manifold as a $K(G, 1)$ space but still have finite homotopical height.

A Theorem that will be used frequently is the following strong version of the Lefschetz hyperplane theorem due to Goresky and MacPherson [GM88, Section 2].

Theorem 8.1 ([ABCKT, Theorem 8.8]). *Let V be a smooth (not necessarily closed) complex algebraic variety with $f : V \rightarrow \mathbb{C}\mathbb{P}^N$ a holomorphic map with finite fibers. If $L \subset \mathbb{C}\mathbb{P}^N$ is a generic linear subspace, then $\pi_i(V, f^{-1}(L)) = 0$ for $i \leq \dim_{\mathbb{C}}(f^{-1}(L))$. Equivalently, the inclusion of $f^{-1}(L)$ in V induces an isomorphism of π_i for $i < \dim_{\mathbb{C}}(f^{-1}(L))$ and a surjection for $i = \dim_{\mathbb{C}}(f^{-1}(L))$.*

8.1. Non-uniform lattices in CH^n , $n > 2$. These examples are due to Toledo [Tol90], who showed that a non-uniform lattice Γ in CH^n is the fundamental group of a smooth projective variety if and only if $n > 2$. Roughly, the construction is as follows. Toledo constructs an embedding of the manifold $M = CH^n/\Gamma$ in $\mathbb{C}\mathbb{P}^N$ such that the compactification has just one additional point $P \in \mathbb{C}\mathbb{P}^N$. A generic hyperplane avoids P and cuts M in a (complex) $(n-1)$ -dimensional manifold N . The universal cover \tilde{N} of N is Stein since CH^n is Stein. Also, $\pi_i(N) = \pi_i(M)$ for $i < n-1$ by Theorem 8.1. Therefore, $ht_{\mathcal{S}}(\Gamma) \geq n-1$, and we have a sequence of examples of projective groups of arbitrarily large \mathcal{S} -homotopical height.

The cohomological dimension of Γ is $(2n-1)$ as M is a noncompact $K(\Gamma, 1)$ space. So $ht_{\mathcal{S}}(\Gamma) \leq 2n-1$. In fact Γ is a $(2n-1)$ -dimensional duality group, and so $H^2(\Gamma, \mathbb{Z}\Gamma) = 0$. Further, since N is projective, Proposition 6.10 gives that $b_2(\Gamma) > 0$.

The special case of a non-uniform lattice Γ in CH^3 is interesting. The above discussion gives the crude bounds $2 \leq ht_{\mathcal{S}}(\Gamma) \leq 5$. It would be worth evaluating precisely what value $ht_{\mathcal{S}}(\Gamma)$ has in this case.

8.2. Fundamental groups of circle bundles over products of smooth varieties. Let M and N be two smooth complex projective varieties, of (complex) dimension n , with $\pi_2(M) = \pi_2(N) = 0$ and $n \geq 2$. Then there exists a smooth complex projective variety of dimension n with fundamental group G that fits into an exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow G \rightarrow \pi_1(M) \times \pi_1(N) \rightarrow 1.$$

The construction of these examples is again based on Theorem 8.1 and is detailed in Chapter 8 of [ABCKT]. The above group G appears as the fundamental group of a non-trivial \mathbb{C}^* -bundle \mathcal{L} over $M \times N$. The homotopy exact sequence of the fibration gives $\pi_i(\mathcal{L}) = \pi_i(M) \times \pi_i(N)$ for all $i \geq 2$. In particular, we have $\pi_2(\mathcal{L}) = 0$. Also, if both M and N are aspherical (e.g. if they are products of Riemann surfaces of positive genus) then \mathcal{L} is aspherical and quasi-projective. In such a case, $G \in \mathcal{QPK}$. There exists a complex projective subvariety W of \mathcal{L} of complex dimension n such that $\pi_i(W) = \pi_i(\mathcal{L})$ for $i < n$; this is explained in Chapter 8 of [ABCKT].

If the universal covers \tilde{M} and \tilde{N} of M and N respectively are both Stein, then so is the universal cover $\tilde{\mathcal{L}}$ of \mathcal{L} . Hence the universal cover \tilde{W} of W is also Stein. It follows that $ht_{\mathcal{S}}(G) \geq n$. Also, the cohomological dimension $cd(G) = 4n+1$. So we get the bounds

$$n \leq ht_{\mathcal{S}}(G) \leq 4n+1.$$

A special class of these examples, due to Campana [Cam95], are the Heisenberg groups H^{4n+1} given by

$$1 \rightarrow \mathbb{Z} \rightarrow H^{4n+1} \rightarrow \mathbb{Z}^{4n} \rightarrow 1$$

obtained by taking both M, N to be abelian varieties. Macinic points out that H^{4n+1} is $(2n-1)$ -stage formal, but not $2n$ -stage formal [Mac10]. Hence by Proposition 5.8, we get the better bound

$$n \leq ht_S(G) \leq ht_K(G) \leq 2n$$

in this case.

8.3. Kähler versus Symplectic. We give examples of groups that admit a symplectic $K(G, 1)$ but not a Kähler $K(G, 1)$ and upgrade them to give examples of Kähler groups that admit a symplectic $K(G, 1)$ but not a Kähler $K(G, 1)$.

As usual let H^{2n+1} denote the integral Heisenberg group fitting into the exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow H^{2n+1} \longrightarrow \mathbb{Z}^{2n} \longrightarrow 1.$$

Let M denote the 3-manifold with Nil-geometry having H^3 as its fundamental group. Then $M^3 \times M^3$ admits a symplectic structure. This can be shown directly as follows. Let $M_i, i = 1, 2$ denote two copies of M^3 . Let $p_i : M_i \longrightarrow T_i$ denote the natural projections onto tori T_i . Let α_i, β_i denote closed 1-forms generating $H^1(T_i)$, $a_i = p_i^*(\alpha_i), b_i = p_i^*(\beta_i)$. Also let f_i be a closed 1-form in $H^1(M_i)$ representing the dual to the homology class represented by the circle fiber. Let $C_i = f_i \wedge a_i, D_i = f_i \wedge b_i$. Let $\pi_i : M_1 \times M_2 \rightarrow M_i$ denote the natural projections for $i = 1, 2$. Abusing notation slightly, we denote $\pi_i^* C_i$ by C_i and $\pi_i^* b_i$ by b_i .

Now, define a 2-form on $M_1 \times M_2$ by

$$\omega = C_1 + C_2 + b_1 \wedge b_2.$$

Then a direct computation shows that $\omega^3 = 3C_1 \wedge C_2 \wedge b_1 \wedge b_2$, which is a positive multiple of the volume form and hence $(M_1 \times M_2, \omega)$ is symplectic. The universal cover of $M_1 \times M_2$ is \mathbb{R}^6 and hence $M_1 \times M_2$ is a $K(H^3 \times H^3, 1)$.

However, $H^3 \times H^3$ is not Kähler (see [Mac10] for instance). In particular, there is no Kähler $K(H^3 \times H^3, 1)$. Note that by Theorem 4.3, $M_1 \times M_2$ admits a complex structure. It follows that the complex and symplectic structures cannot be compatible.

We now give an example of a of Kähler group that admits a symplectic $K(G, 1)$ manifold but not a Kähler $K(G, 1)$ manifold. By a result of Campana [Cam95], the group H^{2n+1} is Kähler for $n > 3$. As in the above example let N_1 and N_2 denote 2 copies of the $(2n+1)$ -dimensional Nil-manifold having H^{2n+1} as its fundamental group. Then as before $N_1 \times N_2$ is a symplectic $K(G, 1)$ for the Kähler group $G = H^{2n+1} \times H^{2n+1}$. If $G = H^{2n+1} \times H^{2n+1}$ admits a Kähler $K(G, 1)$, then it must be homotopy equivalent to $N_1 \times N_2$. Hence the cohomology of $N_1 \times N_2$ must satisfy the hard Lefschetz property, i.e. there must be an $\omega \in H^2(N_1 \times N_2)$ such that

$$\omega^{k \cup} : H^{2n+1-k}(N_1 \times N_2) \longrightarrow H^{2n+1+k}(N_1 \times N_2)$$

is an isomorphism. But this is not possible by a Theorem of Benson and Gordon [BG88]. Hence G does not admit a Kähler $K(G, 1)$. In particular, there is no equivariant version of Eliashberg's celebrated Theorem 3.4.

8.4. C-Symplectic Manifolds. Recall that c-symplectic (cohomologically symplectic) manifolds are closed manifolds M^{2n} with $\omega \in H^2(M)$ such that $\omega^n \neq 0$. Let CSP denote the collection of closed symplectic manifolds with a c-symplectic structure.

Proposition 8.2. $ht_{CSP}(\mathbb{Z}^{2n-1}) \geq n - 1$ for $n > 2$.

Proof. Let E be an elliptic curve and E^n denote its n -fold product ($n > 2$) embedded in some projective space. Let X be a hyperplane section of E^n , and let $i : X \hookrightarrow E^{2n}$ be the inclusion. Then $\pi_1(X) = \mathbb{Z}^{2n}$. The natural projection $E^{2n} \rightarrow E^{2(n-1)}$ induces a finite map $f : X \rightarrow E^{2(n-1)}$ of smooth complex projective varieties such that $f^*(\omega) = \eta$ for Kähler forms $\omega \in H^2(E^{2(n-1)}, \mathbb{Z})$ and $\eta \in H^2(X, \mathbb{Z})$. Then $f^*(\omega^{(n-1)}) = \eta^{(n-1)} \neq 0$ gives a (cohomologically) non-vanishing top form on X .

Next $f_* : \pi_1(X) \rightarrow \pi_1(E^{2(n-1)})$ factors through the inclusion isomorphism $i_* : \pi_1(X) \rightarrow \pi_1(E^{2n})$ and hence the kernel K is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Attach a 2-handle $D^2 \times S^{2n-4}$ to an S^1 in X representing a primitive element of K . Let Y denote the manifold obtained from X by this surgery. Then $\pi_1(Y) = \mathbb{Z}^{2n-1}$, and $\pi_i(Y) = \pi_i(X) = 0$ for $1 < i < n - 1$ (the last assertion follows from the Lefschetz hyperplane Theorem and the fact that the surgery does not introduce homotopy groups in dimension less than $2n - 4$). Extending f over the attaching 2-cell and smoothing it out, we obtain a smooth map $g : Y \rightarrow E^{2(n-1)}$ of positive degree (by the local degree formula for instance). Hence $g^*(\omega^{(n-1)}) \in H^{2(n-1)}(Y)$ is non-zero. Choosing $\alpha = g^*(\omega) \in H^2(Y)$, we obtain a c-symplectic structure on Y . (See [IKRT04] for a closely related set of examples, where the starting point is the symplectic category, rather than the class of projective manifolds.) \square

The class of examples given by Proposition 8.2 can be generalized.

Example 8.3. Let W be an n -dimensional smooth variety whose Albanese variety is a smooth n -dimensional abelian variety A . Let $\phi : W \rightarrow A$ be the (finite) Albanese map. As before, let X be a smooth hyperplane section in A . We can arrange such that $V = \phi^{-1}(X)$ is a smooth subvariety of W . Also, $H_1(V) = H_1(A) = \mathbb{Z}^{2n}$.

The natural smooth projection $A \rightarrow T^{2(n-1)}$ induces a finite map $f : V \rightarrow T^{2(n-1)}$, which automatically has positive degree. Then, as before, $f_* : H_1(V) \rightarrow H_1(T^{2(n-1)})$ has kernel K of rank two. Attach, as before, a 2-handle $D^2 \times S^{2n-4}$ to an S^1 in V representing a non-torsion element of K . Let Y denote the manifold obtained from V by this surgery. Then $H_1(Y)$ has rank $2n - 1$. Extend f over the attaching 2-cell as before and smooth it out to obtain a smooth map $g : Y \rightarrow E^{2(n-1)}$ of positive degree. Choosing $\alpha = g^*(\omega) \in H^2(Y)$, we obtain a c-symplectic structure on Y .

If in addition V is constructed in such a way that \tilde{V} is homotopy equivalent to a wedge of $(n - 1)$ -spheres, then $\pi_i(Y) = \pi_i(V) = 0$ for $1 < i < n - 1$ as before and hence $ht_{CSP}(\pi_1(Y)) \geq n - 1$.

8.5. Further Questions. We end with some further questions which we were led to during the course of our study. The following is a special case of the Shafarevich conjecture.

Question 8.4. If a smooth projective variety X is a $K(G, 1)$, is \tilde{X} Stein? This is true for \tilde{X} a product of bounded domains and affine spaces.

Question 8.5. Let $ht_{\mathcal{S}}(G) = n$, and M be a smooth projective variety of (complex) dimension m realizing the \mathcal{S} -homotopical height of G . If in addition the action of $G (= \pi_1(M))$ on $\pi_n(M)$ is the trivial action, and $H^n(G, \mathbb{Z}G) \neq 0$ is G virtually a PD(n) group, with $n = 2m$?

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